MEEG3311 Machine Design

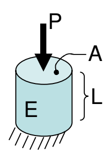
Lecture 3:
Deflection

W Dornfeld
21Sep2023

Fairfield University
School of Engineering

Simple Deformations

- Axial load on a uniform bar



$$\sigma = P/A = \varepsilon E$$

$$\varepsilon = \frac{P}{AE}$$

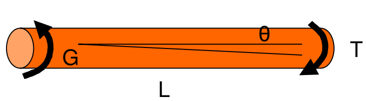
$$\delta = \varepsilon L = \frac{PL}{AE}$$

Stiffness:

$$k = \frac{P}{\delta} = \frac{AE}{L}$$

Hamrock
Section 4.3

- Torsional load (torque) on a uniform round bar



$$\theta = \frac{TL}{JG} Radians$$

$$J = \frac{\pi}{2} r^4$$

Stiffness:

$$k = \frac{T}{\theta} = \frac{JG}{L}$$
$$Nm / Radian$$

Hamrock
Section 4.4.1

Beam Flexure

- For a uniform beam in pure bending,

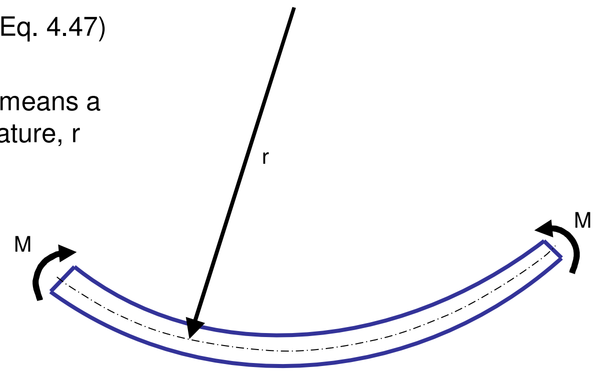
$$\frac{1}{r} = \frac{M}{EI} \quad \text{(Eq. 4.47)}$$

So a large Moment means a small radius of curvature, r

$$r = \frac{EI}{M}$$

Because

$$\frac{d^2y}{dx^2} = \frac{-M}{EI}$$



We can integrate our way from Moment, M, to the deflection, y.

3

Example Beam Loading

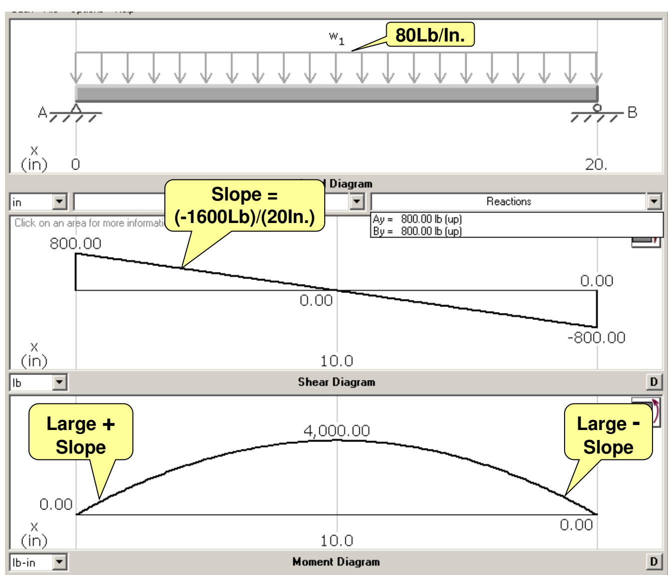
20 inch long beam with w = 80lb/in load

Shear, V (Lb)

$$\frac{dV}{dX} = -w$$

Moment, M (In.Lb.)

$$\frac{dM}{dX} = V$$



4

Beam Deflection

Moment, M
(In.Lb.)

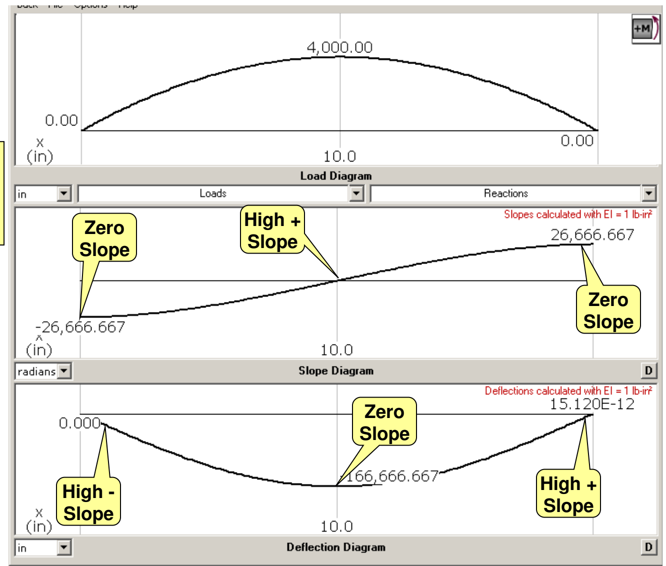
Θ & y depend on:
1. Material
2. Beam Section

Slope, EIθ
(Rad; EI=1)

$$\frac{dEI\theta}{dX} = M$$

Deflection,
EIy
(In; EI=1)

$$\frac{dEIy}{dX} = EI\theta$$



5

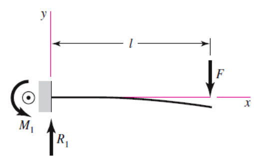
MEEG3311 Approach

1. You must be able to draw the V & M diagrams to find the max bending and transverse shear stresses in beams. (This is really stress, but it is the basis of deflection.)
2. Understand the slope and deflection concept, but because it is tedious, use tables like Hamrock Appendix D or a handbook to determine beam deflections.
3. Use Superposition to handle combined loadings (including loads in different planes, like Horiz & Vert).
4. Understand how to use slopes and rotations.
5. Use a program (like MDSolids or Excel etc.) to solve the deflection.
6. For complicated structures, use Finite Element or Castigliano.

6

Hamrock Appendix D Beams

1 Cantilever—end load

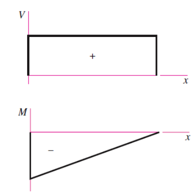


$$R_1 = V = F \qquad M_1 = Fl$$

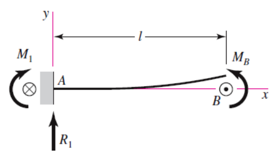
$$M = F(x - l)$$

$$y = \frac{Fx^2}{6EI}(x - 3l)$$

$$y_{max} = -\frac{Fl^3}{3EI}$$

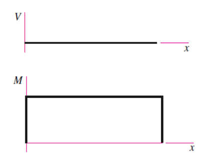


4 Cantilever—moment load



$$R_1 = V = 0 \qquad M_1 = M = M_B$$

$$y = \frac{M_B x^2}{2EI} \qquad y_{max} = \frac{M_B l^2}{2EI}$$



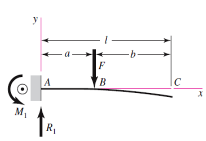
What shape is this deflection?

Hint: See Slide 3

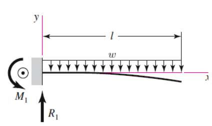
7

Hamrock Appendix D Beams

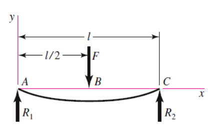
2 Cantilever—intermediate load



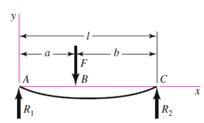
3 Cantilever—uniform load



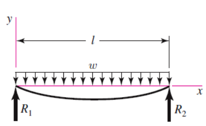
5 Simple supports—center load



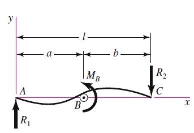
6 Simple supports—intermediate load



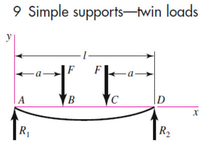
7 Simple supports—uniform load



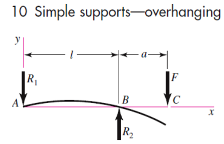
8 Simple supports—moment load



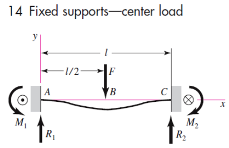
9 Simple supports—twin loads



10 Simple supports—overhanging load



14 Fixed supports—center load



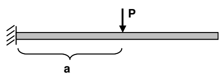
8

Deflection & Slope in Excel: Beam D.2

Hamrock Appendix D.2

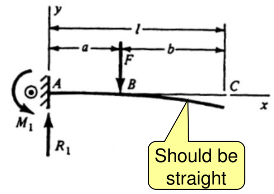
Base	b	0.035	m
Height	h	0.08	m
Length	L	1.7	m
Modulus	E	207	GPa
MomOfInertia	I	1.49333E-06	m^4
Load	P	5000	N
Location	a	1	m
EI	EI	309,120.0	Nm^2

=b*h^3/12



$$y_{AB} = \frac{Fx^2}{6EI}(x - 3a)$$

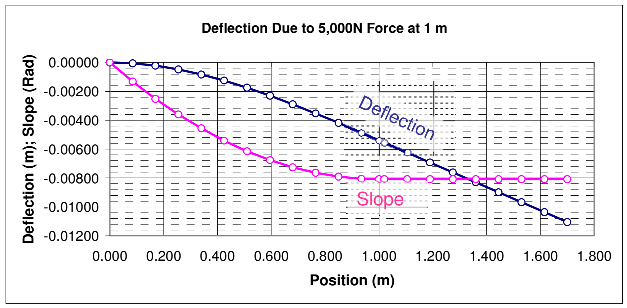
$$y_{BC} = \frac{Fa^2}{6EI}(a - 3x)$$



Defl =IF(X<a,P*X^2/(6*EI)*(X-3*a),P*a^2/(6*EI)*(a-3*X))

Slope =IF(X<a,(P/(EI)*(0.5*X^2-a*X)),-P*a^2/(2*EI))

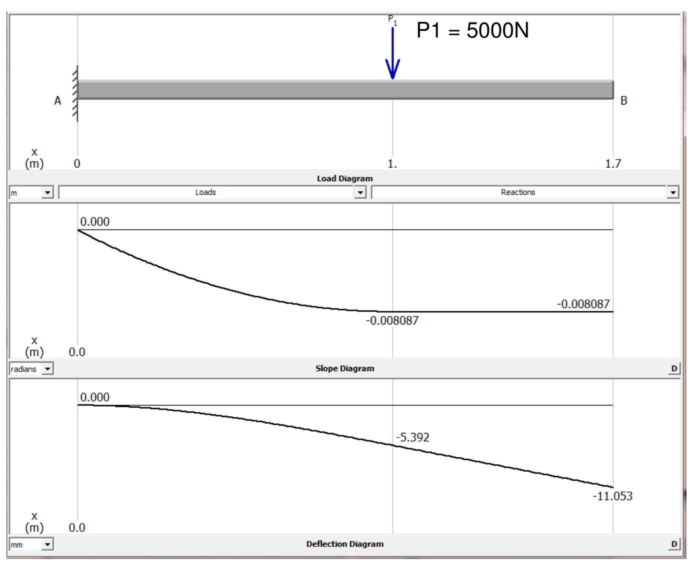
X	Y Defl	Slope
0.000	0.00000	0.00000
0.085	-0.00006	-0.00132
0.170	-0.00022	-0.00252
0.255	-0.00048	-0.00360
0.340	-0.00083	-0.00456
0.425	-0.00125	-0.00541
0.510	-0.00175	-0.00615
0.595	-0.00230	-0.00676
0.680	-0.00289	-0.00726
0.765	-0.00353	-0.00764
0.850	-0.00419	-0.00791
0.935	-0.00487	-0.00805
1.000	-0.00539	-0.00809
1.020	-0.00555	-0.00809
1.105	-0.00624	-0.00809
1.190	-0.00693	-0.00809
1.275	-0.00762	-0.00809
1.360	-0.00830	-0.00809
1.445	-0.00899	-0.00809
1.530	-0.00968	-0.00809
1.615	-0.01037	-0.00809
1.700	-0.011053	-0.008087



9

Deflection & Slope in MDSolids: Beam D.2

EI = 309,120Nm^2

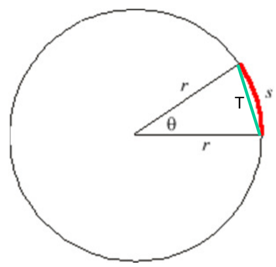


10

Basic Angle Relationship

$$\theta = \frac{S}{r};\ S = r \times \theta$$

Θ is in Radians



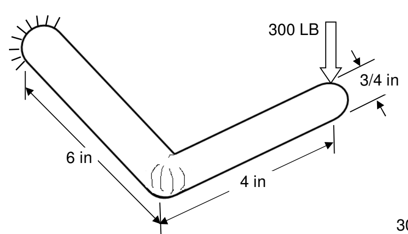
For small angles,
T ≈ S

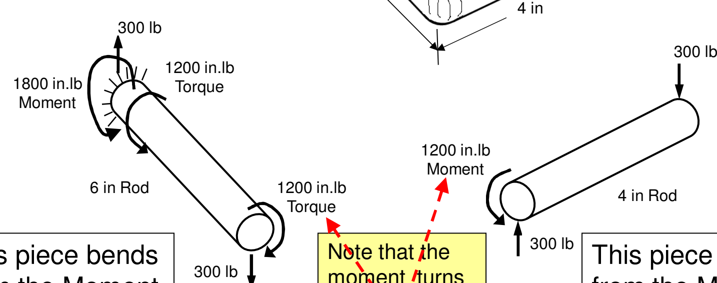
How many degrees is a Radian?

Piece-Wise Deflection of a Crank Arm

How much does the loaded end of the arm deflect?

Breaking it into pieces and looking at the FBD will help.





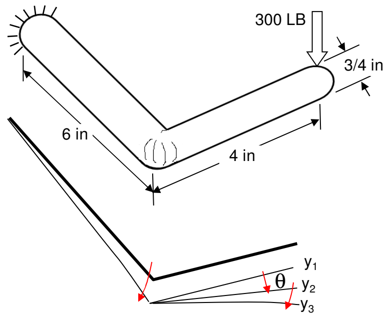
This piece bends from the Moment and twists from the Torque.

Note that the moment turns into a torque where the arm bends the corner.

This piece bends from the Moment.

Piece-Wise Deflection of a Crank Arm

Deflection of the loaded end point is the sum of three deflections:



1. Bending of the 6" Rod due to the 300 lb load.

$$y_1 = \frac{Fl^3}{3EI}$$

2. Twisting of the 6" Rod due to the 1200 in.lb torque, with rotation of the 4" Rod.

$$\theta = \frac{Tl}{JG}, \quad y_2 = r\theta$$

3. Bending of the 4" Rod due to the 300 lb load.

$$y_3 = \frac{Fl^3}{3EI}$$

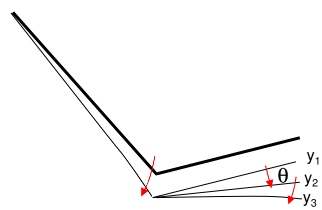
13

Piece-Wise Deflection of a Crank Arm

$$r = 0.375'' \qquad I = \frac{\pi r^4}{4} = 0.01553 in^4$$

$$J = \frac{\pi r^4}{2} = 2I = 0.03106 in^4$$

$$E = 30\times10^6\ psi, \quad G = 11.5\times10^6\ psi$$



$$y_1 = \frac{Fl^3}{3EI} = \frac{(300)(6)^3}{3(30\times10^6)(0.01553)} = 0.0464 in$$ Long arm bending

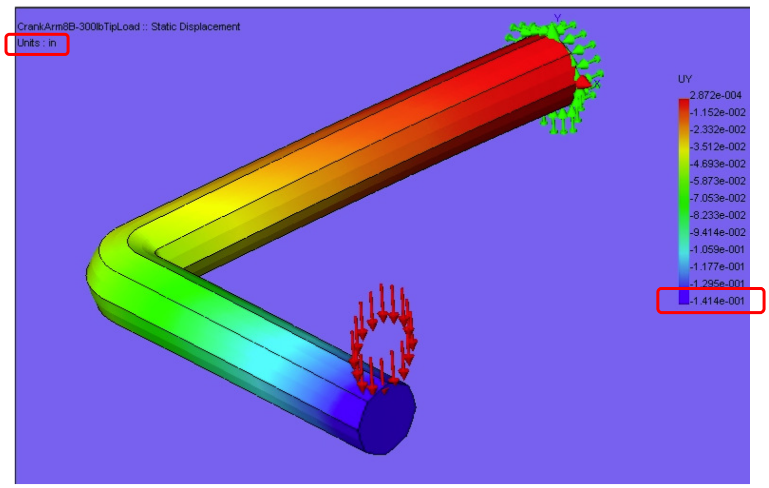
$$y_2 = r\theta = \frac{rTl}{JG} = \frac{(4)(1200)(6)}{(0.03106)(11.5\times10^6)} = 0.0806 in$$ Short arm sweeping as long arm twists

$$y_3 = \frac{Fl^3}{3EI} = \frac{(300)(4)^3}{3(30\times10^6)(0.01553)} = 0.0137 in$$ Short arm bending

$$y_{total} = y_1 + y_2 + y_3 = 0.1407 in$$

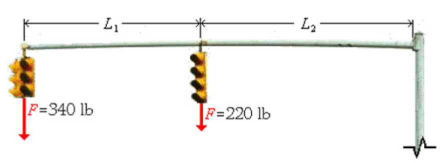
14

Crank Arm Deflection by FEA



15

Traffic Light Pole - Deflection



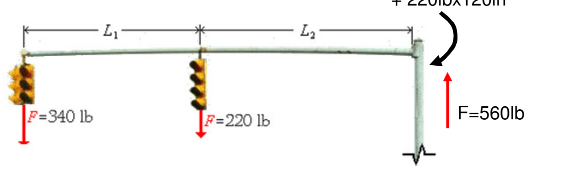
L1 = 100 in.   L2 = 120 in.

From U of Arkansas FEMur

16

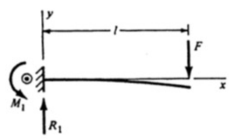
Traffic Light Calculations

Simplified Case With One Beam Size



Use Appendix Formulas and Superposition

1 Cantilever—end load

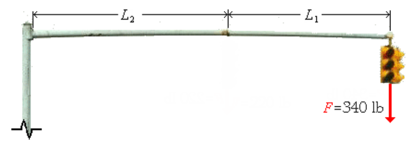


$R_1 = V = F \qquad M_1 = -Fl$

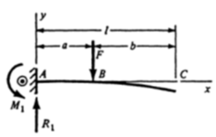
$M = F(x - l)$

$y = \frac{Fx^2}{6EI}(x - 3l)$

$y_{max} = -\frac{Fl^3}{3EI}$



2 Cantilever—intermediate load



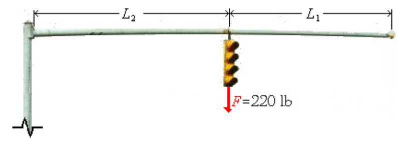
$R_1 = V = F \qquad M_1 = -Fa$

$M_{AB} = F(x - a) \qquad M_{BC} = 0$

$y_{AB} = \frac{Fx^2}{6EI}(x - 3a)$

$y_{BC} = \frac{Fa^2}{6EI}(a - 3x)$

$y_{max} = \frac{Fa^2}{6EI}(a - 3l)$

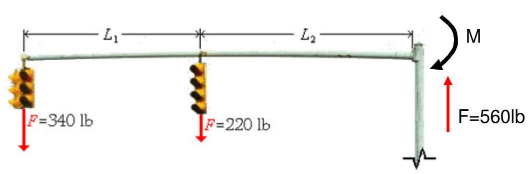


17

Traffic Light Calculations

Simplified Case With One Beam Size

Assume the support beam is AISC Standard Shape 6” Steel Pipe
OD = 6.625”
ID = 6.103”
(0.261” wall)
which has an Area Moment of Inertia
I = 26.4in^4
E = 29 x 10^6 PSI



Tip: $y_{max} = -\frac{Fl^3}{3EI}$

Mid: $y_{max} = \frac{Fa^2}{6EI}(a - 3l)$

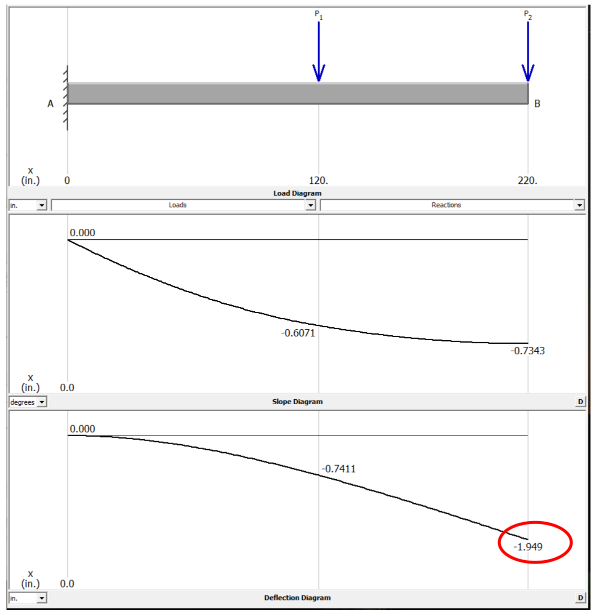
18

How Did We Do?

Compare with MDSolids:

What if we changed to a 5" Pipe, with I = 13.70 in^4 ?

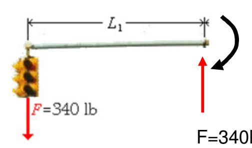
What if we changed to a 6" Aluminum Pipe, with E = 10.5MSI ?

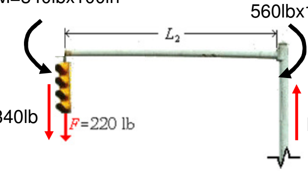


19

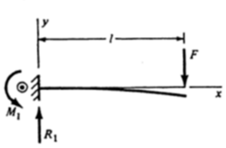
Traffic Light Calculations

Real Case With Two Beam Sizes





1 Cantilever—end load



$R_1 = V = F \qquad M_1 = -Fl$

$M = F(x - l)$

$y = \frac{Fx^2}{6EI}(x - 3l)$

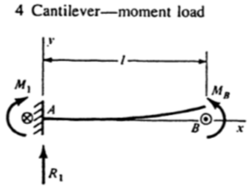
$y_{max} = -\frac{Fl^3}{3EI}$

Slope @ Tip due to F:

$$\frac{dy}{dx} = \frac{3Fx^2}{6EI} - \frac{2Fx3l}{6EI} = \frac{Fx^2}{2EI} - \frac{Fxl}{EI}$$

$$\frac{dy}{dx} = \frac{Fx}{EI}(\frac{x}{2} - l)$$

4 Cantilever—moment load



$R_1 = 0 \qquad M_1 = M_B \qquad M = M_B$

$y = \frac{M_B x^2}{2EI} \qquad y_{max} = \frac{M_B l^2}{2EI}$

Slope @ Tip due to M:

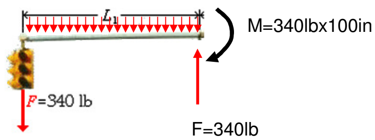
$$\frac{dy}{dx} = \frac{2M_B x}{2EI} = \frac{M_B x}{EI}$$

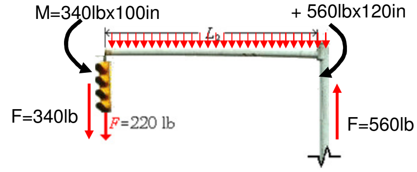
Deflection at tip due to slope is θ x L_1

20

Traffic Light Calculations

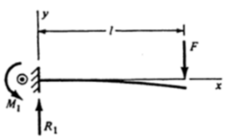
Even More Real Case With Two Beam Sizes and Including the Weight of the Beams





The 6" Steel tube weighs 19lb/ft = 1.58lb/in.

1 Cantilever—end load



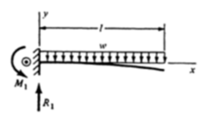
$R_1 = V = F \quad M_1 = -Fl$

$M = F(x - l)$

$y = \frac{Fx^2}{6EI}(x - 3l)$

$y_{max} = -\frac{Fl^3}{3EI}$

3 Cantilever—uniform load



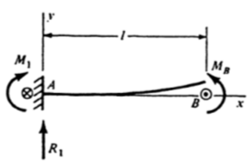
$R_1 = wl \quad M_1 = -\frac{wl^2}{2}$

$V = w(l - x) \quad M = -\frac{w}{2}(l - x)^2$

$y = \frac{wx^2}{24EI}(4lx - x^2 - 6l^2)$

$y_{max} = -\frac{wl^4}{8EI}$

4 Cantilever—moment load



$R_1 = 0 \quad M_1 = M_B$

$y = \frac{M_B x^2}{2EI} \quad y_{max} = \frac{M_B l^2}{2EI}$

Plus, calculate all the end slopes/rotations!

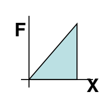
21

Castigliano's Theorem

" ... the partial derivative of the strain energy, considered as a function of the applied forces acting on a linearly elastic structure, with respect to one of these forces, is equal to the displacement in the direction of the force at its point of application."

From intota: *The deflection of an elastic material subjected to a load in the direction of each load equals the partial derivative of the work of deformation with respect to the component of the force in that direction.* This theorem is related to the principle of virtual work, and it applies for elastic material obeying Hooke's law.

This should vaguely make sense, because for a spring, the stored energy is the integrated area under the force vs deflection curve. So it is plausible that differentiating that could get you to the deflection.



22

Castigliano, Continued

1. You write an expression for the total strain energy in your structure, based on each type of loading (see table).
2. If you want to know a deflection where there isn't a load applied, just stick a "fictitious" load, Q, there.
3. Then take partial derivatives of the energy wrt the loads.
4. Then set Q=0 and *voilà*!

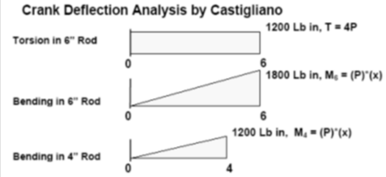
Loading Type	Strain Energy Constant Variables	Strain Energy General Case
Axial	$U = \frac{F^2 l}{2EA}$	$U = \int_0^l \frac{F^2 dx}{2EA}$
Bending	$U = \frac{M^2 l}{2EI}$	$U = \int_0^l \frac{M^2 dx}{2EI}$
Torsion	$U = \frac{T^2 l}{2GJ}$	$U = \int_0^l \frac{T^2 dx}{2GJ}$
Direct Shear	$U = \frac{F^2 l}{2AG}$	$U = \int_0^l \frac{F^2 dx}{2AG}$
Traverse Shear	$U = \frac{K V^2 l}{2GA}$	$U = \int_0^l \frac{KV^2 dx}{2GA}$

Hamrock Table 5.2

Castigliano, Concluded

1. The MEEG3311 web site has an analysis of the Crank Arm by Castigliano (shown here).
2. Hamrock has several examples in section 5.6.
3. All of the formulas for deflection in handbooks and Beer & Johnston were figured out by using Castigliano's theorem.
4. Carlo Alberto Castigliano (1847 – 1884) figured it out when he was 25 years old.
5. There will be no homework or exam <u>problems</u> on using Castigliano's theorem, but maybe a question.

Crank Deflection Analysis by Castigliano



Strain Energy$_{TOTAL}$ = Bending (in 4") + Bending (in 6") + Torsion (in 6")

$$U = \int_{Tip}^{Elbow} \frac{M_4^2}{2EI} dx + \int_{Elbow}^{Base} \frac{M_6^2}{2EI} dx + \int_{Elbow}^{Base} \frac{T^2}{2GJ} dx$$

$$U = \frac{1}{2EI}\int_0^4 (Px)^2 dx + \frac{1}{2EI}\int_0^6 (Px)^2 dx + \frac{(4P)^2}{2GJ}\int_0^6 dx$$

$$U = \frac{P^2}{2EI}\int_0^4 x^2 dx + \frac{P^2}{2EI}\int_0^6 x^2 dx + \frac{16P^2}{2GJ}\int_0^6 dx$$

$$U = \frac{P^2}{2EI}\frac{1}{3}x^3\Big|_0^4 + \frac{P^2}{2EI}\frac{1}{3}x^3\Big|_0^6 + \frac{8P^2}{GJ}x\Big|_0^6$$

$$U = \frac{P^2}{6EI}4^3 + \frac{P^2}{6EI}6^3 + \frac{8P^2}{GJ}6 = \frac{64P^2}{6EI} + \frac{216P^2}{6EI} + \frac{48P^2}{GJ}$$

$$U = P^2\left(\frac{64}{6EI} + \frac{216}{6EI} + \frac{48}{GJ}\right) = P^2\left(\frac{10.67}{EI} + \frac{36}{EI} + \frac{48}{GJ}\right)$$

$$\delta_P = \frac{\partial U}{\partial P} = 2P\left(\frac{10.67}{EI} + \frac{36}{EI} + \frac{48}{GJ}\right)$$

$$\delta_P = 600\left(\frac{10.67+36}{(30\times10^6)(0.01553)} + \frac{48}{(11.5\times10^6)(0.03106)}\right)$$

$$\delta_P = 600(0.00002289 + 0.00007727 + 0.0001344)$$

$$\delta_P = (0.01374 + 0.04636 + 0.08064) = 0.1407\,in.$$