Problem Set 1

1. Let $A = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$. Use the quadratic formula to find the two eignevalues of A. Under what conditions on a, b, c will there be only a single eigenvalue?

2. If A is as in the previous problem and $b = 0$, find the eigenvalues and the two normal eigenvectors of A (as a function of a and c). Check that these eigenvectors are orthonormal.

3. If A is as in the first problem and $a = c = 1$, find the two eigenvalues and the two normal eigenvectors as a function of b and check that these eignevectors are orthonormal.

4. Let n be a natural number and let A be an n by n matrix with $A_{ij} = 0$ for $i \neq j$ and $A_{ii} = c_i$ for some real numbers c_i with $1 \leq i \leq n$. Find the eigenvalues and orthonormal eigenvectors of A.

5. Find the two eigenvalues and orthonormal eigenvectors of the Laplacian of the graph with two vertices and one edge.

6. Use Wolfram Alpha or MatLab or something to find the eigenvalues and orthonormal eigenvectors for the line graphs with (a) three vertices and two edges and (b) four vertices and three edges.

7. For each vertex $v \in V$ of a graph let $d(v)$ be the number of edges connecting to it. Prove that
$$\sum_{v \in V} d(v)$$
is even as long as the graph is finite.

8. Let A be the adjacency matrix of a graph. Recall that $A_{ij} = 1$ if there is an edge between vertex i and j and 0 otherwise. Show that A^2 is a matrix whose i, j entry gives the number of length two paths from vertex i to vertex j. Show by induction that A^n is a matrix whose i, j entry gives the number of length n paths from vertex i to vertex j.