# MEEG3311 Machine Design

## Lecture 10: Springs
(Chapter 17)

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30Nov2023

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School of Engineering

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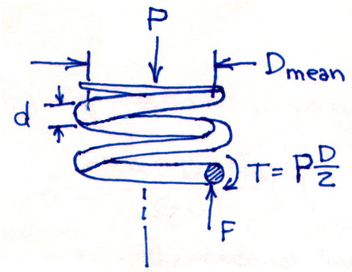
## Compression Springs

A Free Body Diagram of a coil spring (cutting through anywhere on the coil) shows that there must be torsion on the coil to balance the load.

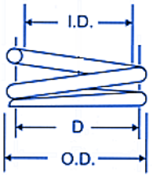


Coil springs have these features:

Wire diameter, d or $d_{wire}$
Coil Mean diameter, D or $D_{mean}$

Coil Inner Diameter, ID = D – d
Coil Outer Diameter, OD = D + d

Spring Index, $C = \frac{D}{d}$



You can think of the OD as the Mean Diameter plus twice the wire radius, so OD = D + 2r = D + d

Hamrock
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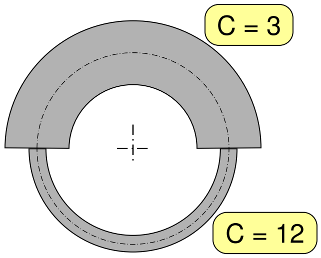
# Spring Index

It is seldom practical to make a spring with an Index, C, less than 3 or greater than 12.

A small Index means a large curvature, and a large index means a small curvature.

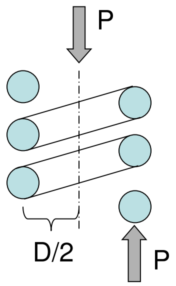


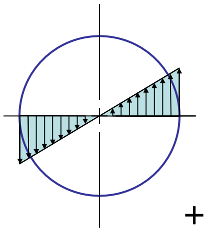
Springs with C in the range of 5 to 10 are preferred.

Springs with small C are hard to manufacture and have large stress concentrations due to the tight curvature.

Hamrock
Eqn. 17.7

# Stress In Springs

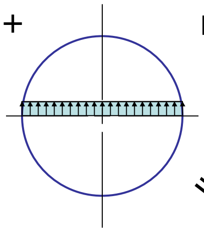




Torsional Shear

$$\tau_{tors,\max} = \frac{Tr}{J} = \frac{Td(32)}{2\pi d^4} = \frac{8PD}{\pi d^3}$$

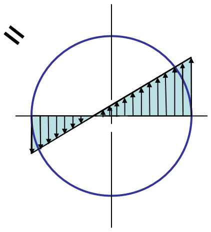
+



Direct Shear

$$\tau_{dir,\max} = \frac{P}{A} = \frac{4P}{\pi d^2}$$

=



Total Shear

$$\tau_{tot,\max} = \frac{8DP}{\pi d^3}\left(1 + \frac{d}{2D}\right)$$

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# Stress In Springs

$$\tau_{tot,\max} = \frac{8DP}{\pi d^3}\left(1 + \frac{d}{2D}\right)$$

The term in parentheses is a constant, so this can be rewritten:

$$\tau_{tot,\max} = \frac{8D\boxed{K_d}P}{\pi d^3} \qquad \text{[Eqn. 17.8]}$$

Where Kd, the Transverse Shear Factor, is

$$K_d = \left(1 + \frac{d}{2D}\right) = \left(\underbrace{1}_{\text{Torsion}} + \underbrace{\frac{1}{2C}}_{\text{Direct}}\right) = \frac{C + 0.5}{C}$$

Use this for **STATIC** loading.

If the Spring Index, C, ranges from 3 to 12, then the Direct Shear is from 1/6th to 1/24th of the Torsional Shear.

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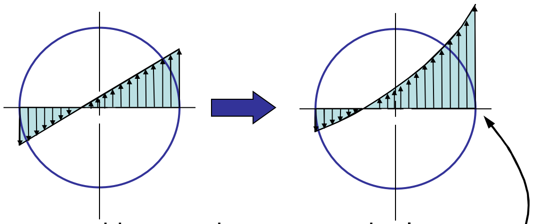
# Spring Stress Exercise

A spring made from 0.1 in. music wire has an outside diameter of 1 inch. If it has a load of 25 Lbs applied to it, what is the maximum shear stress?

# Curvature Effects

Remember curved beams & their stress distribution?



Adding the effect of curvature drives up the stress at the Inner Radius.

For cyclic (dynamic) loading, we include this curvature effect, and write:

$$\tau_{tot,\max} = \frac{8D\bigcirc\!\!\!\!\!K_w P}{\pi d^3}$$ [Eqn. 17.10]

where $$K_w = \frac{4C-1}{4C-4} + \frac{0.615}{C}$$

Curvature

Direct

Use this for **CYCLIC** loading.

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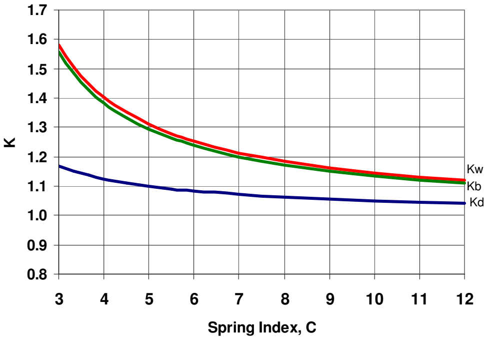
# Shear Factors

Another Cyclic correction factor is the Bergsträsser factor, $K_b$

$$K_b = \frac{4C+2}{4C-3}$$

It is simpler and very close to $K_w$



Clearly, the lower the Index, the higher the curvature, and the higher the max shear.

# Spring Materials

There are a very limited number of materials commonly used for making springs, listed in Table 17.1.

The allowable shear yield of these materials Ssy = 0.40 Sut, [Eqn. 17.3] and the <u>Sut varies with wire size</u>!

The variation is:

$$S_{ut} = \frac{A_p}{d^m} \qquad \text{[Eqn. 17.2]}$$

where Ap and m come from Table 17.2.

Caution: Use Ap in <u>ksi</u> with d in inches, and Ap in <u>MPa</u> with d in mm.

A plot of Sut versus wire diameter for the material in Table 17.2 is shown on the next two slides, first in Metric and then in English units.

How much factor of safety did our spring have?

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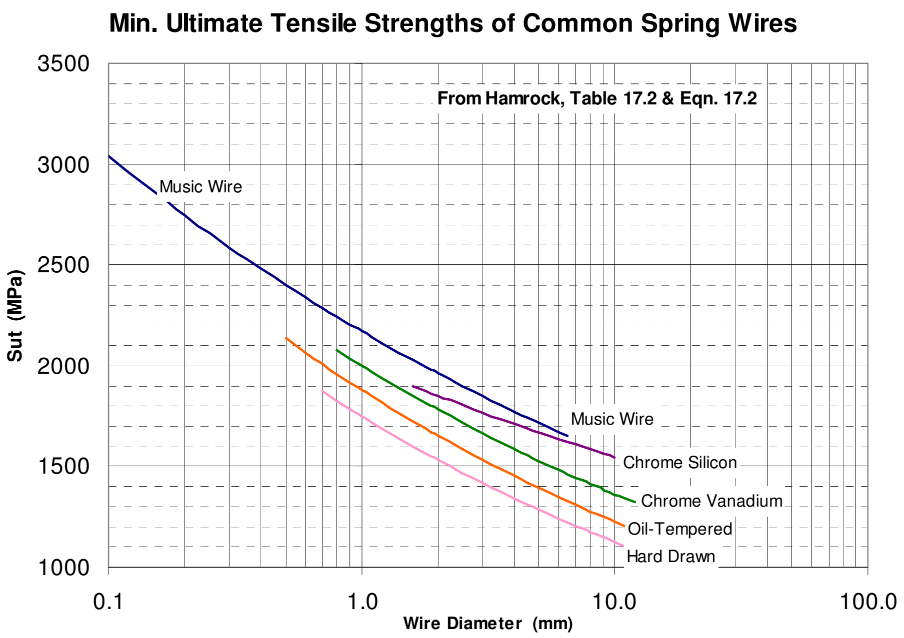

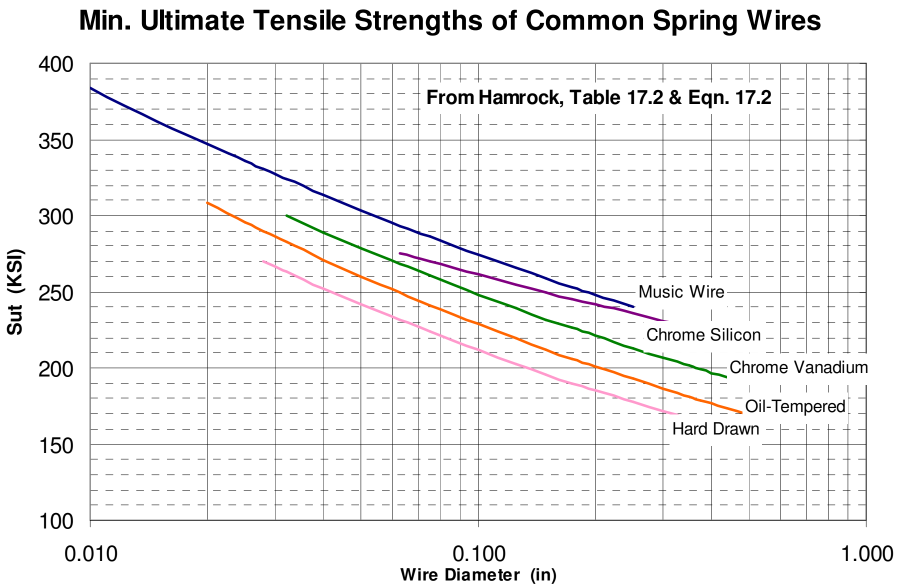

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# Spring Deflection

Castigliano's theorem gives spring deflection as

$$\delta = \frac{8PC^3N_a}{Gd}\left(1+\boxed{\frac{0.5}{C^2}}\right) \qquad \text{[Eqn. 17.17]}$$

Because C is usually between 3 and 12, the second term would be between 0.056 and 0.0035, and the equation is often shortened to

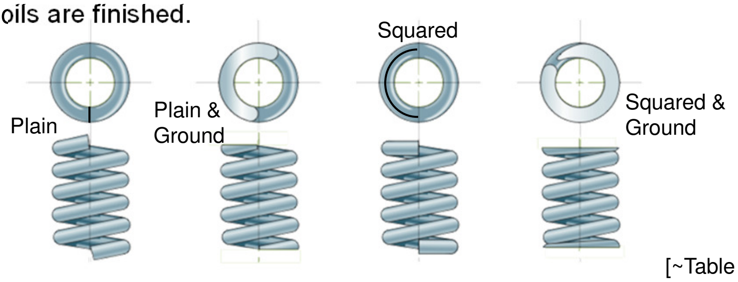
$$\delta = \frac{8PC^3N_a}{Gd} \qquad \text{[Eqn. 17.15]}$$

thereby ignoring between 0.35% and 5.6% of the deflection. I suggest that you generally use Eqn. 17.17, dropping the second term only if C is large.

In both equations, P is the applied load on the spring, G is the material Shear Modulus $= \frac{E}{2(1+\nu)}$ , and Na is the number of active coils.

# Active Coils

The number of active coils depends on how the ends of the coils are finished.



[~Table 17.3]

| Ends | $N_{end}$ | $N_{active}$ | $L_{solid}$ | Pitch* |
|---|---|---|---|---|
| Plain | 0 | $N_{tot}$ | $d(N_{tot} + 1)$ | $(L_{free} - d)/N_a$ |
| Plain & Ground | 1 | $N_{tot} - 1$ | $d \times N_{tot}$ | $L_{free}/(N_a+1)$ |
| Squared | 2 | $N_{tot} - 2$ | $d(N_{tot} + 1)$ | $(L_{free} - 3d)/N_a$ |
| Squared & Ground | 2 | $N_{tot} - 2$ | $d \times N_{tot}$ | $(L_{free} - 2d)/N_a$ |

* Pitch is ONLY measured when spring is UNLOADED! ($L = L_{free}$)

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# Spring Stiffness

The stiffness, k, is the force per deflection.

$$k = \frac{Gd}{8C^3 N_a (1 + 0.5/C^2)}$$ [Eqn. 17.18]

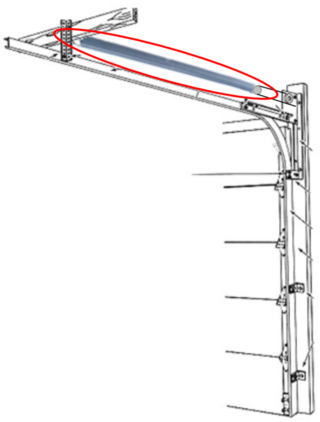
k is also known as the Spring Rate or Spring Constant.

If C is large, this can be reduced to

$$k = \frac{Gd}{8C^3 N_a} = \frac{Gd^4}{8D^3 N_a}$$

Note: k, δ, and τ are functions of d, D, Na, G, and P, but NOT of pitch and therefore not $L_{free}$.

# Spring Example



My Garage Door spring (yes, it's an extension spring, but it is close enough).

$d_{wire}$ = 0.155 in.

OD = 1.400 in.

$N_{tot}$ = 160 turns

$L_{free}$ = 26.3 in.

$L_{extended}$ = 62.5 in.

G = 11.5 x $10^6$ psi

$$D_{mean} = OD - d = 1.400 - 0.155 = 1.245\,in.$$

$$C = \frac{D}{d} = \frac{1.245}{0.155} = 8.032$$

$$k = \frac{Gd}{8C^3 N_a} = \frac{(11.5\times10^6)(0.155)}{8(8.032)^3(160)} = 2.687\,lb/in$$

$$Force = k\Delta l = (2.687)(62.5-26.3) = 97.3\,lb$$

$$K_w = \frac{4C-1}{4C-4} + \frac{0.615}{C} = 1.183$$

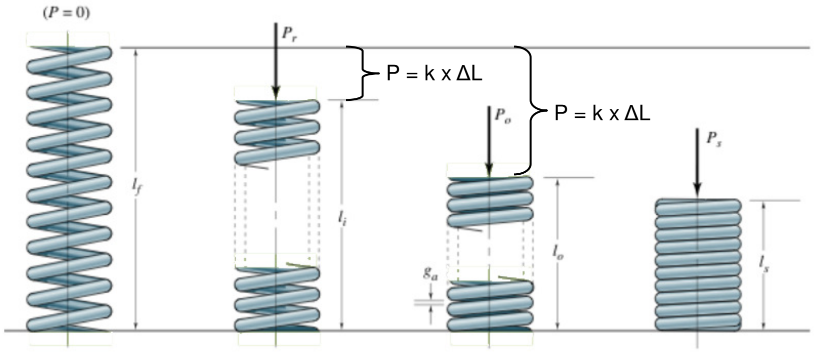
$$\tau = \frac{8D\,K_w\,P}{\pi d^3} = \frac{8(1.245)(1.183)(97.3)}{\pi(0.155)^3} = 87.97\,ksi$$

$$\frac{0.5}{C^2} = \frac{0.5}{8.032^2} = \frac{0.5}{64.51} = 0.0078$$

So we missed <1%

# Spring Force and Deflection

Two things to (almost) always count on.



P = k x ΔL

P = k x ΔL

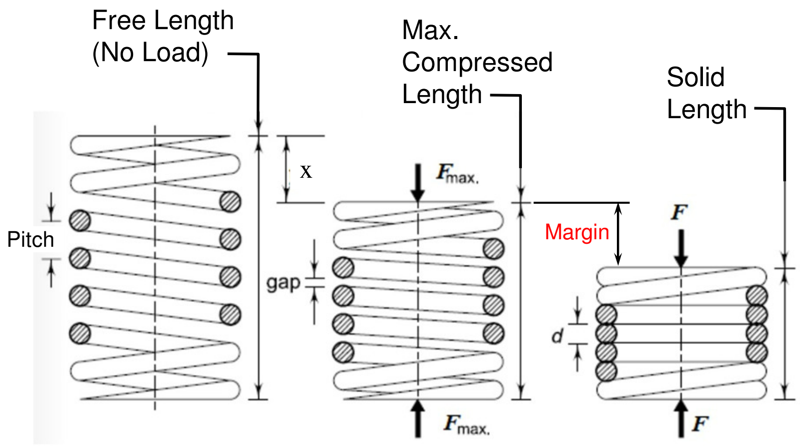
❶ At Free Length

P = 0

❷ A change in spring length is always accompanied by a change in spring force = k x ΔL, up until the spring bottoms out.

At Solid Length

# Spring Free Length



Free Length = Solid Length + Margin + Max Compressed Length

Margin keeps you safe from bottoming out.

x = Fmax / Stiffness, k

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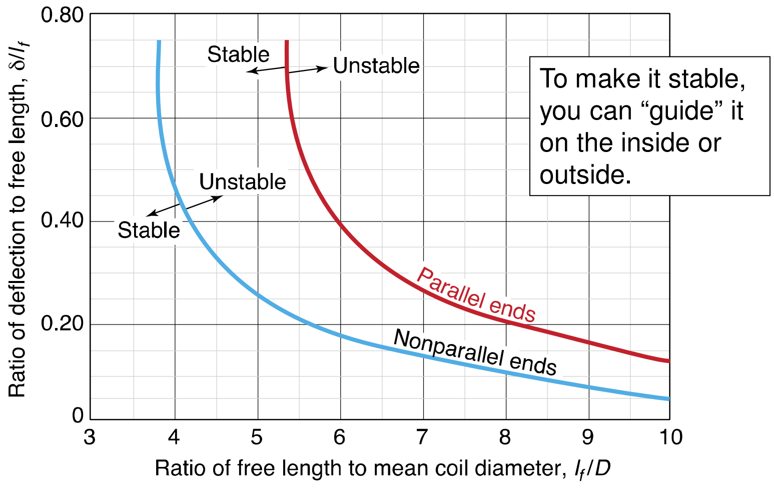
# Spring Buckling

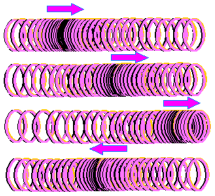


To make it stable, you can "guide" it on the inside or outside.

Like all long, skinny things with a load on them, springs can buckle.
Buckling is related to the Free Length of the spring, and to the End Conditions.

# Spring Vibration

Springs can vibrate longitudinally (or surge) just like a Slinky:



The frequency is

$$f_n = \frac{2}{\pi N_a}\frac{d}{D^2}\sqrt{\frac{G}{32\rho}}\quad Hz$$

[Eqn. 17.20]

Here G is the Shear Modulus, and ρ is the mass density (or weight density divided by g).

To avoid resonances, avoid cyclic loading a spring near integral multiples of $f_n$.

For Information Only

For steel springs where G and ρ are constants, this can be simplified to:

$$f_n = \frac{13{,}900d}{ND^2}\ Hz\quad (d\ and\ D\ in\ inches)$$

$$f_n = \frac{353{,}000d}{ND^2}\ Hz\quad (d\ and\ D\ in\ mm)$$

For the garage door spring, $f_n$=8.7Hz.

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# Fatigue / Cyclic Loading of Helical Springs

- Helical springs are NEVER used as both compression and extension springs (Hamrock, top of Section 17.3.7).

- Therefore, loading is never fully reversing, so we will use the modified Goodman diagram instead of an S-N plot.

1. Get the steady (mean) and alternating loads, $P_{mean}$ and $P_{alt}$.
2. Compute the mean and alternating shears, using $K_{Wahl}$ for BOTH:

$$\tau_{mean} = \frac{8DK_W P_{mean}}{\pi d^3} \qquad \tau_{alt} = \frac{8DK_W P_{alt}}{\pi d^3}$$

3. FOS against yielding: $$n_s = \frac{S_{Sy}}{\tau_{alt}+\tau_{mean}} = \frac{0.4S_{UT}}{\tau_{\max}}$$

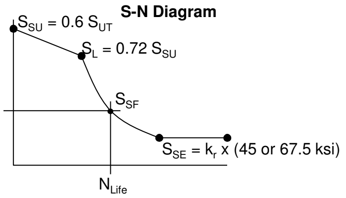
4. FOS against fatigue (Infinite life): $$n_s = \frac{S_{SE}}{\tau_{alt}}$$

# Design for Finite Life

If a finite life is specified, use the S-N diagram to compute the allowable shear stress for N cycles of life, to use in the Goodman diagram ($S_{se}$):

- Use $S_L = 0.72\ S_{SU}$ because this is Torsional loading [Eq. 7.7]
- $S_{SU}$ is the shear ultimate strength:
  $S_{SU} = 0.60\ S_{UT}$ [Eq. 17.29]
- Use $S'_{SE} = 45$ KSI for unpeened springs, and [Eq. 17.28]
  $S'_{SE} = 67.5$ KSI for peened springs
  for materials in Table 17.2 with wire diameter d < 3/8” (10mm).
- Note that these $S'_{SE}$ are corrected for ALL modification factors EXCEPT reliability, $k_r$.

See Figures on next slide.



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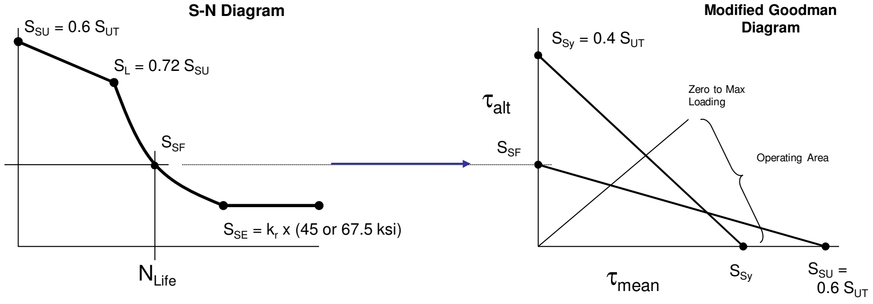
# Design for Finite Life



FOS against fatigue (Finite life): $$n_s = \frac{S_{SF}}{\tau_{alt}}$$

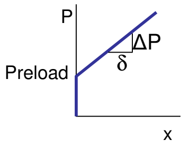
See Example 17.4, page 500.

# Helical Extension Springs

A. All coils are active. One coil is typically added to the number of active coils to obtain the body length.

B. The Free Length is measured between the insides of the end loops.

C. They are often close-wound with some initial preload.



$$k = \frac{\Delta P}{\delta}$$

once P > preload.

D. Spring rate and shear stress are the same as for compression springs.

E. Critical stresses can be in the end hooks.

> See Eqns. 17.36 and 17.37

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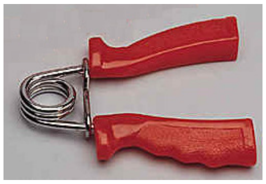
# Helical Torsion Springs

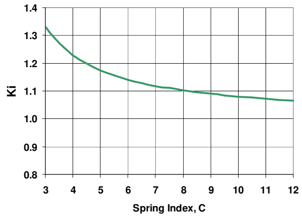
Similar to unwinding a garden hose from a reel, these springs work in bending.



$$\sigma_{\max} = \frac{K_i M c}{I} = K_i \frac{32M}{\pi d^3}$$

M = Moment
C = Index

where $K_i = \dfrac{4C^2 - C - 1}{4C(C-1)}$



Compare the max stress to Bending yield $= \sqrt{3} S_{sy}$

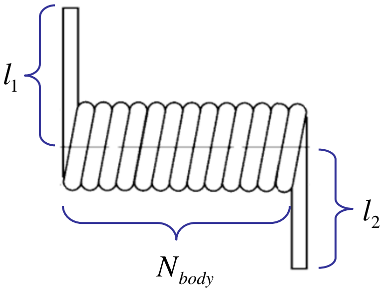
and Bending endurance $= \sqrt{3} S_{se}$

# Helical Torsion Spring Deflection

Torsional spring stiffness has different units from compression or extension springs:

$$k_\theta = \frac{d^4 E}{10.186\, D N_a} \quad \frac{Torque}{\mathrm{Re}\, v}$$

$$k_\theta = \frac{d^4 E}{64\, D N_a} \quad \frac{Torque}{Radian}$$



D = $D_{mean}$

Active Turns: $N_a = N_{body} + \frac{(l_1 + l_2)}{3\pi D}$

Note: The ID changes as the spring is loaded: $ID_{loaded} = \frac{N_a}{N_{a_{loaded}}} ID$

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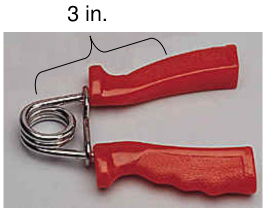
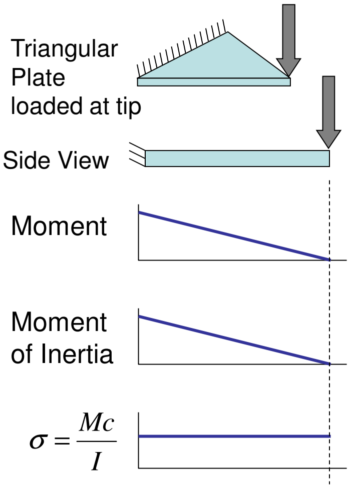
# Exercise: Hand Grip

$d_{wire}$ = 0.20 in.
ID = 0.90 in.
$L_1 = L_2$ = 3 in.
Nb = 2.4 turns of steel wire

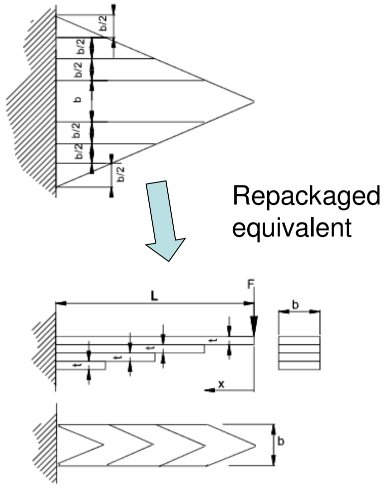
What is $K_\theta$?
How much force does it take to squeeze the handles 1.5” together (measured at a 3 in. radius from the coil center)?



# What's the deal with Leaf Springs?



Material has same stress everywhere!



Rubbing between plates damps out motion.