Spectral Graph Theory - Problem Set 2

1. Suppose that G is a graph with an even number of vertices and with a *Hamiltonian cycle*, a path of edges that visits every vertex once except it ends where it started). Prove that the largest eignevalue of the Laplacian of G is at least 4. Hint: use the Udell notes Sect. 3.3

2. If A is a matrix with the entries in each column summing to 1 and $\mathbf{w}$ is a probability vector show that $A\mathbf{w}$ is a probability vector.

3. Suppose $\mathbf{w}$ is an eigenvalue for the *connected* graph Laplacian of graph G with eigenvalue 0. Show that $\mathbf{w}$ has all entries equal by considering the vertex with the largest entry and looking at what the effect of the Laplacian is on its value.

4. Prove that every connected graph has a one-dimensional eigenspace with the 0 eigenvalue, and that there is always an eigenvector which is a probability vector (all entries are ≥ 0 and they sum to 1).

5. Suppose $\mathbf{w} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdot + c_n\mathbf{v}_n$ where the c_i are scalars and each $\mathbf{v_i}$ is an eigenvalue of the Laplacian with eigenvalue λ_i. Show that the solution to the heat equation for $\mathbf{w}$ with parameter α is

$$c_1 e^{-\alpha\lambda_1 t}\mathbf{v}_1 + c_2 e^{-\alpha\lambda_2 t}\mathbf{v}_2 + \cdots + c_n e^{-\alpha\lambda_n t}\mathbf{v}_n.$$

6. Show that for $\mathbf{w}$ as in the previous problem the limit as $t \to \infty$ of the solution of the heat equation with initial condition $\mathbf{w}$ is $c_1\mathbf{v}_1$.

7. If $\mathbf{w}$ is a vector and $\mathbf{v}_1, \ldots, \mathbf{v}_n$ is a complete set of orthogonal eigenvectors for the Laplacian (remember we can always choose eigenvectors orthogonal because it is symmetric) show that we can write $\mathbf{w} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdot + c_n\mathbf{v}_n$ where each $c_i = \langle \mathbf{w}, \mathbf{v}_i \rangle \,/\, \langle \mathbf{v}_i, \mathbf{v}_i \rangle$. Hint, observe you can write $\mathbf{w}$ in terms of the $\mathbf{v}_i$ and then take the inner product.