When it's new, things will be flat and the pressure will be uniform.

\[ F = p \times A = p \times \pi (r_o^2 - r_i^2) \quad \text{[Eqn. 18.4]} \]

Can also integrate \( dA = 2\pi r dr \):

\[
\int p dA = \int_{r_i}^{r_o} p 2\pi r dr = 2\pi p \left( \frac{1}{2} r^2 \right) \bigg|_{r_i}^{r_o} = \pi p (r_o^2 - r_i^2)
\]

This is useful because we need to integrate to get the Torque:

\[ dT = p dA \mu r \], where \( \mu \) is the coefficient of friction

\[ dT = 2\pi r^2 p \mu dr \]

\[
T = \int_{r_i}^{r_o} 2\pi p \mu r^2 dr = 2\pi p \mu \left( \frac{1}{3} r^3 \right) \bigg|_{r_i}^{r_o} = \frac{2}{3} \pi p \mu (r_o^3 - r_i^3)
\quad \text{[Eqn. 18.5]}

Since \[ p = \frac{F}{\pi (r_o^2 - r_i^2)} \]

\[
T = \frac{2}{3} \mu F \frac{(r_o^3 - r_i^3)}{(r_o^2 - r_i^2)} \quad \text{[Eqn. 18.5]}
\]

If the clutch has multiple plates, multiply this Torque by the \( N \) friction interfaces.
Since wear is proportional to pressure times surface velocity, and surface velocity increases linearly with radius, $r$, wear is maximum at the outer radius. Wear reduces the pressure at the rim, and the geometry changes to a uniform wear condition.

If $pv$ is constant and $v$ increases linearly with $r$, then $p$ must decrease with $r$:

$$p = p_{\text{max}} \frac{r_i}{r}$$

Now computing the normal force

$$F = \int_{r_i}^{r_o} 2\pi \ p \ r \ dr = \int_{r_i}^{r_o} 2\pi \ p_{\text{max}} \frac{r_i}{r} \ r \ dr = 2\pi p_{\text{max}} \ r_i (r_o - r_i)$$

[Eqn. 18.10]

Similarly,

$$T = \int_{r_i}^{r_o} 2\pi \ p \mu \ r^2 \ dr = \int_{r_i}^{r_o} 2\pi \ p_{\text{max}} \mu \frac{r_i}{r} \ r^2 \ dr = \pi \mu p_{\text{max}} \ r_i (r_o^2 - r_i^2)$$

[Eqn. 18.12]

Or, in terms of applied force, $F$

$$T = N\mu F \left(\frac{r_o + r_i}{2}\right)$$, where $N$ is the number of friction faces.  

[Eqn. 18.13]

Example of multiplate axial clutch with $N = 4$.
- Four gaps close up on actuation.
- Components slide axially so that each contact surface transmits the actuating force, $F$.
- $N$ is usually even.
For a given applied force, $F$, there is an optimal ratio of $r_o$ to $r_i$.

\[ T = \pi \mu p_{\text{max}} r_i \left( r_o^2 - r_i^2 \right) \]

\[ T = k \times r_i \left( r_o^2 - r_i^2 \right) \quad \text{where} \quad k = \pi \mu p_{\text{max}} \]

\[ T = kr_i r_o^2 - kr_i^3 \]

\[ \frac{d}{dr_i} T = kr_o^2 - 3kr_i^2 = 0 \quad \text{at max or min} \]

\[ kr_o^2 = 3kr_i^2 \]

\[ r_o^2 = 3r_i^2 \]

\[ r_o = \sqrt{3} r_i \quad \text{or} \quad r_i = 0.577 r_o \]

Generally $r_i$ is between .45 and .80 of $r_o$.

**CONE CLUTCHES**

Hamrock §18.3

### Torque Equations

\[ T_{\text{UnifPr}} = \frac{1}{\sin(\alpha)} \left( \frac{2}{3} \mu F \left( r_o^3 - r_i^3 \right) \right) \frac{1}{\left( r_o^2 - r_i^2 \right)} \]

\[ T_{\text{UnifWear}} = \frac{\mu F}{\sin(\alpha)} \frac{r_o + r_i}{2} \]

- Simply take the torque equations for Axial clutches, and multiply them by $1/\sin(\alpha)$
- Since $8^\circ < \alpha < 15^\circ$ usually, the multiplier $\sim 7.2$ to $3.9$.
- $N$ usually is 1 -- cone clutches are not typically ganged.
DRUM BRAKE - SHORT SHOE

DRUM WIDTH = b
FRICITION COEF. = \( \mu \)

FOR A LINING PRESSURE OF \( P_a \),
THE NORMAL FORCE IS \( P = P_a r b \) RADIANS
THE TORQUE IS \( T = \mu P r \)

CASE 1: PIVOT AT ₁
ΣM ABOUT PIVOT:
\[
WD + \mu P A = PC
\]
\[
WD = PC - \mu P A = P (C - \mu A)
\]
\[
W = P \frac{(C - \mu A)}{D}
\]

THE FRICTION TORQUE \( \mu P A \) ADDS TO THE ACTUATION TORQUE.
IF \( \mu A > C \) THEN IS SELF-LOCKING
\[
T = \frac{\mu P WD}{(C - \mu A)} \quad \text{[EQ. 18.28]}
\]

CASE 2: PIVOT AT ₂
ΣM:
\[
WD = PC + \mu P B = P (C + \mu B)
\]
\[
W = P \frac{(C + \mu B)}{D}
\]
\[
T = \frac{\mu P WD}{(C + \mu B)} \quad \text{[EQ. 18.30]}
\]
Problem 18.27 Done As SHORT Shoe
\[ \Sigma M_L \quad \text{cw} \quad \text{ccw} \]
\[ (1000 \times 36) + \mu F_L(15) = (200)(4) + F_L(17) \]
\[ F_L(15\mu - 17) = 800 - 36000 \]
\[ F_L(-13,25) = -35,200 \]
\[ F_L = 2656.6 \text{ lb.} \]

\[ \Sigma M_R \quad \text{cw} \quad \text{ccw} \]
\[ F_R(17) + \mu F_R(15) = (1000)(40) \]
\[ F_R(17 + 3.75) = 40,000 \]
\[ F_R = 1927.7 \text{ lb.} \]

PAD: \[ L = \frac{1}{3}(\pi)(30) = 31.42'' \text{ contact} \]
\[ P_L = \frac{F_L}{L \cdot b} = \frac{2656.6}{3192} = 100 \text{ psi} \]
\[ b = \frac{2656.6}{3192} = 0.845 \text{ in.} \]
\[ P_R = \frac{F_R}{L \cdot b} = \frac{1927.7}{(3192)(0.845)} = 72.6 \text{ psi} \]

TORQUE = \[ \mu r (F_L + F_R) = (0.25)(15)(2656.6 + 1927.7) \]
\[ T = 17,191 \text{ in. lb.} \]
**Drum Brake - Long Shoe**

**Fig. 18.8**

\[ a = d_7 \]
\[ C = d_6 \]
\[ P_a = P_{\text{max}} \]

**Eq. 18.35**

\[ M_f = \frac{M_p abr}{\sin \theta_a} \left[ -r (\cos \theta_2 - \cos \theta_1) - \frac{a}{2} (\sin^2 \theta_2 - \sin^2 \theta_1) \right] \]

Moment on shoe from circumferential friction

**Eq. 18.34**

\[ M_p = \frac{P_ab r a}{\sin \theta_a} \left[ \frac{1}{2} (\theta_2 - \theta_1) \frac{\pi}{180} - \frac{1}{4} (\sin 2\theta_2 - \sin 2\theta_1) \right] \]

Moment from perpendicular forces

\[ \Sigma M \text{ @ pivot} \]

\[ F_C = M_p \pm M_f \]

Sign dependent on direction of rotation

**Torque:**

\[ T = \frac{ \mu P_ab r^2}{\sin \theta_a} (\cos \theta_1 - \cos \theta_2) \] [18.38]
Figure 3.7 Free-body force diagrams for two shoes of an internal brake. Actuating force $Q$, anchor-pin reactions $R$, and distribution and relative magnitude of pressure and friction forces are shown.

Prob. 18.27
AS LONG SHOE

\[ M_f = \frac{MP_a br}{\sin \Theta_a} \left[ -r \left( \cos \Theta_2 - \cos \Theta_1 \right) - \frac{a}{2} \left( \sin^2 \Theta_2 - \sin^2 \Theta_1 \right) \right] \]
\[ = (0.25)(100)b(15) \left[ -15(-0.866 - 0.866) - 8.5(0.25 - 0.25) \right] \]
\[ = 375 \times 25.98 \]
\[ M_f = 9792.5 \text{ in. lb.} \]

\[ M_p = \frac{P_a b r a}{\sin \Theta_a} \left[ \left( \frac{120}{360} \right) \frac{\pi}{2} - \frac{1}{4} \left( \sin 2 \Theta_2 - \sin 2 \Theta_1 \right) \right] \]
\[ = (100)b(15)(17) \left[ 1.0472 - (0.25)(-0.866 - 0.866) \right] \]
\[ = 25,500 \times 1.0472 + 0.433 \]
\[ M_p = 37745.1 \text{ in. lb.} \]

\[ FC = M_p - M_f \]
\[ 35,200 = 37745.1b - 9792.5b \]
\[ b = \frac{35,200}{28,000} = 1.26 \text{ in.} \]

\[ T_{left} = \frac{MP_a b r^2}{\sin \Theta_a} \left( \cos \Theta_1 - \cos \Theta_2 \right) \]
\[ = (0.25)(100)(31.26)(15)^2(0.866 - (-0.866)) \]
\[ T_{left} = 12,276 \text{ lb. in.} \]
Solve for Pressure on Non-Self Actuating Shoe

\[ M_F = \frac{(9742.5)(1.26)}{100} P_{\text{right}} \]

\[ M_P = 122.76 \, P_{\text{right}} \]

\[ M_P = \frac{(37795.1)(1.26)}{100} \, P_{\text{right}} \]

\[ M_P = 475.59 \, P_{\text{right}} \]

\[ F_C = M_P + M_F \]

\[ (40)(1000) = 475.59 \, P_{\text{right}} + 122.76 \, P_{\text{right}} \]

\[ P_{\text{right}} = \frac{40,000}{(475.59 + 122.76)} = \frac{40,000}{598.35} \]

\[ P_{\text{right}} = 66.85 \text{ PSI} \]

\[ T_{\text{right}} = \mu P_{\text{right}} b w^2 (\cos \Theta_1 - \cos \Theta_2) \]

\[ = (0.125)(66.85)(1.26)(15)^2 (2)(0.866) \]

\[ T_{\text{right}} = 8206 \text{ Lb. in.} \]

\[ T_{\text{tot}} = T_{\text{left}} + T_{\text{right}} = 20,482 \text{ Lb. in.} \]
Automotive-type disk clutch. (Courtesy Borg-Warnar Corporation.)