Simple Deformations

- **Axial load on a uniform bar**
  \[ \sigma = \frac{P}{A} = \varepsilon E \]
  \[ \varepsilon = \frac{P}{AE} \]
  \[ \delta = \varepsilon L = \frac{PL}{AE} \]
  Stiffness:
  \[ k = \frac{P}{\delta} = \frac{AE}{L} \]
  Hamrock Section 4.3

- **Torsional load (torque) on a uniform round bar**
  \[ \theta = \frac{TL}{JG} \text{ Radians} \]
  \[ J = \frac{\pi}{2} r^4 \]
  Stiffness:
  \[ k = \frac{T}{\theta} = \frac{JG}{L} \text{ Nm / Radian} \]
  Hamrock Section 4.4.1
Beam Flexure

• For a uniform beam in pure bending, 

\[ \frac{1}{r} = \frac{M}{EI} \quad (\text{Eq. 4.47}) \]

So a large Moment means a small radius of curvature, \( r \)

Because

\[ \frac{d^2 y}{dx^2} = -\frac{M}{EI} \]

We can integrate our way from Moment, \( M \), to the deflection, \( y \).
Example Beam Loading

20 inch long beam with \( w = 80 \text{lb/in load} \)

Shear, \( V \) (Lb)

\[
\frac{dV}{dX} = -w
\]

Moment, \( M \) (In.Lb.)

\[
\frac{dM}{dX} = V
\]

\[ \text{Slope} = \frac{-1600 \text{lb}}{20 \text{in.}} \]

Large + Slope

Large - Slope
Beam Deflection

Moment, M
(In.Lb.)

Θ & y depend on:
1. Material
2. Beam Section

Slope, EIΘ
(Rad; EI=1)

\[ \frac{dEI\theta}{dX} = M \]

Deflection, EIy
(In; EI=1)

\[ \frac{dEIy}{dX} = EI\theta \]
ME311 Approach

1. You must be able to draw the V & M diagrams to find the max bending and transverse shear stresses in beams. (This is really stress, but it is the basis of deflection.)

2. Understand the slope and deflection concept, but because it is tedious, use tables like Hamrock Appendix D or a handbook to determine beam deflections.

3. Use Superposition to handle combined loadings (including loads in different planes, like Horiz & Vert).

4. Understand how to use slopes and rotations.

5. Use a program (like MDSolids or Excel etc.) to solve the deflection.

6. For complicated structures, use Finite Element or Castigliano.
Hamrock Appendix D Beams

1. Cantilever—end load

\[ R_1 = V = F \quad M_1 = Fl \]
\[ M = F(x - l) \]
\[ y = \frac{Fx^2}{6EI} \quad (x - 3l) \]
\[ y_{\text{max}} = -\frac{Fl^3}{3EI} \]

4. Cantilever—moment load

\[ R_1 = V = 0 \quad M_1 = M = M_B \]
\[ y = \frac{MBx^2}{2EI} \]
\[ y_{\text{max}} = \frac{MBl^2}{2EI} \]

What shape is this deflection?
Hamrock Appendix D Beams

2 Cantilever—intermediate load

3 Cantilever—uniform load

5 Simple supports—center load

6 Simple supports—intermediate load

7 Simple supports—uniform load

8 Simple supports—moment load

9 Simple supports—twin loads

10 Simple supports—overhanging load

14 Fixed supports—center load
### Hamrock Appendix D.2

<table>
<thead>
<tr>
<th>Base</th>
<th>b</th>
<th>0.035 m</th>
</tr>
</thead>
<tbody>
<tr>
<td>Height</td>
<td>h</td>
<td>0.08 m</td>
</tr>
<tr>
<td>Length</td>
<td>L</td>
<td>1.7 m</td>
</tr>
<tr>
<td>Modulus</td>
<td>E</td>
<td>207 GPa</td>
</tr>
<tr>
<td>Moment of Inertia</td>
<td>I</td>
<td>1.49333E-06 m^4</td>
</tr>
</tbody>
</table>

Load | P | 5000 N |
Location | a | 1 m |

**Equations:**

Deflection & Slope in Excel: Beam D.2

\[ \text{Deflection } = \begin{cases} \frac{P}{EI} \left(\frac{0.5X^2}{2} - aX\right) & \text{if } X < a \\ \frac{P}{EI} \left(\frac{a^2}{2} - X\right) & \text{otherwise} \end{cases} \]

Position (m) | Deflection (m) | Slope (Rad) |
--- | --- | --- |
0.000 | 0.00000 | 0.00000 |
0.085 | -0.00006 | -0.00132 |
0.170 | -0.00022 | -0.00252 |
0.255 | -0.00048 | -0.00360 |
0.340 | -0.00083 | -0.00456 |
0.425 | -0.00125 | -0.00541 |
0.510 | -0.00175 | -0.00615 |
0.595 | -0.00230 | -0.00676 |
0.680 | -0.00289 | -0.00726 |
0.765 | -0.00353 | -0.00764 |
0.850 | -0.00419 | -0.00791 |
0.935 | -0.00487 | -0.00805 |
1.000 | -0.00539 | -0.00809 |
1.020 | -0.00555 | -0.00809 |
1.105 | -0.00624 | -0.00809 |
1.190 | -0.00693 | -0.00809 |
1.275 | -0.00762 | -0.00809 |
1.360 | -0.00830 | -0.00809 |
1.445 | -0.00899 | -0.00809 |
1.530 | -0.00968 | -0.00809 |
1.615 | -0.01037 | -0.00809 |
1.700 | -0.01105 | -0.00809 |

---

**Diagram:**

Deflection Due to 5,000N Force at 1 m

- **Deflection:**
  - Blue line
- **Slope:**
  - Magenta line

Position (m) vs. Deflection (m); Slope (Rad)
Deflection & Slope in MDSolids: Beam D.2

\[ P_1 = 5000 \text{N} \]

\[ EI = 309,120 \text{Nm}^2 \]
Basic Angle Relationship

\[ \theta = \frac{S}{r}; \quad S = r \times \theta \]

Θ is in Radians

For small angles, \( T \approx S \)

How many degrees is a Radian?
How much does the loaded end of the arm deflect?

Breaking it into pieces and looking at the FBD will help.

This piece bends from the Moment and twists from the Torque.

Note that the moment turns into a torque where the arm bends the corner.

This piece bends from the Moment.
Deflection of the loaded end point is the sum of three deflections:

1. Bending of the 6” Rod due to the 300 lb load.
   \[ y_1 = \frac{Fl^3}{3EI} \]

2. Twisting of the 6” Rod due to the 1200 in.lb torque, with rotation of the 4” Rod.
   \[ \theta = \frac{Ll}{JG}, \quad y_2 = r\theta \]

3. Bending of the 4” Rod due to the 300 lb load.
   \[ y_3 = \frac{Fl^3}{3EI} \]
Piece-Wise Deflection of a Crank Arm

\[ r = 0.375'' \quad I = \frac{\pi r^4}{4} = 0.01553 \text{in}^4 \]

\[ J = \frac{\pi r^4}{2} = 2I = 0.03106 \text{in}^4 \]

\[ E = 30 \times 10^6 \text{ psi}, \quad G = 11.5 \times 10^6 \text{ psi} \]

\[ y_1 = \frac{Fl^3}{3EI} = \frac{(300)(6)^3}{3(30 \times 10^6)(0.01553)} = 0.0464 \text{in} \]

\[ y_2 = r\theta = \frac{rTl}{JG} = \frac{(4)(1200)(6)}{(0.03106)(11.5 \times 10^6)} = 0.0806 \text{in} \]

\[ y_3 = \frac{Fl^3}{3EI} = \frac{(300)(4)^3}{3(30 \times 10^6)(0.01553)} = 0.0137 \text{in} \]

\[ y_{total} = y_1 + y_2 + y_3 = 0.1407 \text{in} \]
Crank Arm Deflection by FEA
Traffic Light Pole - Deflection

What’s going on Here

L1 = 96 in.  L2 = 116 in.

From U of Arkansas FEMur
Traffic Light Calculations
Simplified Case With One Beam Size

Use Appendix Formulas and Superposition

1. Cantilever—end load
   \[ R_1 = V = F \quad M_1 = -Fl \]
   \[ M = F(x - l) \]
   \[ y = \frac{Fx^2}{6EI} (x - 3l) \]
   \[ y_{\text{max}} = -\frac{Fl^3}{3EI} \]

2. Cantilever—intermediate load
   \[ R_1 = V = F \quad M_1 = -Fa \]
   \[ M_{AB} = F(x - a) \quad M_{BC} = 0 \]
   \[ y_{AB} = \frac{Fx^2}{6EI} (x - 3a) \]
   \[ y_{BC} = \frac{Fa^2}{6EI} (a - 3x) \]
   \[ y_{\text{max}} = \frac{Fa^2}{6EI} (a - 3l) \]
Traffic Light Calculations
Simplified Case With One Beam Size

Assume the support beam is AISC Standard Shape 6” Steel Pipe
OD = 6.625”
ID = 6.065”
(0.28” wall)
which has an Area Moment of Inertia of
28.14in$^4$
E = 29 x $10^6$ PSI

Tip: $y_{max} = -\frac{Fl^3}{3EI}$

Mid: $y_{max} = \frac{Fa^2}{6EI}(a - 3l)$
How Did We Do? Compare with MDSolids:

What if we changed to a 5” Pipe, with \( I = 15.166 \text{ in}^4 \) ?

What if we changed to a 6” Aluminum Pipe, with \( E = 10.5 \text{MSI} \) ?
Traffic Light Calculations
Real Case With Two Beam Sizes

Slope @ Tip due to F:
\[ \frac{dy}{dx} = \frac{3Fx^2}{6EI} - \frac{2Fx3l}{6EI} = \frac{Fx^2}{2EI} - \frac{Fxl}{EI} \]
\[ y_{max} = -\frac{Fx^3}{3EI} \]
\[ dy = \frac{F(x)(x - l)}{EI} \]

Slope @ Tip due to M:
\[ \frac{dy}{dx} = \frac{2M_Bx}{2EI} = \frac{M_Bx}{EI} \]

Deflection at tip due to slope is \( \theta \times L_1 \)
Traffic Light Calculations
Even More Real Case With Two Beam Sizes and Including the Weight of the Beams

1. Cantilever—end load

2. Cantilever—uniform load

3. Cantilever—moment load

The 6” Steel tube weighs 19lb/ft = 1.58lb/in.

Plus, calculate all the end slopes/rotations!
Castigliano’s Theorem

“ ... the partial derivative of the strain energy, considered as a function of the applied forces acting on a linearly elastic structure, with respect to one of these forces, is equal to the displacement in the direction of the force at its point of application.”

From intota: The deflection of an elastic material subjected to a load in the direction of each load equals the partial derivative of the work of deformation with respect to the component of the force in that direction. This theorem is related to the principle of virtual work, and it applies for elastic material obeying Hooke's law.

This should vaguely make sense, because for a spring, the stored energy is the integrated area under the force vs deflection curve. So it is plausible that differentiating that could get you to the deflection.
1. You write an expression for the total strain energy in your structure, based on each type of loading (see table).

2. If you want to know a deflection where there isn’t a load applied, just stick a “fictitious” load, \( Q \), there.

3. Then take partial derivatives of the energy wrt the loads.

4. Then set \( Q=0 \) and voilà!

<table>
<thead>
<tr>
<th>Loading Type</th>
<th>Strain Energy Constant Variables</th>
<th>Strain Energy General Case</th>
</tr>
</thead>
<tbody>
<tr>
<td>Axial</td>
<td>( u = \frac{F^2 I}{2EA} )</td>
<td>( u = \int_{0}^{l} \frac{F^2 dx}{2EA} )</td>
</tr>
<tr>
<td>Bending</td>
<td>( u = \frac{M^2 I}{2EI} )</td>
<td>( u = \int_{0}^{l} \frac{M dx}{2EI} )</td>
</tr>
<tr>
<td>Torsion</td>
<td>( u = \frac{T^2 I}{2GJ} )</td>
<td>( u = \int_{0}^{l} \frac{T dx}{2GJ} )</td>
</tr>
<tr>
<td>Direct Shear</td>
<td>( u = \frac{F^2 I}{2AG} )</td>
<td>( u = \int_{0}^{l} \frac{F dx}{2AG} )</td>
</tr>
<tr>
<td>Traverse Shear</td>
<td>( u = \frac{KV^2 I}{2GA} )</td>
<td>( u = \int_{0}^{l} \frac{KV dx}{2GA} )</td>
</tr>
</tbody>
</table>
1. The ME311 web site has an analysis of the Crank Arm by Castigliano (shown here).

2. Hamrock has several examples in section 5.6.

3. All of the formulas for deflection in handbooks and Beer & Johnston were figured out by using Castigliano’s theorem.

4. Carlo Alberto Castigliano (1847 – 1884) figured it out when he was 25 years old.

5. There will be no homework or exam problems on using Castigliano’s theorem, but maybe a question.