The New Link and Three Manifold Invariants

Steve Sawin
Geometric Topology 935
MIT

Spring '94

Contents

1 The Jones Polynomial

1.1 Preliminaries

The material in the first two subsections is quite standard. A good reference is [BZ85]. The third subsection is part of the well known folklore of the field. It is implied by the stronger results of [Tur].

1.1.1 Links and Knots

We will work in the piecewise-linear category unless explicitly stated otherwise.

Definition 1 *A* link *is a one dimensional submanifold of S^3 (i.e., a disjoint union of circles). Two links are* equivalent *if there is an isotopy of S^3 which* *takes one to the other. That is, there is a map $f_t(x)$, for $x \in S^3$ and $t \in [0,1]$, such that for each t the map f_t is a homeomorphism of S^3 to itself, f_0 is the identity and f_1 takes one link homeomorphically to the other.*

There are two other equivalent definitions of link equivalence, as is proven in [BZ85, Prop. 1.10] (the proposition is stated for knots, but applies as well to links). The first is that two links are equivalent if there exists an orientation preserving homeomorphism of S^3 which takes one to the other. The second is that two links are equivalent if one can get from one to another by a sequence of Δ moves and their inverse. A Δ move replaces a straight segment AB of the link with two line segments AC and CB, where C is such that the triangle ABC does not intersect the link except along AB.

Notice one can remove a simplex which does not intersect the link (after taking a subdivision if necessary) and identify the remainder with $\mathbf{R}^3$. Thus in its most concrete form, a link is a collection of polygons in $\mathbf{R}^3$ modulo the $\Delta^{\pm 1}$ moves.

A knot is a link with one component. The unlink is the unique link with no components. The unknot is the knot which bounds a disk. The distant union of two links L_1 and L_2, written $L_1 \amalg L_2$, is their union in the connect sum of their two respective copies of S^3.

An oriented link is just one in which the one dimensional manifold is equipped with an orientation, and equivalence is the same except the isotopy / homeomorphism / sequence of moves is required to take one link to the other in an orientation preserving fashion.

A link L is the connect sum (or product) of two links L_1 and L_2, written $L_1 \# L_2$, if there is a plane which intersects it in two points, and the link on one side of the plane (connecting those two points along the plane) is equivalent to L_1, and the link on the other side is equivalent to L_2. One can form the connect sum of two oriented links in only one way if one chooses a component of each to connect sum along, two ways if they are unoriented.

1.1.2 Link Projections

Consider a link as a polygon in $\mathbf{R}^3$, and consider a plane through the origin. The projection of the link onto the plane is called regular if all but finitely many points have a one point preimage, and those that don't have two points in their preimage, neither of which is a vertex. Given such a regular projection and a choice of normal directions to the plane, the image of the projection plus an indication at each double point (called crossings) of which line segment goess over the other completely determines the link equivalence class. We draw these as in Figure (1), where the lower strand at each crossing is indicated by a short gap. Orientations are indicated by an arrow on each component.

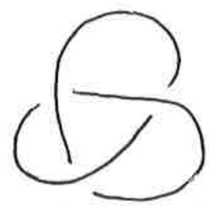


Figure 1: A typical link projection

Theorem 1 (Reidemeister) *[BZ85, Prop. 1.12, 1.14] Every projection can be rotated by an arbitrarily small amount to become a regular projection. Two regular projections are of equivalent links if and only if they can be connected by planar isotopy and a sequence of Reidemeister moves I-III pictured below in Figure (2) and their mirror images.*

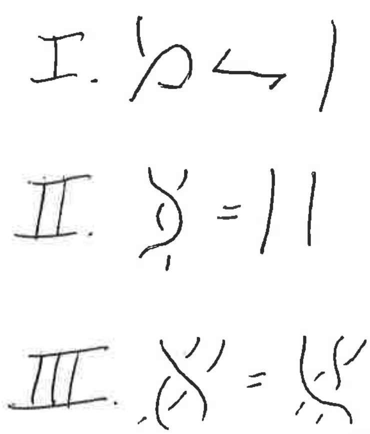


Figure 2: Reidemeister moves

Pf: *(Sketch)* The set of projections with normal direction can be identified with S^2. It is easy to see that the set of nonregular projections corresponds to a finite collection of curves on S^2. Thus the first statement is clear.

The if part of the second statement is also clear. Now two different projections of the same link can be connected by a path in S^2 crossing the bad curves transversely. Each such crossing amounts to one of the moves I-III, and the rest of the movement is planar isotopy. Going from one link to another with the projection fixed involves a sequence of $\Delta^{\pm 1}$ moves, which look like planar isotopy and moves II and III on the projection. ■

1.1.3 Framed Links

A framed link is defined just like a link, except it is homeomorphic to a union of copies of $S^1 \times I$ rather than of S^1. The underlying link of a framed link is the image of all copies of $S^1 \times \{0\}$ under such a homeomorphism. Given a regular projection of the underlying link and an explicit homeomorphism of each component with $S^1 \times I$, one can clearly isotope the framed link so that $\{\theta\} \times I$ is parallel to the plane for all θ of each component but a little segment, which looks like Figure (3). Thus a framed link is determined by a

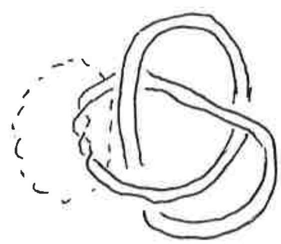


Figure 3: A projection of a framed link with the twist confined

projection of the underlying ordinary link together with the winding number for each component with respect to the normal to the projection. Since a type I Reidemeister move changes this winding number by ± 1, it is always possible to find a projection for which this winding number is zero. Such a projection is called a regular projection of the framed link.

Theorem 2 *Two regular framed link projections come from the same framed links if and only if they can be connected by planar isotopy and a sequence of the three moves framed Reidemeister pictured in Figure (4) and their mirror images. The same applies as before to oriented framed links.*

Lemma 1 *Let L be a link projection with two pieces A and B as pictured in Figure (5) and let L' be the same projection with A and B interchanged.*

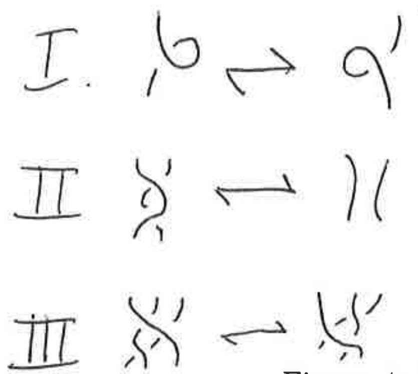


Figure 4: Framed Reidemeister moves

Then L and L′ can be connected by a sequence of framed Reidemeister moves (in fact by a sequence of moves II and III).

Figure 5: Interchangeable pieces of a framed link projection

Lemma 2 *The move pictures in Figure (6) and its mirror image are the composition of a sequence of framed Reidemeister moves (in fact, of moves II and III).*

Figure 6: Canceling opposite twists

Proof of Lemmas: *an exercise.*

Pf:(of the Theorem)

The if part is clear. For the other direction, consider two projections of the same framed link. They are in particular projections of the same ordinary link. Thus they are connected by a sequence of ordinary Reidemeister moves. Choose such a sequence from one to the other. Replace each application of move I which creates a twist with an application of Lemma (2) creating two twists. If an application of move I would delete a twist, do not do it. The result is a sequence of framed Reidemeister moves going from the first projection to a projection which looks just like the second except that it has some additional twists. But this projection with the additional twists has the same framing as the first projection (being connected by a sequence of framed Reidemeister moves) which is in turn assumed to have the same framing as the second projection, so therefore the number of extra positive twists (those corresponding to a counterclockwise rotation) minus the number of negative twists must be zero.

Now, if there is at least one extra positive twist, then there is at least one extra negative twist, so an application of Lemma (1) can put them next to each other. An application of framed Reidemeister move I if necessary can make them look as in Figure (6), and an application of Lemma (2) will cancel them. Repeating this until there are no more twists gives a sequence of framed Reidemeister moves connecting the first and second projections. ■

1.2 The Bracket and Jones Polynomial

The Kauffman bracket was introduced by Kauffman in [Kau]. All of the material in the first subsection appears in [Kau88]. The Jones polynomial appears first in [Jon85].

1.2.1 The Kauffman Bracket

Given a projection $\mathcal{L}$ of a framed (unoriented) link L, a *state* is a replacement of each crossing in the projection with two parallel strands in one of two possible ways. That is, replacing **X** with **H** or **V** in Figure (7)

Thus a state is a projection of a distant union of unknots. If s is a state, let $n(s)$ be the number of such unknots, and let $m(s)$ be the number resolved into horizontal strands (labeled **H** in Figure (7)) minus the number of crossing resolved into vertical strands (labeled **V** in the figure), where horizontal and vertical are defined with respect to a viewpoint that makes the upper strand go from the upper left to the lower right. Now define the Kauffman bracket of a projection, $\langle \mathcal{L} \rangle$, to be a polynomial in variables A and

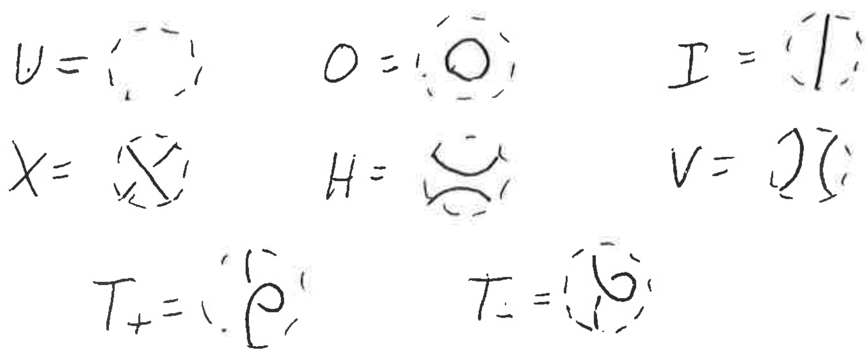


Figure 7: Some fragments of projections

A^{-1} by

$$\langle \mathcal{L} \rangle = \sum_s A^{m(s)}(-A^2 - A^{-2})^{n(s)} \tag{1}$$

where the sum is over all states of $\mathcal{L}$.

Proposition 1 *The Kauffman bracket satisfies the equations below, where they are understood to relate the bracket of projections which are identical except in a small disk, where they look as pictured in Figure (7).*

$$\langle \mathbf{X} \rangle = A\langle \mathbf{H} \rangle + A^{-1}\langle \mathbf{V} \rangle \tag{2}$$

$$\langle \mathbf{O} \rangle = (-A^2 - A^{-2})\langle \mathbf{U} \rangle \tag{3}$$

$$-A^{-3}\langle \mathbf{T}_+ \rangle = \langle \mathbf{I} \rangle = -A^3\langle \mathbf{T}_- \rangle \tag{4}$$

Pf: To prove Equation (2), there is a one-to-one correspondence between the states of the left hand projection and the union of the states of the two right hand projections. Specifically, a state which resolves the pictured crossing horizontally gets sent to the state of the first right hand projection with all the same resolutions outside the dotted circle. A state which is resolved vertically gets sent to the corresponding state of the second projection. This map preserves $n(s)$ and changes $m(s)$ by ± 1, depending on whether it's going to the second or first projection. From this the result follows.

For Equation (3) the one-to-one map sends a state of the left side to the state of the right side with the same resolutions. The map preserves $m(s)$ and decreases $n(s)$ by 1. From this the result follows.

Equation (4) follows by applying the above two equations to the crossing shown.

■

Proposition 2 *The Kauffman bracket is the unique function on framed link projection satisfying Equations (2) and (3) of the previous proposition whose value on the unlink is* 1. *In particular, these facts give an effective procedure for computing the bracket of any framed link projection.*

Pf: By induction on the number of crossings. If there are no crossings, it is a standard projection of a distant union of unknots, and thus its value can be computed from Equation (3) and the value of the unlink. If the projection has $n+1$ crossings, choose one and apply Equation (2) to relate its bracket to the bracket of two projections with n crossings. By induction, this determines its bracket. ■

Theorem 3 *The Kauffman bracket has the same value on all projections of a single framed link L. Thus it is a framed link invariant, and we shall henceforth use $\langle L \rangle$ to refer to this value common to all its projections.*

Pf: We shall use Theorem (2), and show that this quantity does not change under one of the three framed Reidemeister moves. Of course, it does not change under planar isotopy.

To prove invariance under move I, simply apply Equation (4) to both sides of the equation.

To prove invariance under move II, apply Equation (2) to both crossings on the left hand side to get four projections, one of which has an extra component. Apply Equation (3) to this projection, and then notice that all the projections cancel out except the one which corresponds to the right hand projection.

To prove invariance under move III, apply Equation (2) to resolve the bottom crossing on one side and the top crossing on the other. The equality of the two sides now follows from the invariance under move II. ■

Proposition 3 *If L and L' are mirror images, then $\langle L \rangle$ is $\langle L' \rangle$ with A^{-1} substituted for A.*

Pf: If we define a new invariant $\langle L \rangle'$ to be $\langle L' \rangle$, Notice that $\langle \cdot \rangle'$ is one on the unlink. Furthermore, the fact that $\langle \cdot \rangle$ satisfies Equations (2) and (3) means that $\langle \cdot \rangle'$ satisfies these equations with A^{-1} substituted for A. By Proposition (2), this means $\langle L \rangle'$ is $\langle L \rangle$ with A^{-1} substituted for A. ■

Proposition 4 $\langle L_1 \amalg L_2 \rangle = \langle L_1 \rangle \cdot \langle L_2 \rangle$.

Pf: We shall prove that the bracket is multiplicative on projections which are disconnected as graphs, by induction on the number of crossings in one of the components.

If $\mathcal{L}_1$ has no crossings, it is a projection of a distant union of unknots, and by Equation (3) and the value of the bracket on the unlink the result follows. If $\mathcal{L}_1$ has $n+1$ crossings, use Equation (2) to resolve one. The result follows by induction. ■

Proposition 5 *If L can be written in some way as the connect sum of links L_1 and L_2, then $\langle L \rangle = \langle L_1 \rangle \cdot \langle L_2 \rangle / (-A^2 - A^{-2})$.*

Pf: Choose a projection of L so that there is a line which crosses the projection in exactly two points and divides it into a projection of L_1 and a projection of L_2. As above we shall proceed by induction on the number of crossings in the projection of L_1.

If it has no crossings, L_1 is a distant union of unknots, and the result follows from Equation (3) and the normalization of the unlink. If L_1 has $n+1$ crossings, apply Equation (2) to one of them. The result follows by induction. ■

1.2.2 The Definition of the Jones Polynomial

Let L be an oriented link, and let $\mathcal{L}$ be a projection of L. Of course $\mathcal{L}$ determines a framing on L. Every crossing of $\mathcal{L}$ can be labeled as positive or negative according to whether it looks like L_+ or L_- respectively in Figure (8). That is, a crossing is positive if one strand is pointed counterclockwise with respect to the other. We define the writhe of $\mathcal{L}$, $w(\mathcal{L})$, to be the number of positive crossings minus the number of negative crossings. The reader may check that the writhe is unchanged by any of the framed Reidemeister moves, and thus is a framed link invariant.

Figure 8: Fragments of projections occurring in the Jones skein relation

Theorem 4 *The quantity* $(-A)^{-3w(\mathcal{L})}\langle\mathcal{L}\rangle$ *is an invariant of* L*, where* $\langle\mathcal{L}\rangle$ *means bracket of the unoriented projection.*

Pf: Since the bracket and the writhe are framed link invariants, so is this quantity. Thus we only have to check invariance under the first unframed Reidemeister move. The left hand side as pictured in Figure (2) has writhe one more than the right hand side, but by Equation (4), the bracket of the left hand side is $-A^3$ times that of the right. The combined quantity is thus unchanged. The same argument applies to the mirror image. ■

Definition 2 *The* Jones polynomial *of a link* L*,* $V_L(t)$*, is the polynomial in* $t^{1/4}$ *and* $t^{-1/4}$ *given by substituting* $t^{1/4}$ *in for* A *in* $(-A)^{-3w(\mathcal{L})}\langle\mathcal{L}\rangle$

Corollary 1 *The Jones polynomial is an invariant of oriented links.*

Proposition 6

(a) The Jones polynomial satisfies the skein relation

$$t^{-1}V_{\mathbf{L}_-} - tV_{\mathbf{L}_+} = (t^{1/2} - t^{-1/2})V_{\mathbf{L}_0} \qquad (5)$$

where $\mathbf{L}_+$*,* $\mathbf{L}_-$*, and* $\mathbf{L}_0$ *have projections as pictured in Figure (8) and are identical outside the dotted circle. Also it satisfies*

$$\begin{aligned} V_{\mathbf{U}} &= 1 \\ V_{\mathbf{O}} &= -t^{1/2} - t^{-1/2}, \end{aligned} \qquad (6)$$

where $\mathbf{U}$ *is the unlink and* $\mathbf{O}$ *is the unknot.*

(b) The above skein relation and normalization conventions, Equations (5) and (6), completely determine the Jones polynomial and give an effective algorithm for computation from a link projection.

Pf:

(a) By taking a linear combination of Equation (2) and Equation (2) rotated 90°, we see the Kauffman bracket satisfies

$$-A^{-1}\langle \mathbf{L}_- \rangle + A\langle \mathbf{L}_+ \rangle = (A^2 - A^{-2})\langle \mathbf{L_0} \rangle.$$

hence

$$\begin{aligned} & -A^{-1}(-A)^{-3w(\mathbf{L_0})}\langle \mathbf{L}_- \rangle + A(-A)^{-3w(\mathbf{L_0})}\langle \mathbf{L}_+ \rangle \\ = \; & (A^2 - A^{-2})(-A)^{-3w(\mathbf{L_0})}\langle \mathbf{L_0} \rangle \end{aligned}$$

or

$$\begin{aligned} & A^{-4}(-A)^{-3w(\mathbf{L}_-)}\langle \mathbf{L}_- \rangle - A^{4}(-A)^{-3w(\mathbf{L}_+)}\langle \mathbf{L}_+ \rangle \\ = \; & (A^2 - A^{-2})(-A)^{-3w(\mathbf{L_0})}\langle \mathbf{L_0} \rangle. \end{aligned}$$

Substituting $t^{1/4}$ gives the correct equation.

Equations (6) follow from the definition.

(b) We will first argue that there is an algorithm to turn any projection of any link into a projection of a distant union of unknots by changing a subset of the crossings. To do this, start at some point on some component, and travel along the component until you return to that point. Then move to a point on a different component and travel along it. Repeat this until you have traveled along the entire link. Each time you come to a crossing for the first time, change the sign of the crossing if necessary to make the strand you are on go under. When you are done, the link projection will be *layered.* In particular, the link of which it is a projection may be isotoped so that each component gets monotonically closer to the viewer in the direction you traveled, except for immediately before each starting point, and so that each succeeding component is closer to the viewer than all of each preceding component (in the order in which they were traveled). Thus the link is a distant union of unknots.

Define the unlinking number of a projection to be the least number of crossings which must be changed to make it a projection of a distant union of unknots. Define the complexity of the projection to be the unlinking number plus the square of the number of crossings. We shall prove the claim by induction on the complexity of a projection.

If a projection has unlinking number zero (which includes the base case when the complexity is zero), it is a projection of a distant union of n unknots. But if $\mathbf{O}_n$ is the distant union of n unknots, and $\mathcal{O}_+$, $\mathcal{O}_-$, and $\mathcal{O}_0$ are the projections of $\mathbf{O}_n$, $\mathbf{O}_n$ and $\mathbf{O}_{n+1}$ shown in Figure (9), then the skein relation Equation (5) says $t^{-1}V_{\mathbf{O}_n} - tV_{\mathbf{O}_n} = (t^{1/2} - t^{-1/2})V_{\mathbf{O}_{n+1}}$ so $V_{\mathbf{O}_{n+1}} = (-t^{1/2} - t^{-1/2})V_{\mathbf{O}_n}$. Thus by induction (using Equation (6) as the base case), $V_{\mathbf{O}_n} = (-t^{1/2} - t^{-1/2})^n$.

If the complexity is nonzero and the unlinking number is zero, apply the previous paragraph. If the complexity is nonzero and the unlinking number is nonzero, there is a crossing which when changed reduces the unlinking number. Call this link $L_\pm$, call the link with this crossing reversed $\mathbf{L}_\mp$, and call the link with that crossing smoothed to parallel strands L_0. Then the skein relation, Equation (5), says

$$V_{L_\pm} = \mp t^{\mp 2} V_{L_\mp} \mp t^{\mp 1}(t^{1/2} - t^{-1/2})V_{L_0}. \qquad (7)$$

Therefore $V_{L_\pm}$ is determined by $V_{L_\mp}$ and V_{L_0}. The first of these links has smaller complexity because it has the same number of crossings and smaller unlinking number, the second because it has one fewer crossings and certainly its unlinking number is less than its number of crossings. Thus by inductive hypothesis their Jones polynomials, and hence $V_{L_\pm}$, can be computed from these relations.

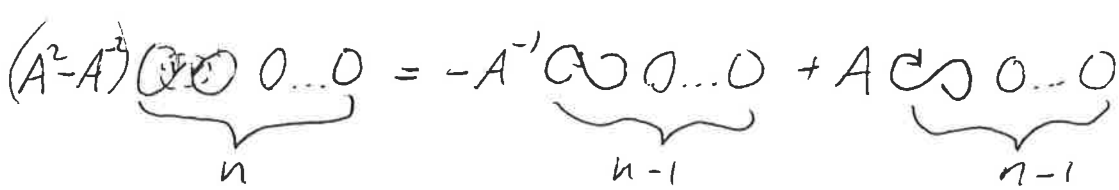


Figure 9: Inductive computation of $V_{\mathbf{O}_n}$

■

1.2.3 Properties of the Jones Polynomial

Proposition 7 *The Jones polynomial is of the form* $(t^{1/2})^n P(t, t^{-1})$, *where* n *is the number of components and* $P(t, t^{-1})$ *is a polynomial in* t *and* t^{-1}. *Thus* $V_L(t)$ *is 'almost' a Laurent polynomial in* t.

Pf: Again, by induction on the complexity of the projection. If the unlinking number is zero, the link is a distant union of n unknots, and as noted above, its Jones polynomial is $t^{n/2}(-1 - t^{-1})^n$, so it is of the correct form. If the unknotting number is not zero, pick a crossing which will reduce it and apply the skein relation. Both the resulting projections have lower complexity and thus the claim applies to them. Thus writing $V_{L_\mp} = t^{n_1/2}P_1(t)$ and $V_{L_0} = t^{n_2/2}P_2(t)$, Equation (7) says

$$V_{L_\pm} = \mp t^{n_1/2 \mp 2} P_1(t) \mp t^{(n_2+1)/2}[t^{\mp 1}(1 - t^{-1})P_2(t)].$$

Since $L_\mp$ has the same number of components as $L_\pm$, $n = n_1$, and since L_0 has one more or fewer, $n = (n_2 + 1) \bmod 2$. Thus $V_{L_\pm}$ is of the correct form. ■

Proposition 8 $V_{L_1 \amalg L_2}(t) = V_{L_1}(t) \cdot V_{L_2}(t)$.

Pf: The Kauffman bracket is multiplicative under distant unions by Proposition (4) and the writhe is additive. ■

Proposition 9 *If* L *can be written in any way as a connect sum of* L_1 *and* L_2 *then* $V_L(t) = V_{L_1}(t) \cdot V_{L_2}(t)/(-t^{1/2} - t^{-1/2})$.

Pf: By Proposition (5) and the additivity of writhe under connect sum. ■

Proposition 10 *Reversing the orientation of every component of* L *does not change* $V_L(t)$.

Pf: Let L' be L with every component reversed. Considering $V_{L'}(t)$ as an invariant of L, notice that the skein relation and the two normalization relations hold true. Thus by Proposition (6b), it is equal to $V_L(t)$. ■

Proposition 11 *If* L' *is the mirror image of* L, $V_L(t) = V_{L'}(t^{-1})$.

Pf: By Proposition (3) and the fact that the mirror image of a projection has the negative of its writhe. ■

2 Tangle Representations

2.1 Tangles

Tangles are discussed in [FY89] and [RT90].

2.1.1 Tangles and Their Projections

Let the variables $\hat{n}$ and $\hat{m}$ represent arbitrary sequences of $+$ and $-$, let D be the unit disk in $\mathbf{C}$, and let $D_{\hat{n}}$ be the unit disk with a sequence of evenly spaced distinguished points along the x-axis from -1 to 1 with orientations corresponding to $\hat{n}$. An $(\hat{n}, \hat{m})$ tangle is an oriented one dimensional submanifold with boundary of the cylinder $D \times I$, which intersects $D \times \{0\}$ and $D \times \{1\}$ exactly at the boundary points, and such that $D \times \{0\}$ and $D \times \{1\}$ with these boundary points distinguished are exactly $D_{\hat{n}}$ and $D_{\hat{m}}$ respectively. The distinguished points on $D_{\hat{n}}$ and $D_{\hat{m}}$ are called *strands*. Two $(\hat{n}, \hat{m})$ tangles are considered the same if an isotopy of $D \times I$ takes one in an orientation preserving fashion to the other, such that it continues to be an $(\hat{n}, \hat{m})$ tangle throughout the isotopy.

Unoriented tangles are defined the same way, except that $\hat{n}$ and $\hat{m}$ represent nonnegative integers and $D_{\hat{n}}$ contains just an evenly spaced sequence of unoriented points. Framed tangles are also defined the same way, except that the components are homeomorphic to $S^1 \times I$ and $I \times I$, and $\{0\} \times I$ and $\{1\} \times I$ are required to lie on the x-axis at the top or bottom (with $\{0\} \times [0,1]$ and $\{1\} \times [0,1]$ imbedded positively into the x-axis).

Tangles form a category. Recall that a category consists of a set of objects and a set of morphisms. Each morphism has associated two objects called its domain and range, and if domain(A) = range(B) there exists a composition morphism $A \circ B$ with domain that of B and range that of A. Composition must be associative, i.e., $(A \circ B) \circ C = A \circ (B \circ C)$ where defined, and to each object a there must correspond an identity morphism $\mathbf{1}_a$, with domain and range a and $\mathbf{1}_a \circ A = A$, $B \circ \mathbf{1}_a = B$ where defined.

The objects for the tangle category are sequences $\hat{n}$, morphisms are tangles (up to equivalence). The domain of an $(\hat{n}, \hat{m})$ tangle is $\hat{n}$, the range is $\hat{m}$. The composition $A \circ B$ is formed by mapping B's $D \times I$ into $D \times [0, 1/2]$ and A's into $D \times [1/2, 1]$. It is easy to check that this product is well defined and associative, and that $D_{\hat{n}} \times I$ is $\mathbf{1}_{\hat{n}}$. We will draw tangles with $[0, 1]$ going from the bottom of the page to the top, so that $A \circ B$ has A on the top, and if $\hat{n}$ and $\hat{m}$ have all positive entries, then an $(\hat{n}, \hat{m})$ tangle has all strands going up.

Tangles in fact form a strict monoidal category. That is, they admit a tensor product functor. The tensor product of $\hat{n}$ and $\hat{m}$ is the composition $\hat{n}$ followed by $\hat{m}$. The tensor product of two tangles A and B is formed by gluing A and B into $X \times I$ and $Y \times I$ respectively, where X and Y are tangent circles in D with centers on the x-axis, sized so that the distinguished points are equally spaced. See Figure (10).

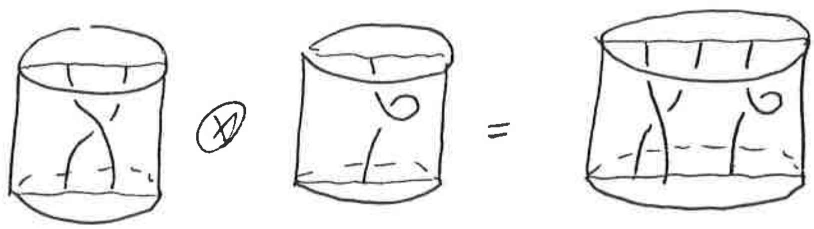


Figure 10: An example of the tensor product functor

As before it is easy to check that the tensor product is well defined and associative. Also, $(A \otimes B) \circ (C \otimes D) = (A \circ C) \otimes (B \circ D)$ when both sides are well defined. The empty object $\hat{0}$, and the empty $(\hat{0}, \hat{0})$ tangle are the identities for tensor product.

Links in S^3 can be naturally identified with $(\hat{0}, \hat{0})$ tangles, by imbedding $D \times I$ in S^3. Tensor product and composition both correspond to distant union.

We will find it convenient to consider labeled tangles. These are tangles which have an element of some label set Λ associated to each component (closed or open). Labeled tangles form a strict monoidal category, with objects being sequences of labeled oriented points and composition and tensor product working as before.

Imbedding $D \times I$ in $\mathbf{R}^3$, one can project a tangle onto any plane, except that we require that the normal chosen have positive inner product with the vector $(i, 0)$ in $D \times I$, and that the image of the tangle lie between the image of the x-axis $\times\{0\}$ and the x-axis $\times\{1\}$. A regular projection is defined as before.

Theorem 5 *Any tangle projection can be rotated slightly to a regular projection. Two regular (oriented, unoriented, framed oriented, framed unoriented)* *tangle projections come from the same tangle class if and only if they can be connected by a sequence of (oriented, unoriented, framed oriented, framed unoriented) Reidemeister moves of Figures (2) and (4) and their mirror images, as well as planar isotopy which keeps the image of the tangle between the images of the two x-axes.*

Pf: *(Sketch)* The argument is the same as in Theorem (1). The one difference is the allowable projections do not form a sphere, they form a subspace of it which is topologically a disk. ■

2.1.2 Generators and Relations

Viewed as projections modulo the Reidemeister moves, it is clear that every tangle can be written as a composition of very simple tangles, which are tensor products of finitely many generating tangles. Furthermore, a complete set of relations can easily be written. Let $\mathcal{A}$ through $\mathcal{E}$ be the projections of unoriented tangles shown in Figure (11).

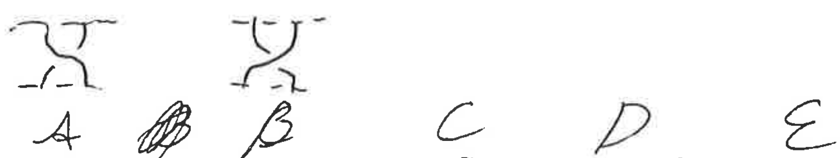


Figure 11: Generating tangles

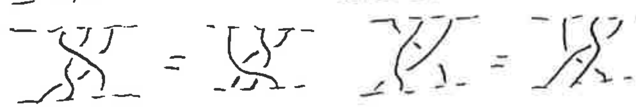
Theorem 6

(a) Every (framed or unframed) unoriented tangle is a composition of tangles of the form $X \otimes Y \otimes Z$, where X and Z are tensor products of some number of factors $\mathcal{E}$ and Y is one of $\mathcal{A}$ through $\mathcal{E}$.

(b) Two products of generators are in the same unframed tangle class if and only if they can be related by a sequence of the following moves, where the moves are understood to apply if both are tensored on the left and right by any number of factors $\mathcal{E}$.

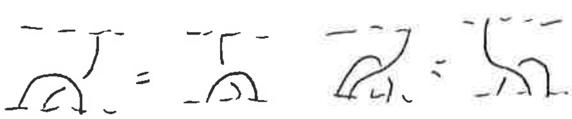
I 

II 

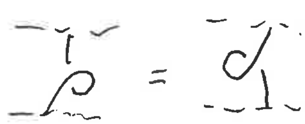
III 

IV $X \circ I_1 = X = I_2 \circ X$ *where* X *is any generator and* I_1*and*I_2 *are the identities of its domain and range respectively.* $(X \otimes I_1) \circ (I_2 \otimes W) = (I_3 \otimes W) \circ (X \otimes I_4)$ *if* I_2 *and* I_3 *are the identities of* X*'s domain and range and* I_4 *and* I_1 *are the identities of* W*'s domain and ranges.*

V 

VI 

(c) The above claim applies to unoriented framed tangles if I is replaced by

I^f 

(d) replacing $\mathcal{A}$ *through* $\mathcal{E}$ *with the fourteen oriented tangles gotten by giving each of them orientations gives a set of generators for (framed or unframed) oriented tangles. The above relations for framed and unframed unoriented tangles gives a complete set of relations for framed and unframed oriented tangles if in each one every possible orientation is substituted.*

Pf:

(a) Consider a projection of a tangle, and view it as sitting inside $I \times I$, with the top and bottom boundaries identified with the range and domains of the tangle respectively (just as we always draw it). Suppose the projection has the property that each vertex and crossing occurs at a different height. Separating the projection along a horizontal line

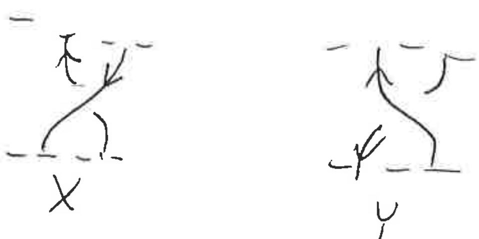


Figure 18: Definition of X and Y

as is Equation (10), and the last equation of Relation III now translates easily to Equation (12). ■

Remark 1 *The identification of $V_{-\lambda}$ with $V_{+\lambda}^*$ is convenient for computation, but it should not be mistaken for a canonical identification. In particular, One could have chosen the bilinear form which gives that identification to be $\mathcal{F}(D)$ instead of $\mathcal{F}(C)$ (in this case the operator associated with $\mathcal{F}(C)$ would have been ϕ_λ^{-1}). One may take ϕ_λ to measure the failure of these two identifications to agree.*

2.2.3 Extending the Label Set

Let $\mathcal{F}$ be a tangle representation with label set Λ and let Λ' be another label set. Suppose $\mathcal{G}$ is a function which takes each sequence of signed elements of Λ' to a sequence of signed elements of Λ in such a way that $\hat{0}$ goes to $\hat{0}$ and tensor products are preserved, and takes each tangle labeled by Λ' to a tangle labeled by Λ whose domain and range are the image of the domain and range of the original tangle, in such a way that $\mathcal{G}(T \circ S) = \mathcal{G}(T) \circ \mathcal{G}(S)$ and $\mathcal{G}(T \otimes S) = \mathcal{G}(T) \otimes \mathcal{G}(S)$ and $\mathcal{G}(\mathbf{1}_{\hat{n}})$ is the appropriate identity morphism (i.e., it is a functor between these two monoidal categories). Then it is immediate from the definition that $\mathcal{FG}$ is a tangle representation with label set Λ'.

For example, Let Λ' be Λ together with a new label 1, and let $\mathcal{G}$ take a tangle T labeled by Λ' to the tangle formed by deleting every component labeled by 1. It is clear that this is well defined on tangle classes and is a functor. See Figure (19) for an example. The label 1 is called the trivial label. One can check that $V_{\pm 1} = \mathbf{F}$.

Similarly, let Λ' be a label set which contains every label in Λ as well as a new label λ^* for each label $\lambda \in \Lambda$. Let $\mathcal{G}$ send each tangle T labeled by Λ' to the tangle formed by reversing the orientation of each component labeled by λ^* and relabeling it by λ. See Figure (19) for an example. Again this is a functor and produces a tangle representation labeled by Λ'. The label λ^* is called the dual label to λ. Notice that $V_{\pm\lambda^*} = V_{\mp\lambda}$.

between each of these heights, we have written the tangle as a product of tangles each one of which clearly corresponds to one of the generators given. But any projection which does not have this property can clearly be isotoped slightly so that it does (by an argument similar to that used in Theorem (1)). Thus every tangle is equivalent to a product of the generators.

(b) If T is a product of tangles, than it is clear that we can find a projection of T with every crossing and vertex at a different height, and such that the product of generators gotten from this projection is exactly the product of generators started with, perhaps with copies of the identity tangle interposed. So we need only show that one can connect any two projections of a given tangle class by a sequence of the given relations, where they are interpreted as saying that there is one vertex between each dotted line. This involves showing that these relations generate all planar isotopy, and that they generate all Reidemeister moves.

A planar isotopy will only change the sequence of generators if a vertex is added or removed from a straight line segment or the order of the heights of two consecutive vertices or crossings is switched. A vertex being added or removed corresponds to adding or removing the identity, which is the first half of relation IV. If the two points passing each other do not share a common line segment, then the change in generators is just the second half of relation IV (see Figure (12) for an example). If they share a common line segment, then that segment must be flat

Figure 12: Commuting generators being moved past each other

at that point. Thus it corresponds to a line segment moving from downward to upward or vice versa, as pictured in Figure (13). This is easily seen to be a composition of relations V and VI.

Given any Reidemeister move, it is clearly possible by planar isotopy to make the two projections it connects look like two of the products of generators pictured in relations I-III. This completes the proof.

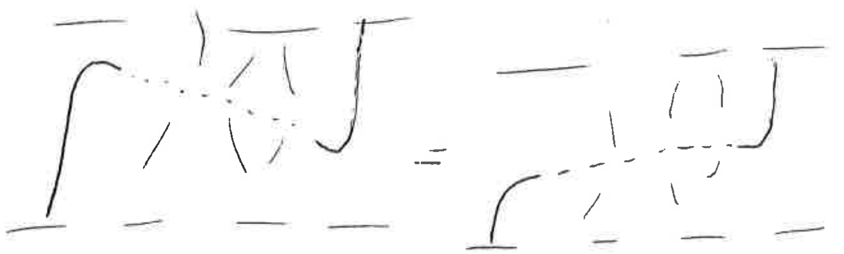


Figure 13: A line segment going from downward to upwards

(c) Immediate from the definition.

■

Corollary 2

(a) The tangles $\mathcal{A}$ and $\mathcal{C}$-$\mathcal{E}$ form a minimal set of generators for the unoriented tangle category, with the following relations, I^f replacing I in the framed category.

I

I^f

II

III

IV $X \circ I_1 = X = I_2 \circ X$ *where X is any generator and $I_1 and I_2$ are the identities of its domain and range respectively.* $(X \otimes I_1) \circ (I_2 \otimes W) = (I_3 \otimes W) \circ (X \otimes I_4)$ *if I_2 and I_3 are the identities of X's domail and range and I_4 and I_1 are the identities of W's domain and ranges.*

V

VI

(b) The tangles given in Figure (14) are a minimal set of generators for the oriented tangle category, with the following relations, I^f replacing I in the framed category.

I

I^f

II

III

IV $X \circ I_1 = X = I_2 \circ X$ *where* X *is any generator and* $I_1 and I_2$ *are the identities of its domain and range respectively.* $(X \otimes I_1) \circ (I_2 \otimes W) = (I_3 \otimes W) \circ (X \otimes I_4)$ *if* I_2 *and* I_3 *are the identities of* X*'s domail and range and* I_4 *and* I_1 *are the identities of* W*'s domain and ranges.*

V

VI

Pf:

(a) Notice $\mathcal{B}$ is equal to either of the products of minimal generators occurring in relation VI. It is the easy to check that relations V and VI of

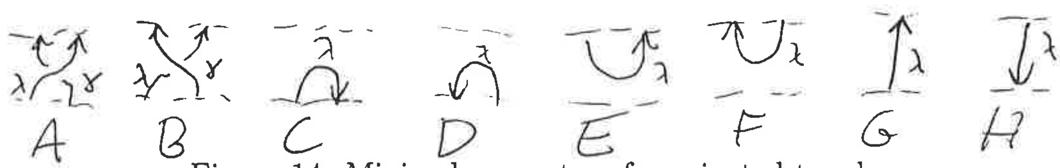


Figure 14: Minimal generators for oriented tangles

Theorem (6a) follow from relations V and VI here. Thus these relations suffice to show the equivalence of any product of minimal generators related by planar isotopy. Now it is easy to show that relations I-III of the theorem follow from relations I-III above and planar isotopy. The one special case is the second equation in relation III, which follows from the first by a rotation and an application of relation II.

(b) In Figure (15) each of the generators not given in Figure (14) is expressed in two different ways in terms of the minimal set of generators (mirror images are defined by the mirror image of the corresponding definition). The equality of those expressions follows from V and VI. With this in hand, Relations V and VI of Theorem (6b) follow from V and VI above. Thus these relations suffice to show the equivalence of any product of minimal generators related by planar isotopy. But now any relation in part a. of this corollary with any orientation is equivalent to one of these relations under planar isotopy.

■

Figure 15: Expressing the other generators in terms of the minimal set

2.2 Tangle Representations

Tangle representations are mentioned in [FY89], but not much of what we do with them appears there. The material here is quite standard as applied to the specific tangle representations coming from quasitraingular Hopf algebras. The best source for this is one of the originals, [RT90]. The notation used here is meant to dovetail with the language of quasitriangular Hopf algebras (i.e., of quantum groups), and thus is sometimes slightly awkward in itself.

An approach more like the one done here in technique is used in [Wen90, Wen93], except he works with braids instead of tangles, and only uses the subalgebra of the commutant spanned by tangles.

2.2.1 Definition

The category with vector spaces as objects and linear maps as morphisms is also a strict monoidal category, with the obvious sense of tensor product. It is thus natural to look for functors from labeled tangles to vector spaces.

Definition 3 *A* tangle representation *with label set* Λ *is a map which sends each object* $\hat{n}$ *to a finite dimensional vector space* $V_{\hat{n}}$ *in such a way that tensor product is taken to vector tensor product and* $V_{\hat{0}}$ *is explicitly identified with* $\mathbf{F}$*, and which sends each* $(\hat{n}, \hat{m})$ *tangle* T *to a linear map* $\mathcal{F}(T)$ *from* $V_{\hat{n}}$ *to* $V_{\hat{m}}$ *such that* $\mathcal{F}(T \circ S) = \mathcal{F}(T)\mathcal{F}(S)$*,* $\mathcal{F}(T \otimes S) = \mathcal{F}(T) \otimes \mathcal{F}(S)$*,and* $\mathcal{F}(\mathbf{1}_{\hat{n}}) = id_{V_{\hat{n}}}$*.*

Of course, if T is a $(\hat{0}, \hat{0})$ tangle, i.e. a link, then $\mathcal{F}(T)$ is a linear map from the ground field to itself, and thus can be identified with an element of the ground field. This is then a numerical link invariant, and we may think of tangle representations as a generalization of link invariants.

It is clear that the value of an oriented tangle representation on objects is determined by the vector spaces $V_{\pm\lambda}$ for each $\lambda \in \Lambda$. Specifically, if $\hat{n}$ is the sequence $(n_1, \ldots, n_k)$ and each n_i is $\pm\lambda$ for some λ, then $V_{\hat{n}} = \bigotimes_{i=1}^{k} V_{n_i}$. Similarly, an unoriented tangle representation on objects is determined by the vector spaces V_λ for $\lambda \in \Lambda$, and if $\hat{n} = (\lambda_1, \ldots \lambda_k)$ then $V_{\hat{n}} = \bigotimes_{i=1}^{k} V_{\lambda_i}$.

An unoriented tangle representation $\mathcal{F}$ can be turned into an oriented tangle representation with the same label set as follows. Defining $V_{\pm\lambda} = V_\lambda$, let T be an oriented $(\hat{n}, \hat{m})$ tangle with labels in Λ, and let T' be the labeled unoriented $(\hat{n}', \hat{m}')$ tangle gotten by removing the orientations from T. Clearly we can identify $V_{\hat{n}}$ and $V_{\hat{m}}$ with $V_{\hat{n}'}$ and $V_{\hat{m}'}$ respectively in a canonical way. Then $\mathcal{F}'(T) = \mathcal{F}(T')$ as a linear map from $V_{\hat{n}}$ to $V_{\hat{m}}$ is easily seen to be an oriented tangle representation. Thus from now on we will deal only with oriented tangle representations.

Suppose $\mathcal{F}$ is a tangle representation on Λ, and consider vector spaces $W_{\pm\lambda}$ and invertible linear maps $f_{\pm\lambda} : W_{\pm\lambda} \to V_{\pm\lambda}$ for each $\lambda \in \Lambda$. Define $W_{\hat{n}} = \bigotimes_{i=1}^{k} W_i$, and define $f_{\hat{n}} : W_{\hat{n}} \to V_{\hat{n}}$ by $f_{\hat{n}} = \bigotimes_{i+1}^{k} f_{n_i}$. Then it is easy to check that the assignment of the map $f_{\hat{m}}^{-1}\mathcal{F}(T)f_{\hat{n}}$ to each tangle T gives a new tangle representation which may be considered a simple renaming of the first. This is said to be a conjugate tangle representation to $\mathcal{F}$.

Finally, let $\lambda, \gamma \in \Lambda$ be two labels for a tangle representation $\mathcal{F}$, and let $f_\pm : V_{\pm\lambda} \to V_{\pm\gamma}$ be invertible maps. Let T be an $(\hat{n}, \hat{m})$ tangle with a component c labeled by λ and let T' be the $(\hat{n}', \hat{m}')$ tangle gotten by changing the label on c to γ (of, course, one or both of $\hat{n}'$, $\hat{m}'$ may be the same as $\hat{n}$, $\hat{m}$ respectively). Let $f_{\hat{n}} : V_{\hat{n}} \to V_{\hat{n}'}$ be defined by putting the identity in every tensor factor in which $\hat{n}$ and $\hat{n}'$ have the same entries and $f_\pm$ in every factor in which $\pm\lambda$ changes to $\pm\gamma$. Define $f_{\hat{m}}$ similarly. Then $f_\pm$ is said to be an *isomorphism* if for every such T we have $\mathcal{F}(T) = f_{\hat{m}}^{-1}\mathcal{F}(T')f_{\hat{n}}$. We say λ and γ are *equivalent*, written $\lambda \sim \gamma$, if there exists an isomorphism between them.

Of course, equivalence just means that one can replace any λ labels by γ (or vice versa) and no information is lost. If one eliminates all but one label in Λ for each equivalence class, and restricts the tangle representation to this smaller label set, one gets a tangle representation in which no two labels are equivalent. Such a tangle representation is called *reduced*, and we will reduce any tangle representations we encounter before working with them. Of course, the value of the original tangle representation on any tangle is determined by the value of the reduced tangle on the appropriately relabeled tangle.

2.2.2 Minimal Data and Relations

Let $\mathcal{F}$ be a tangle representation and let $\{V_{\pm\lambda}\}_\Lambda$ be the associated vector spaces. Consider the named tangles in Figure (14). If the upper component of A is labeled by λ and the lower component is labeled by γ, then $\mathcal{F}(A)$ is a map from $V_{+\lambda} \otimes V_{+\gamma}$ to $V_{+\gamma} \otimes V_{+\lambda}$. If P is the map from $V_{+\gamma} \otimes V_{+\lambda}$ to $V_{+\lambda} \otimes V_{+\gamma}$ given by $P(v \otimes w) = w \otimes v$, then define the operator $R_{\lambda,\gamma} = P\mathcal{F}(A)$ on the space $V_{+\lambda} \otimes V_{+\gamma}$. This is sometimes called the R-matrix. The first part of Relation II of Corollary (2b) then implies that $\mathcal{F}(B)$ is the map $R_{+\gamma,+\lambda}^{-1}P$

if the upper component is labeled by λ and the lower by γ.

If C is labeled by λ, then $\mathcal{F}(C)$ is a linear functional on $V_{+\lambda}\otimes V_{-\lambda}$. Thus it may be thought of as a bilinear form on these two spaces, $\langle\cdot,\cdot\rangle$. If E is labeled by λ, then $\mathcal{F}(E)$ may be thought of as an element $\sum_i a_i \otimes b_i \in V_{-\lambda}\otimes V_{+\lambda}$ (namely, the element that $1 \in \mathbf{F}$ gets mapped to). Relation V of Corollary (2b) forces two relations among these. Namely, for all $x \in V_{+\lambda}$ we have $\sum_i \langle x, a_i\rangle b_i = x$, and for each $y \in V_{-\lambda}$ we have $\sum_i \langle b_i, y\rangle a_i = y$. The first equation in particular says that for no nonzero x is the linear functional on $V_{-\lambda}$ given by $\langle x,\cdot\rangle$ identically zero, and the second says that for no nonzero y is the linear functional $\langle\cdot, y\rangle$ on $V_{+\lambda}$ indentically zero. Thus $\langle\cdot,\cdot\rangle$ is a nondegenerate bilinear form and the map from $V_{-\lambda}$ to $V_{+\lambda}^*$ given by the bracket is a linear isomorphism. With this identification $\mathcal{F}(E)$ corresponds to the element of $V_{+\lambda}^*\otimes V_{+\lambda}$ given by $\sum_i a_i^*\otimes a_i$, where a_i is any basis of $V_{+\lambda}$ and a_i^* is its dual basis of $V_{+\lambda}^*$ (it is a worthwhile exercise to check that this element of the tensor product space does not depend on the basis chosen).

The same argument applies to give a linear functional on $V_{-\lambda}\otimes V_{+\lambda}$ corresponding to $\mathcal{F}(D)$. This may be interpreted as a map which sends each element of $V_{+\lambda}$ linearly to an element of the dual of $V_{-\lambda}$. But using the bilinear form of the previous paragraph to identify $V_{-\lambda}$ with $V_{+\lambda}^*$, we have a linear map from $V_{+\lambda}$ to its double dual, which since $V_{+\lambda}$ is finite dimensional is just $V_{+\lambda}$ again. That is, there is a map ϕ_λ on $V_{+\lambda}$ such that $\mathcal{F}(D)(y\otimes x) = \langle\phi_\lambda(x), y\rangle$. An application of relation V of Corollary (2b) shows that ϕ_λ must be invertible and $\mathcal{F}(F)$ as an element of $V_{+\lambda}\otimes V_{+\lambda}^*$ is just $\sum_i \phi_\lambda^{-1}(a_i)\otimes a_i^*$, where a_i and a_i^* are as above.

All of these identifications are summarized in Table (1), where a_i is any basis of $V_{+\lambda}$ and a_i^* is its dual basis of $V_{-\lambda}$ under the bilinear form given by $\langle\cdot,\cdot\rangle$.

We will need some notation dealing with linear maps among tensor product spaces. Let $A_1, A_2, \ldots, A_k$ and $B_1, B_2, \ldots, B_j$ be finite dimensional vector spaces and let

$$f : \bigotimes_{i=1}^{k} A_i \to \bigotimes_{i=1}^{j} B_i$$

be a linear map. Let 1_{B_i} be the identity map on B_i, let B_i^* be the set on linear maps from B_i to the ground field $\mathbf{F}$, let v_n be a basis for A_1 and v_n^* be the dual basis for A_1^* (so that $v_m^*(v_n) = \delta_{n,m}$) and let w_n, w_n^* be dual bases

$\mathcal{F}$ of	takes	to
A	$x \otimes y \in V_{+\lambda} \otimes V_{+\gamma}$	$P \cdot R_{\lambda,\gamma}(x \otimes y) \in V_{+\gamma} \otimes V_{+\lambda}$
B	$x \otimes y \in V_{+\lambda} \otimes V_{+\gamma}$	$R^{-1}_{\gamma,\lambda} \cdot P \in V_{+\gamma} \otimes V_{+\lambda}$
C	$x \otimes y \in V_{+\lambda} \otimes V_{-\lambda}$	$\langle x, y \rangle \in \mathbf{F}$
D	$y \otimes x \in V_{-\lambda} \otimes V_{+\lambda}$	$\langle \phi_\lambda(x), y \rangle \in \mathbf{F}$
E	$c \in \mathbf{F}$	$c \sum_i a_i^* \otimes a_i \in V_{-\lambda} \otimes V_{+\lambda}$
F	$c \in \mathbf{F}$	$c \sum_i \phi^{-1}(a_i) \otimes a_i^* \in V_{+\lambda} \otimes V_{-\lambda}$
G	$x \in V_{+\lambda}$	$x \in V_{+\lambda}$
H	$y \in V_{-\lambda}$	$y \in V_{-\lambda}$

Table 1: Identifying tangle generators with linear maps

for A_k and A_k^*. Then define maps

$$\begin{aligned} f^\flat : A_1 \otimes \cdots \otimes A_k \otimes B_j^* &\rightarrow B_1 \otimes \cdots \otimes B_{j-1} \\ {}^\flat f : B_1^* \otimes A_1 \otimes \cdots \otimes A_k &\rightarrow B_2 \otimes \cdots \otimes B_j \\ f^\sharp : A_2 \otimes \cdots \otimes A_k &\rightarrow A_1^* \otimes B_1 \otimes \cdots \otimes B_j \\ {}^\sharp f : A_1 \otimes \cdots \otimes A_{k-1} &\rightarrow B_1 \otimes \cdots \otimes B_j \otimes A_k^* \end{aligned}$$

defined by the following formulae:

$$\begin{aligned} f^\flat(a_1 \otimes \cdots \otimes a_k \otimes b_j^*) &= (1_{B_1} \otimes \cdots \otimes 1_{B_{j-1}} \otimes b_j^*) f(a_1 \otimes \cdots \otimes a_k) \\ {}^\flat f(b_1^* \otimes a_1 \otimes \cdots \otimes a_k) &= (b_1^* \otimes 1_{B_2} \otimes \cdots \otimes 1_{B_j}) f(a_1 \otimes \cdots \otimes a_k) \\ f^\sharp(a_1 \otimes \cdots \otimes a_{k-1}) &= \sum_n f(a_1 \otimes \cdots \otimes a_{k-1} \otimes w_n) \otimes w_n^* \\ {}^\sharp f(a_2 \otimes \cdots \otimes a_k) &= \sum_n v_n^* \otimes f(v_n \otimes a_2 \otimes \cdots \otimes a_k). \end{aligned}$$

Notice that ${}^\flat({}^\sharp f) = {}^\sharp({}^\flat f) = f = (f^\sharp)^\flat = (f^\flat)^\sharp$.

Lemma 3 *Let T be a tangle and $T^\flat$, ${}^\flat T$, $T^\sharp$, and ${}^\sharp T$ be as pictured in Figure (16). Then the following equations hold:*

$$\begin{aligned} \mathcal{F}(T^\flat) &= \mathcal{F}(T)^\flat \\ \mathcal{F}({}^\flat T) &= {}^\flat[(\phi_\lambda \otimes 1 \otimes \cdots \otimes 1)\mathcal{F}(T)] \\ \mathcal{F}(T^\sharp) &= [\mathcal{F}(T)(1 \otimes \cdots \otimes 1 \otimes \phi_\lambda^{-1})]^\sharp \\ \mathcal{F}({}^\sharp T) &= {}^\sharp \mathcal{F}(T) \end{aligned}$$

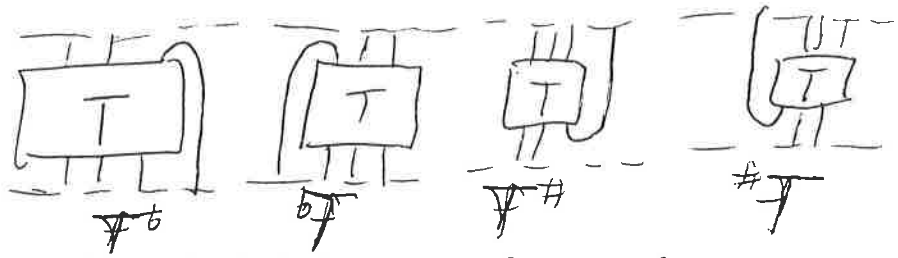


Figure 16: Rotating one strand on a tangle

Given an operator $f : V_{+\lambda} \to V_{+\lambda}$, define the quantum trace of f, $\mathrm{qtr}(f)$ to be $\sum_i \langle \phi_\lambda f(a_i), a_i^* \rangle$, where a_i and a_i^* are as above. Define the inverse quantum trace, $\mathrm{iqtr}(f)$, to be $\langle \phi_\lambda^{-1} f(a_i), a_i^* \rangle$. If f is instead a map from $V_{+\lambda} \otimes A$ to $V_{+\lambda} \otimes B$, write it as $\sum_i f_i \otimes f^i$, where each f_i is an operator on $V_{+\lambda}$ and each f^i is a map from A to B, and define $(\mathrm{qtr} \otimes 1)[f]$ to be the map from A to B given by $\sum_i \mathrm{qtr}(f_i) f^i$. Similar definitions apply to $1 \otimes \mathrm{qtr}$, $\mathrm{iqtr} \otimes 1$, and $1 \otimes \mathrm{iqtr}$.

Lemma 4 *Let* T *be a tangle and* ${}^cT$, *and* T^c *be as pictured in Figure (17). Then*

$$\begin{aligned} \mathcal{F}({}^cT) &= (qtr \otimes 1)[\mathcal{F}(T)] \\ \mathcal{F}(T^c) &= (1 \otimes iqtr)[\mathcal{F}(T)] \end{aligned}$$

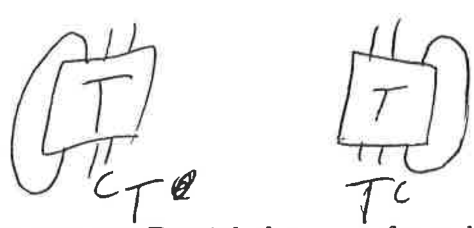


Figure 17: Partial closures of tangles

Pf:(*of the lemmas*) From the definitions above and Table (1). ■

Finally, if $R_{\lambda,\gamma}$ is as defined above, then we can define $R_{\lambda,\gamma}$, $R_{\gamma,\delta}$ and $R_{\lambda,\delta}$ on the space $V_\lambda \otimes V_\gamma \otimes V_\delta$ by $R_{\lambda,\gamma} \otimes 1$, $1 \otimes R_{\gamma,\delta}$ and $(1 \otimes P)(R_{\lambda,\delta} \otimes 1)(1 \otimes P)$ respectively.

Theorem 7 *Any tangle representation with a label set* Λ *yields invertible linear operators* ϕ_λ *and* $R_{\lambda,\gamma}$ *for every* $\lambda, \gamma \in \Lambda$ *satisfying the following equations:*

$$R_{\lambda,\gamma}(\phi_\lambda \otimes \phi_\gamma) = (\phi_\gamma \otimes \phi_\lambda) R_{\lambda,\gamma} \tag{8}$$

$$
\begin{aligned}
(qtr \otimes 1)[PR_{\lambda,\lambda}] &= (1 \otimes iqtr)[PR_{\lambda,\lambda}] && (9)\\
\bar{R}_{\lambda,\gamma} &= (\tilde{R}_{\lambda,\gamma})^{-1} && (10)\\
R_{\gamma,\delta}R_{\lambda,\delta}R_{\lambda,\gamma} &= R_{\lambda,\gamma}R_{\lambda,\delta}R_{\gamma,\delta} \ \ on\ \ V_{+\lambda}\otimes V_{+\gamma}\otimes V_{+\delta} && (11)\\
\bar{R}_{\delta,\gamma}R_{\lambda,\delta}\tilde{R}_{\lambda,\gamma} &= \tilde{R}_{\lambda,\gamma}R_{\lambda,\delta}\bar{R}_{\delta,\gamma} \ \ on\ \ V_{+\lambda}\otimes V_{-\gamma}\otimes V_{+\delta} && (12)
\end{aligned}
$$

where

$$\tilde{R}_{\lambda,\gamma} = P^{\flat}[(\phi_\lambda \otimes 1)R^{-1}_{\lambda,\gamma}P(1\otimes \phi_\lambda^{-1})]^{\sharp}$$

and

$$\bar{R}_{\lambda,\gamma} =^{\sharp} (PR_{\lambda,\gamma})^{\flat}P.$$

Furthermore, any set of vector spaces $\{V_{+\lambda}\}_{\lambda\in\Lambda}$ *and maps* ϕ_λ *and* $R_{\lambda,\gamma}$ *satisfying these equations yields a tangle representation defined by Table (1), where* $V_{-\lambda}$ *is taken to be* $V^*_{+\lambda}$ *and the pairing mentioned in the table is the pairing of a space with its dual.*

Pf: First note that the identifications of Table (1) are forced on us by the definition of a tangle representation. The values of G and H follow immediately from the definition, the value of B comes from Relation II of Corollary (2), and the values of E and F follow from Relation V. It is clear that the behavior of tangle representations under tensor products then determines the value on generators, and that the behavior under composition determines its value on compositions of generators. Thus all that remains in order to prove both halves of the theorem are that the above equations imply and are implied by the relations given in Corollary (2) translated to the tangle representation.

Relation I is exactly Equation (9), by Lemma (4). The first two equations of Relation II are equivalent to the definition of B. The first part of Relation III is exactly Equation (11). Relation IV is satisfied automatically, and Relation V is equivalent to the definition of E and F. Relation VI requires that ${}^{\flat}[{}^{\flat}((\phi_\gamma \otimes \phi_\lambda)PR_{\lambda,\gamma}(\phi_\lambda{-}1 \otimes \phi_\gamma^{-1}))^{\sharp}]^{\sharp} =^{\sharp} [{}^{\sharp}(PR_{\lambda,\gamma})^{\flat}]^{\flat}$. But an easy calculation shows that if f and g are maps between $A\otimes B$ and $C\otimes D$, then ${}^{\flat}[{}^{\flat}f^{\sharp}]^{\sharp} =^{\sharp} [{}^{\sharp}g^{\flat}]^{\flat}$ if and only if $f=g$. From this it is easy to see that Relation VI is equivalent to Equation (8).

Finally, notice that if X and Y are as pictured in Figure (18), then by Lemma (3) we have $\mathcal{F}(X) = \tilde{R}_{\lambda,\gamma}$ and $\mathcal{F}(Y) = \bar{R}_{\lambda,\gamma}$. Clearly the last two equations of Relation II are saying exactly that these two maps are inverses, Finally, let Λ' contain every label in Λ as well as a new label $\lambda \otimes \gamma$ for each pair of labels $\lambda, \gamma \in \Lambda$. Define $\mathcal{G}(T)$ to be the tangle gotten by replacing each component labeled by $\lambda \otimes \gamma$ by its 2-cabling, with the cabled components labeled by λ and γ respectively. That is, if a component labeled by $\lambda \otimes \gamma$ is identifed with $I \times I$ (or $S^1 \times I$ respectively), erase the image of $I \times (1/3, 2/3)$ (or $S^1 \times (1/3, 2/3)$ respectively) to leave two components, and label one with λ and the other with γ. This is illustrated in Figure (19). Once again this gives a functor. The label $\lambda \otimes \gamma$ is called the tensor product label of λ and γ. Of course it only makes sense when dealing with framed tangles. One may check that $V_{+\lambda\otimes\gamma} = V_{+\lambda} \otimes V_{+\gamma}$ and $V_{-\lambda\otimes\gamma} = V_{-\gamma} \otimes V_{-\lambda}$.

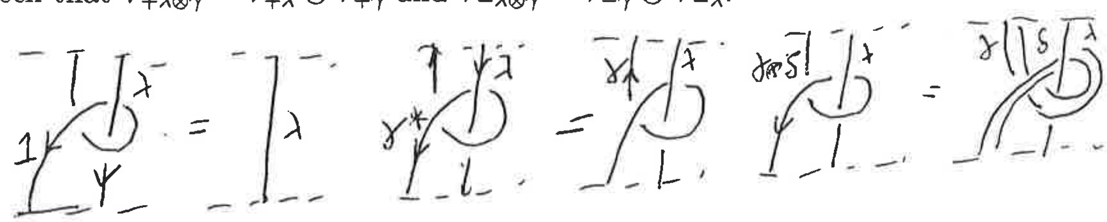


Figure 19: Adding new labels to the label set

We will also define a nontopological operation on the label set. Let $\mathcal{F}$ be a tangle representation with label set Λ, and let Λ' contain every label in Λ as well as a new label $\lambda \oplus \gamma$ for each $\lambda, \gamma \in \Lambda$. Define $V_{\pm\lambda\oplus\gamma} = V_{\pm\lambda} \oplus V_{\pm\gamma}$ to $V_\lambda \oplus V_\gamma$. If $\hat{n}$ is a sequence of signed labels in Λ', identify $V_{\hat{n}}$ with $\bigoplus_\alpha V_{\hat{n}_\alpha}$, where α runs over every way of replacing each direct sum label $\lambda \oplus \gamma$ occurring with sign in the sequence by either λ or γ with the same sign, and $\hat{n}_\alpha$ is the sequence gotten by such a replacement. If T is a tangle with labels in Λ', let α run through all the ways of replacing any label $\lambda \oplus \gamma$ of a component of T by either λ or γ, let T_α be the resulting tangle with labels in Λ, and let $\hat{n}_\alpha$ and $\hat{m}_\alpha$ be its domain and range respectively. Then $\mathcal{F}(T_\alpha)$ is a linear map from $V_{\hat{n}_\alpha}$ to $V_{\hat{m}_\alpha}$, and may be thought of as a linear map from $V_{\hat{n}}$ to $V_{\hat{m}}$ by making its value zero on the other direct summands. Define $\mathcal{G}(T)$ to be $\sum_\alpha \mathcal{F}(T_\alpha)$. This is easily seen to be a tangle representation.

Proposition 12 *The following equivalences of labels hold in the appropriate label sets:* $1^* \sim 1$, $(\lambda^*)^* \sim \lambda$, $(\lambda \otimes \gamma)^* \sim \lambda^* \otimes \gamma^*$, $(\lambda \oplus \gamma)^* \sim \lambda^* \oplus \gamma^*$, $(\lambda \otimes \gamma) \otimes \delta \sim \lambda \otimes (\gamma \otimes \delta)$, $(\lambda \oplus \gamma) \oplus \delta \sim \lambda \oplus (\gamma \oplus \delta)$, $\lambda \otimes \gamma \sim \gamma \otimes \lambda$, $\lambda \oplus \gamma \sim \gamma \oplus \lambda$, $1 \otimes \lambda \sim \lambda$ *and* $\lambda \otimes (\gamma \oplus \delta) \sim \lambda \otimes \gamma \oplus \lambda \otimes \delta$.

Pf: In each case the isomorphism is the obvious map between the two representations, and the fact that it is an isomorphism follows easily from the definitions. The one exception is the commutativity of tensor product. Here define f_+ to be $PR_{\lambda,\gamma}$ and f_- is $\mathcal{F}$ of the tangle which is a negative crossing with all strands oriented downwards. This is easily seen to be an isomorphism. ∎

Definition 4 *A tangle representation on a label set* Λ *is called closed if it is reduced and if there exists the following: an element of* Λ *equivalent to* 1*, for each* $\lambda \in \Lambda$ *an element of* Λ *equivalent to* λ^**, and for each* $\lambda, \gamma \in \Lambda$ *elements equivalent to* $\lambda \otimes \gamma$ *and* $\lambda \oplus \gamma$*. By abuse of notation we will refer to these unique elements of* Λ *as* 1, λ^*, $\lambda \otimes \gamma$ *and* $\lambda \oplus \gamma$ *respectively.*

Proposition 13 *For every reduced tangle representation* $\mathcal{F}$ *with label set* Λ *there is a closed tangle representation* $\mathcal{G}$ *with label set* $\Lambda' \supset \Lambda$ *such that* $\mathcal{F}(T) = \mathcal{G}(T)$ *for every tangle labeled by* Λ.

Pf: Apply the four constructions of this section to extend the label set. Apply them to the resulting enlarged label set. Repeat this process and take the union of all the label sets and the tangle representation defined on it. This will be closed under all the given operations. Now reduce the label set to eliminate equivalent labels. This is now a closed tangle representation. ∎

Corollary 3 *The label set of a closed tangle representation satisfies the axioms of a commutative ring with identity with an involution, except addition has no inverse.*

2.3 The Category of Intertwiners

This material continues to follow [RT90] in spirit and [Wen90, Wen93] in technique. The decomposition according to the commutant (which is simply the decomposition into stable subspaces) and the category of intertwiners is the central idea that makes the quantum field theories of the next chapter work in this general context. Irreducible representations are deemphasized in favor of what are called here nonreducible representations, at the cost of some very strong structure for the sake of generality and simplicity. The notation of relevant and irrelevant labels is highly nonstandard and should be taken with a grain of salt. As far as I know the observations about irrelevant labels do not occur eleswhere.

2.3.1 The Commutant Algebra

Fix a tangle representation $\mathcal{F}$ with label set Λ, and choose a $\lambda \in \Lambda$.

Definition 5 *The* commutant algebra of λ, C_λ, *is the algebra of all operators x on $V_{+\lambda}$ which commute with ϕ_λ and such that $(x \otimes 1)R_{\lambda,\gamma} = R_{\lambda,\gamma}(x \otimes 1)$ and $(1 \otimes x)R_{\gamma,\lambda} = R_{\lambda,\gamma}(1 \otimes x)$.*

The point is that, since ϕ_λ and the R-matrix determine the tangle representation, x commutes with every tangle in a sense which we will now make more precise.

Let T be a tangle and let c be an open component of T. The component c can lie in T in four essentially different ways.

(a) It intersects the top and bottom of the tangle at positively oriented strands: i.e., c goes from bottom to top.

(b) It intersects the top and bottom of the tangle at negatively oriented strands: i.e., c goes from top to bottom.

(c) It intersects the bottom in two strands, one positively and one negatively oriented.

(d) It intersects the top in two strands, one positively and one negatively oriented.

Let x be an operator on $V_{+\lambda}$, and let $\langle\cdot,\cdot\rangle$ be the bilinear form associated with $\mathcal{F}(C)$. Since it is nondegenerate, we can define $x^\dagger$ to be the dual operator to x on $V_{-\lambda}$. That is, $x^\dagger$ is the unique operator on $V_{-\lambda}$ such that $\langle xv, w\rangle = \langle v, x^\dagger w\rangle$, for every $v \in V_{+\lambda}$ and every $w \in V_{-\lambda}$.

Now let T be an $(\hat{n}, \hat{m})$ tangle and c be an open component of T labeled by λ. If c is of type (a) above, define $x_{\hat{n}}$ to be the operator on $V_{\hat{n}}$ which is the identity on every tensor factor except the one corresponding to the strand where c intersects the bottom, where it is x. Define $x_{\hat{m}}$ the same way, except it is an operator on $V_{\hat{m}}$ and x goes in the tensor factor corresponding to the strand where c meets the top of c. If c is of type (b), define $x^\dagger_{\hat{n}}$ and $x^\dagger_{\hat{m}}$ the same way, with all factors having the identity except the one corresponding to the appropriate end of c, which has $x^\dagger$. If c is of type (c), define the operator $x_{\hat{n}}$ on $V_{\hat{n}}$ to be the identity in each tensor factor except the one corresponding to the positively oriented strand of c. Define $x^\dagger_{\hat{n}}$ likewise except that an $x^\dagger$ is put in the factor where c is oriented negatively. If c is if type (d), define $x_{\hat{m}}$ and $x^\dagger_{\hat{m}}$ in the same manner. In short, define the operators so that they can be multiplied on the left or right by $\mathcal{F}(T)$.

Proposition 14 *Suppose $x \in C_\lambda$, $x^\dagger$ is as above, and T is an $(\hat{n}, \hat{m})$ tangle with an open component c labeled by λ. We have the following relationships, depending on which of the four cases above applies:*

(a) $\mathcal{F}(T)x_{\hat{n}} = x_{\hat{m}}\mathcal{F}(T)$

(b) $\mathcal{F}(T)x^\dagger_{\hat{n}} = x^\dagger_{\hat{m}}\mathcal{F}(T)$

(c) $\mathcal{F}(T)x_{\hat{n}} = \mathcal{F}(T)x^\dagger_{\hat{n}}$

(d) $x_{\hat{m}}\mathcal{F}(T) = x^\dagger_{\hat{m}}\mathcal{F}(T)$.

Conversely, if x satisfies the equations (a) – (b) for all tangles, then $x \in C_\lambda$.

Pf: The converse is quite straightforward. If T is the tangle labeled by A in Figure (14), then the equation in (a) says exactly that $x \otimes 1$ and $1 \otimes x$ commute with $R_{\lambda,\gamma}$ and $R_{\gamma,\lambda}$ respectively. If T is the tangle labeled by D, then (c) gives immediately that x commutes with ϕ_λ. The first statement requires a little more work.

First we note that the proposition is true if T is one of the tangles appearing in Figure (14). A and B follow from the fact that x commutes with the R-matrix, C and E are immediate, D and F follow from the fact that x commutes with ϕ_λ, and G and H are trivial. Also, it is clear that if this statement is true of a tangle, it is still true of the tensor product of this tangle with any other. Thus it is true on generators, and we have only to show that it is true on products of generators. We do this by induction on the number of generators.

Suppose (a) – (d) are true of all products of n generators and let T be a product of $n+1$ generators. Write T as $T_1 \circ T_2$, where T_1 is a generator. If the component c of T does not pass though the part of T_1 which looks like $A - F$ in Figure (14) then the conclusion is obvious, so assume it does. If that part looks like A or B, then the component c must be of type (a), and we conclude

$$\mathcal{F}(T_1)\mathcal{F}(T_2)x_{\hat{n}} = \mathcal{F}(T_1)x_{\hat{k}}\mathcal{F}(T_2) = x_{\hat{m}}\mathcal{F}(T_1)\mathcal{F}(T_2),$$

assuming T_2 is a $(\hat{n}, \hat{k})$ tangle and T_1 is a $(\hat{k}, \hat{m})$ tangle.

If the interesting part of T_1 looks like the generator labeled by C, then the component c is either of type (a) or (c). If it is of type (c) then T_2 contains components of type (a) and (b). Then we have

$$\begin{aligned} \mathcal{F}(T_1)\mathcal{F}(T_2)x_{\hat{n}} &= \mathcal{F}(T_1)x_{\hat{k}}\mathcal{F}(T_2) \\ &= \mathcal{F}(T_1)x^{\dagger}_{\hat{k}}\mathcal{F}(T_2) \\ &= \mathcal{F}(T_1)\mathcal{F}(T_2)x^{\dagger}_{\hat{n}}. \end{aligned}$$

If the component c is of type (a), then it is made up of a component c_1 of type (a) in T_2, the tangle C in T_1, a component c_2 of type (d) in T_2, and the tangle G in T_1. Then we have

$$\begin{aligned} \mathcal{F}(T_1)\mathcal{F}(T_2)x_{\hat{n}} &= \mathcal{F}(T_1)x_{\hat{k}}\mathcal{F}(T_2) \\ &= \mathcal{F}(T_1)x^{\dagger}_{\hat{k}}\mathcal{F}(T_2) \\ &= \mathcal{F}(T_1)x_{\hat{k}}\mathcal{F}(T_2) \\ &= x_{\hat{m}}\mathcal{F}(T_1)\mathcal{F}(T_2), \end{aligned}$$

where the first occurrence of $x_{\hat{k}}$ above refers to that associated to c_1 and the second to that associated to c_2. A more visual but less precise argument is illustrated in Figure (20), where an x at point 1 is moved to an x at position

2 by (a), then to an $x^\dagger$ at position 3 by (c), then to an x at position 4 by (d), and then to an x at position 5 by (a).

If T_1 contains the tangle D, the argument of the previous paragraph applies exactly. If it contains E or F, then T is of type (d) and we are done. ■

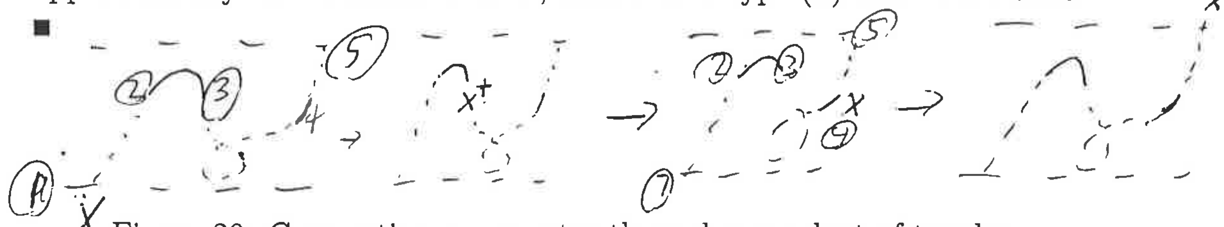


Figure 20: Commuting an operator through a product of tangles

Corollary 4 *If $x \in \mathcal{C}_\lambda$, then $x^\dagger \in \mathcal{C}_{\lambda^*}$. In fact, $\mathcal{C}_{\lambda^*} = (\mathcal{C}_\lambda)^\dagger$.*

Proposition 15 *Let $\hat{n} = (+\lambda_1, \ldots, +\lambda_k)$, and let T be an $(\hat{n}, \hat{n})$ tangle. Then $\mathcal{F}(T) \in \mathcal{C}_\lambda$, where $\lambda = \bigotimes_i \lambda_i$.*

Pf: Geometrically obvious by the above proposition once you have observed that $\mathcal{F}(T)^\dagger$ is just $\mathcal{F}(T')$, where T' is T rotated in the plane 180°. ■

Proposition 16 *$\mathcal{C}_\lambda$ contains the semisimple and nilpotent parts of every operator it contains.*

Pf: Let $v \in V_{+\gamma}$ and $w \in V_{-\gamma}$, and write $R_{\lambda,\gamma}$ as $\sum_i f_i \otimes f^i$, where each f_i and f^i is an operator on $V_{+\lambda}$ or $V_{+\gamma}$ respectively. Then an operator x commutes with the operator $\sum_i \langle f^i(v), w\rangle f_i$ on $V_{+\lambda}$ for all v and w if and only if $x \otimes 1$ commutes with $R_{\lambda,\gamma}$. Likewise one can construct a set of operators on $V_{+\lambda}$ such that x commutes with these if and only if $1 \otimes x$ commutes with $R_{\gamma,\lambda}$. Thus $\mathcal{C}_\lambda$ is literally the set of operators which commute with all these operators and with ϕ_λ. But anything which commutes with an operator commutes with its semisimple and nilpotent parts, by [Hum72, p. 17]. ■

Proposition 17 *$\mathcal{C}_\lambda$ contains a set of idempotents p_i such that $p_i p_j = \delta_{i,j} p_i$, $\sum_i p_i = 1$, and $p_i \mathcal{C}_\lambda p_i$ consists only of multiples of p_i and nilpotent operators.*

Pf: Consider the set of all idempotents in $\mathcal{C}_\lambda$, and order them by $p \prec q$ if Range(p) $\subset$ Range(q). Choose a minimal p. The operator $1-p$ is also in $\mathcal{C}_\lambda$, so choose a minimal projection less than $1-p$. Repeating, we get a resolution of the identity into minimal idempotents, which thus satisfy all the conditions but the last.

Let p be one of these minimal idempotents and let $x \in p\mathcal{C}_\lambda p$. Let x_s be its semisimple part. Since x commutes with p (in fact $xp = x = px$), so does x_s. But if x_s is in $\mathcal{C}_\lambda$, so are the idempotents onto its eigenspaces. If any eigenspace is a proper subspace of the range of p, its idempotent would be a smaller idempotent than p. Thus there is only one eigenspace, namely the range of p, and x_s is a multiple of p. ■

Let p be an idempotent in $\mathcal{C}_\lambda$, and let V_{+p} be the range of p in $V_{+\lambda}$. Likewise let V_{-p} be the range of the idempotent $p^\dagger$ in $V_{-\lambda}$. Define $\phi_p = \phi_\lambda p$. Notice that this equals $p\phi_\lambda$ and thus its range is contained in V_{+p}. Define $R_{p,\gamma} = R_{\lambda,\gamma}(p\otimes 1)$, $R_{\gamma,p} = R_{\gamma,\lambda}(1\otimes p)$, and $R_{p,p} = R_{\lambda,\lambda}(p\otimes p)$. Notice in each case the commutation relations guarantee that the range of the operator is what it should be.

Proposition 18 *If the label p is added to the label set Λ and the tangle representation $\mathcal{F}$ is extended by the definitions given above, the result is a tangle representation.*

Pf: All that is required is to prove that every instance of Equations (8 – 12) with one or more of the labels replaced by p still holds. For example, Equation (8):

$$\begin{aligned}
R_{p,\gamma}(\phi_p \otimes \phi_\gamma) &= R_{\lambda,\gamma}(p\otimes 1)(\phi_\lambda \otimes \phi_\gamma)(p\otimes 1)\\
&= R_{\lambda,\gamma}(\phi_\lambda \otimes \phi_\gamma)(p\otimes 1)\\
&= (\phi_\lambda \otimes \phi_\gamma)R_{\lambda,\gamma}(p\otimes 1)\\
&= (\phi_\lambda \otimes \phi_\gamma)(p\otimes 1)R_{\lambda,\gamma}(p\otimes 1\\
&= \phi_p \otimes \phi_\gamma)R_{p,\gamma}.
\end{aligned}$$

Each equation follows this pattern. Namely, the left hand side can be written as a product of the same generators with λ replacing p, interposed with operators which are a tensor product of identity operators, except with p or $p^\dagger$ in factors of $V_{+\lambda}$ or $V_{-\lambda}$ respectively. By the fact that p commutes with the generators, these can be written as a simple product of generators, preceded (and if necessary followed) by an operator which is a tensor product of identity operators and powers of p and $p^\dagger$. But since these are both idempotents we may take the exponents to be all ones. The product of generators is now the left hand side of the original equation with p's replaced by λ's, and thus can be replaced by the left hand side. Reversing the procedure, p's and $p^\dagger$'s can be interposed wherever necessary to replace each generator by the same generator with p instead of λ. The reader should try an example or two to convince herself that all the equations follow in this way. ■

Proposition 19 *Let p be an idempotent in C_λ, and let $q = 1 - p$. Then $p \oplus q \sim \lambda$.*

Pf: Clearly the isomorphism is $f_+ : V_{+p} \oplus V_{+q} \to V_{+\lambda}$ defined by $f_+(x \oplus y) = x + y$, thinking of V_{+p} and V_{+q} as the ranges of p and q respectively and thus already sitting inside of $V_{+\lambda}$. $f_- : V_{-p} \oplus V_{-q} \to V_{-\lambda}$ is defined similarly. Now $p + q = 1$, so $\phi_\lambda = \phi_\lambda(p+q) = \phi_p + \phi_q$. Similarly, $R_{\lambda,\gamma} = R_{p,\gamma} + R_{q,\gamma}$, with the obvious identification of the spaces on which they act. Also, if x is in range of p and y is in range of $q^\dagger$, then $\langle x, y\rangle = \langle px, q^\dagger y\rangle = \langle qpx, y\rangle = 0$. Similarly if x is in the range of q and y is in the range of $p^\dagger$, then $\langle x, y\rangle = 0$. Thus if T is any one of the tangles pictured in Figure (14) with one or more components labeled by $p \otimes q$, T' is the same tangle with some or all of those labels replaced by λ, and $f_{\hat{n}}$ and $f_{\hat{m}}$ are as given in the definition of isomorphism in Section 2.2.1, then $\mathcal{F}(T) = f_{\hat{m}}^{-1}\mathcal{F}(T')f_{\hat{n}}$. But then this is clearly also true for tensor products and compositions of these operators, so it is true for all tangles. ■

Definition 6 *A label λ is called* nonreducible *if C_λ consists only of nipotent operators and multiples of the identity. A closed tangle representation is said to be* complete *if every label is the direct sum of nonreducible labels.*

Theorem 8 *Every closed tangle representation can be extended to a complete tangle representation.*

Pf: For each $\lambda \in \Lambda$, choose a resolution of the identity into minimal idempotents of C_λ, as in Proposition (17). Make each of these a new label, as in Proposition (18). Each such V_p is nonreducible, because every element of the commutant of p is certainly an element of the commutant of λ, so $C_p = pC_\lambda p$, which is just multiples of the identity and nilpotents by Proposition (17). But

λ is the direct sum of the labels p coming from its commutant by Proposition (19) because they were a resolution of the identity. Alternating the process of closing this label set with adding all the new nonreducibles, in the limit we get a complete tangle representation (actually, this is not necessary: the reader may check that after closing under direct sum and reducing the tangle representation is already complete. ■

2.3.2 Intertwiners

Definition 7 *An* intertwiner *between $\hat{n}$ and $\hat{m}$ is a linear map f from $V_{+\lambda}$ to $V_{-\gamma}$ such that $f\phi_\lambda = \phi_\gamma f$, $(f \otimes 1)R_{\lambda,\delta} = R_{\gamma,\delta}(f \otimes 1)$ and $(1 \otimes f)R_{\delta,\lambda} = R_{\delta,\gamma}(1 \otimes f)$.*

If f is an intertwiner from $\hat{n}$ to $\hat{m}$, define its dual $f^\dagger$ to be the unique map from $V_{-\gamma}$ to $V_{-\lambda}$ such that $\langle f(x), y \rangle = \langle x, f^\dagger(y) \rangle$ for all $x \in V_{+\lambda}$ and $y \in V_{-\gamma}$.

Proposition 20 *Let T be any $(\hat{n}, \hat{m})$ tangle with an open component c labeled by λ, and let T' be the same tangle with c labeled by γ. The map f is an intertwiner from λ to γ if and only if one of the following holds for all T, according to which type of open component c is:*

(a) $\mathcal{F}(T')f_{\hat{n}} = f_{\hat{m}}\mathcal{F}(T)$

(b) $\mathcal{F}(T)f^\dagger_{\hat{n}} = f^\dagger_{\hat{m}}\mathcal{F}(T')$

(c) $\mathcal{F}(T')f_{\hat{n}} = \mathcal{F}(T)f^\dagger_{\hat{n}}$

(d) $f_{\hat{m}}\mathcal{F}(T) = f^\dagger_{\hat{m}}\mathcal{F}(T')$

where $f_{\hat{n}}$, $f^\dagger_{\hat{n}}$ etc. are defined as in the previous subsection as a tensor product of the identity map, except in one factor where it is f or $f^\dagger$ respectively, in such a way that the products make sense.

Pf: The proof is exactly the same as the proof of Proposition (14). ■

Corollary 5 *If f is an intertwiner from λ to γ, then $f^\dagger$ is an intertwiner from γ^* to λ^*.*

Figure 21: All tangles commute with the R- matrix

Figure 22: All tangles commute with ϕ

Proposition 21 *If T is an $(\hat{n}, \hat{m})$ tangle, with $\hat{n} = (n_1, \ldots, n_k)$ and $\hat{m} = (m_1, \ldots, m_j)$, then $\mathcal{F}(T)$ is an intertwiner from λ to γ, where $\lambda = \bigotimes_{i=1}^{k} \lambda_i$ with $\lambda_i = \delta$ if $n_i = +\delta$ and $\lambda_i = \delta^*$ if $n_i = -\delta$, and γ is similarly defined in terms of $\hat{m}$.*

Pf: The proof is essentially contained in Figures (21) and (22). $\mathcal{F}$ of the crossing on the left side of Figure (21) is clearly $PR_{\delta,\gamma}$ and $\mathcal{F}$ of the right is clearly $PR_{\delta,\lambda}$, so the equality implies that $R_{\delta,\gamma}(1 \otimes \mathcal{F}(T)) = (1 \otimes \mathcal{F}(T))R_{\delta,\lambda}$. Likewise for $R_{\gamma,\delta}$ and $R_{\lambda,\delta}$. In Figure (22), T' is meant to be T rotated in the plane by 180°, and thus the first equation yields that, for any $x \in V^{+\lambda}$ and any $y \in V_{-\gamma}$, $\langle \mathcal{F}(T)x, y\rangle = \langle x, \mathcal{F}(T')y\rangle$, and thus that $\mathcal{F}(T') = \mathcal{F}(T)^\dagger$. But then the second equation gives that $\langle \phi_\gamma \mathcal{F}(T)x, y\rangle = \langle \phi_\lambda x, \mathcal{F}(T')y\rangle$. But since $\mathcal{F}(T') = \mathcal{F}(T)^\dagger$, this means $\langle \phi_\gamma \mathcal{F}(T)x, y\rangle = \langle \mathcal{F}(T)\phi_\lambda x, y\rangle$. Since this is true for all x and y, it follows that $\phi_\gamma \mathcal{F}(T) = \mathcal{F}(T)\phi_\lambda$. ∎

Proposition 22

(a) If f is an intertwiner from λ to γ, and g is an intertwiner from δ to λ, then fg is an intertwiner from δ to γ.

(b) If f is an intertwiner from λ_1 to γ_1 and g is an intertwiner from λ_2 to γ_2, then $f \otimes g$ is an intertwiner from $\lambda_1 \otimes \lambda_2$ to $\gamma_1 \otimes \gamma_2$.

(c) Intertwiners for a closed tangle representation $\mathcal{F}$ form a strict monoidal category with objects being labels and morphisms being intertwiners.

(d) $\mathcal{F}$ gives a morphism from the category of tangles to the category of intertwiners.

Pf:

(a) $R_{\mu,\gamma}(1\otimes fg) = (1\otimes f)R_{\mu,\lambda}(1\otimes g) = (1\otimes fg)R_{\mu,\delta}$. The same argument applies to $R_{\gamma,\mu}$. Similarly, $\phi_\gamma fg = f\phi_\lambda g = fg\phi_\delta$.

(b) Recall that $R_{\delta,\lambda_1\otimes\lambda_2} = (P\otimes 1)(1\otimes R_{\delta,\lambda_2})(P\otimes 1)(R_{\delta,\lambda_1}\otimes 1)$ and $\phi_{\lambda_1\otimes\lambda_2} = \phi_{\lambda_1}\otimes\phi_{\lambda_2}$ with the identification $V_{+\lambda_1\otimes\lambda_2} = V_{+\lambda_1}\otimes V_{+\lambda_2}$. Thus

$$\begin{aligned}
&R_{\delta,\gamma_1\otimes\gamma_2}(1\otimes f\otimes g)\\
&= (P\otimes 1)(1\otimes R_{\delta,\gamma_2})(P\otimes 1)(R_{\delta,\gamma_1}\otimes 1)(1\otimes f\otimes g)\\
&= (P\otimes 1)(1\otimes R_{\delta,\gamma_2})(P\otimes 1)(1\otimes f\otimes 1)(R_{\delta,\lambda_1}\otimes 1)(1\otimes 1\otimes g)\\
&= (1\otimes f\otimes 1)(P\otimes 1)(1\otimes R_{\delta,\gamma_2})(P\otimes 1)(R_{\delta,\lambda_1}\otimes 1)(1\otimes 1\otimes g)\\
&= (1\otimes f\otimes 1)(P\otimes 1)(1\otimes R_{\delta,\gamma_2})(1\otimes 1\otimes g)(P\otimes 1)(R_{\delta,\lambda_1}\otimes 1)\\
&= (1\otimes f\otimes g)(P\otimes 1)(1\otimes R_{\delta,\lambda_2})(P\otimes 1)(R_{\delta,\lambda_1}\otimes 1)\\
&= (1\otimes f\otimes g)R_{\delta,\lambda_1\otimes\lambda_2}.
\end{aligned}$$

The same argument applies for $R_{\gamma_1\otimes\gamma_2,\delta}$. Likewise

$$\begin{aligned}
\phi_{\gamma_1\otimes\gamma_2}(f\otimes g) &= (\phi_{\gamma_1}\otimes\phi_{\gamma_2})(f\otimes g)\\
&= (f\otimes g)(\phi_{\lambda_1}\otimes\phi_{\lambda_2})\\
&= (f\otimes g)\phi_{\lambda_1\otimes\lambda_2}.
\end{aligned}$$

(c) The fact that $\mathcal{F}$ is a closed tangle representation means that the set of objects is closed under tensor product and has a unit. Tensor product of objects is associative by Proposition (12). Associativity of composition and tensor product of morphisms as well as the fact that $(A\otimes B)\circ(C\otimes D) = (A\circ C)\otimes(B\circ D)$ all follow from the corresponding facts about linear maps. The identity operator on a given $V_{+\lambda}$ is always an intertwiner, and is the identity morphism.

(d) Immediate from Proposition (21).

■

Proposition 23

(a) Every element of C_λ is an intertwiner from λ to λ, and every such intertwiner is in C_λ.

(b) If $f_\pm$ is an isomorphism from λ to γ, then f_+ is an intertwiner, and $f_- = (f_+^{-1})^\dagger$. Conversely, if f is any invertible intertwiner from λ to γ, then $f_+ = f$, $f_- = (f^{-1})^\dagger$ is an isomorphism.

Pf:

(a) By definition.

(b) By Proposition (20) and the definition of isomorphisms.

■

Write $R_{\lambda,\gamma} = \sum_i f_i \otimes f^i$ for some operators f_i on $V_{+\lambda}$ and f^i on $V_{+\gamma}$. Likewise write $R_{\gamma,\lambda} = \sum_i g_i \otimes g^i$. As in the proof of Proposition (16), Consider the algebra of operators on $V_{+\lambda}$ generated by ϕ_λ together with $\sum_i \langle f^i v, w\rangle f_i$ and $\sum_i \langle g_i v, w\rangle g^i$ for all $v \in V_{+\gamma}$ and $w \in V_{-\gamma}$. That is, the algebra with which the commutant is defined as commuting. A label is called *irreducible* if this algebra consists of all the linear operators on $V_{+\lambda}$. Of course, an irreducible label is also nonreducible. But irreducibility is much stronger.

Proposition 24

(a) If f is an intertwiner from λ to γ, with λ irreducible, then its null space (i.e., the set of vectors v such that $f(v) = 0$) is empty or all of $V_{+\lambda}$.

(b) If f is an intertwiner from λ to γ, with γ irreducible, then its range space is empty or all of $V_{+\gamma}$.

(c) There is a nonzero intertwiner between two irreducible representations if and only if they are isomorphic, in which case all intertwiners are multiples of each other.

(d) A direct sum of irreducible labels can be written as such in a unique way. That is, $\sum_i n_i\lambda_i = \sum_i m_i\lambda_i$ only if $m_i = n_i$, where n_i and m_i are the multiplicities of each irreducible representation λ_i.

Pf:

(a) If V is the null space of f, then $fav = afv = 0$ for all $v \in V$ and a in the algebra generated by the initial data. Thus $av \in V$, so each a maps V into V. But the only subspaces of $V_{+\lambda}$ for which this is true for all operators are the trivial subspaces.

(b) If V is the range of f, then every $v \in V$ is fw for some $w \in V_{+\lambda}$. But then $av = afw = faw \in V$. So again every operator takes V into itself, and V must be a trivial subspace of $V_{+\gamma}$.

(c) The if part follows from Proposition (23b). For the only if, notice by (a) and (b) above any intertwiner is invertible, so by Proposition (23b) it is an isomorphism. Given two such isomorphisms, the product of one and the inverse of the other gives an invertible element of the commutant, which must be a multiple of the identity.

(d) Let $f_\pm$ be an isomorphism between $\sum_i n_i\lambda_i$ and $\sum_i m_i\lambda_i$. By part (c) above, f_+ must take each direct summand $n_i\lambda_i$ in a one-to one and onto fashion to $m_i\lambda_i$. But then their dimensions must be the same and hence $n_i = m_i$.

■

2.3.3 Nonreducible Labels

Agree to call a label *relevant* if it is nonreducible and $\mathrm{qtr}(1_{V_{+\lambda}}) \neq 0$. Call it *irrelevant* if it is nonreducible and $\mathrm{qtr}(1_{V_{+\lambda}}) = 0$. If λ is relevant, define $\mathrm{qdim}_\lambda = \mathrm{qtr}(1_{V_{+\lambda}})$.

Proposition 25 *If L is a link with some component labeled by a direct sum of irrelevant labels, the $\mathcal{F}(L) = 0$.*

Pf: It is enough to show this for a component c of L labeled by an irrelevant μ, since $\mathcal{F}(L)$ is the sum of such things. Write L as the closure of a $(+\mu, +\mu)$ tangle T (that is, $L = {}^cT$ as shown in Figure (17)). By Proposition (15), $\mathcal{F}(T) \in \mathcal{C}_\mu$, so $\mathcal{F}(T) = \alpha \cdot \mathbf{1}_\mu + \eta$, where $\alpha \in \mathbf{F}$, $\mathbf{1}_\mu$ is the identity on $V_{+\mu}$ and η is a nilpotent. But by Lemma (4), $\mathcal{F}(L) = \mathrm{qtr}(\mathcal{F}(T)) = \alpha\mathrm{qtr}(\mathbf{1}_\mu) + \mathrm{qtr}(\eta)$. Since $\eta = \mathcal{F}(T) - \alpha\mathbf{1}_\mu \in \mathcal{C}_\mu$, η commutes with ϕ_μ, so $\phi_\mu\eta$ is a nilpotent and therefore its trace $\mathrm{qtr}(\eta) = 0$. But $\mathrm{qtr}(\mathbf{1}_\mu) = 0$ since μ is irrelevant. Thus $\mathcal{F}(L) = 0$. ■

Proposition 26 *If μ is irrelevant, then $\lambda \otimes \mu$ is a direct sum of irrelevant labels for each $\lambda \in \Lambda$.*

Pf: Let p be a minimal idempotent in $\mathcal{C}_{\lambda\otimes\mu}$. We need to show that $\text{qtr}_{\lambda\otimes\mu}(p) = 0$. Write this as $\text{qtr}_\mu((\text{qtr}_\lambda \otimes 1)(p))$, where $(\text{qtr}_\lambda \otimes 1)(p)$ is an operator on $V_{+\lambda}$ as explained in Section 2.2.2. It suffices to show that this operator is in $\mathcal{C}_\mu$. But it is a product $(\mathcal{F}(D)\otimes 1)(1\otimes p)(\mathcal{F}(E)\otimes 1)$ of intertwiners, by Propositions (21) and (22). Thus it is in $\mathcal{C}_\mu$, and by the argument of the previous proposition, its quantum trace must be zero. ■

Proposition 27 *Let L_1 and L_2 be links each with a component labeled by a relevant label λ, and let $L_1\#L_2$ be their connect sum along that component. Then*

$$\mathcal{F}(L_1\#L_2) = \mathcal{F}(L_1)(L_2)/qdim_\lambda.$$

Pf: Write $L_1 = {}^cT_1$ and $L_2 = {}^cT_2$, where T_1 and T_2 are $(+\lambda, +\lambda)$ tangles. Then $L_1\#L_2 = {}^c(T_1 \circ T_2)$. It is geometrically obvious that $T_1 \circ T_2 = T_2 \circ T_1$. But $\mathcal{F}(T_i) = \alpha_i \mathbf{1}_\lambda + \eta_i$, where $\alpha_i \in \mathbf{F}$ and η_i are nilpotents, because λ is nonreducible. The fact that these two operators commute implies that η_1 and η_2 commute and hence that their product is nilpotent. Also, $\mathcal{F}(T_i)$ commutes with ϕ_λ, so η_1, η_2, and $\eta_1\eta_2$ commute with ϕ_λ. In sum,

$$\begin{aligned}
\mathcal{F}(T_1 \circ T_2) &= \alpha_1\alpha_2\mathbf{1}_\lambda + \alpha_1\eta_2 + \alpha_2\eta_1 + \eta_1\eta_2 \quad \text{so} \\
\mathcal{F}(L_1\#L_2) &= \text{qtr}(\mathcal{F}(T_1 \circ T_2)) = \alpha_1\alpha_2 \text{qdim}_\lambda \\
\mathcal{F}(L_1) &= \text{qtr}(\mathcal{F}(T_1)) = \text{qtr}(\alpha_1\mathbf{1}_\lambda + \eta_1) = \alpha_1 \text{qdim}_\lambda \\
\mathcal{F}(L_2) &= \text{qtr}(\mathcal{F}(T_2)) = \text{qtr}(\alpha_2\mathbf{1}_\lambda + \eta_2) = \alpha_2 \text{qdim}_\lambda
\end{aligned}$$

since the quantum trace of a nilpotent which commutes with ϕ_λ is zero. The result is immediate from the last three equations. ■

Proposition 28

(a) If λ is nonreducible, there is at least a one dimensional space of intertwiners from $\lambda \otimes \lambda^*$ to $\mathbf{1}$.

(b) If λ and γ are irreducible, then there is a nonzero intertwiner from $\lambda \otimes \gamma$ to $\mathbf{1}$ if and only if $\gamma \sim \lambda^*$. In this case, the sapce if intertwiners is exactly one dimensional.

Pf:

(a) Any multiple of $\mathcal{F}(C)$ is such an intertwiner, by Proposition (21).

(b) Let $g : V_{+\lambda^*} \otimes V_{+1} \to V_{+\lambda^*}$ be the intertwiner $g(x, c) = cx$ (this is of course the canonical isomorphism between these). If f is the intertwiner from $\lambda \otimes \gamma$ to $\mathbf{1}$, the composition $(g \otimes 1)(1 \otimes f)(\mathcal{F}(E) \otimes 1)$ of intertwiners is thus an intertwiner h from γ to λ^* (see Figure (23)). Since λ is

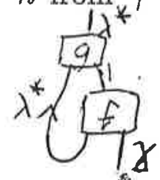


Figure 23: An intertwiner from γ to λ^*

irreducible, so is λ^*, and thus this map must be zero or an isomorphism by Proposition (24).

If p is any intertwiner from $\lambda \otimes \lambda^*$ to $\mathbf{1}$, with λ irreducible, then the intertwiner h defined above must be a multiple of the identity. But then $(1 \otimes p)(\mathcal{F}(E) \otimes 1)$ is a multiple of $(1 \otimes \mathcal{F}(C))(\mathcal{F}(E) \otimes 1)$. Since $\mathcal{F}(C)$ is nondegenerate, it follows that p is a multiple of $\mathcal{F}(C)$.

■

2.4 The Kauffman Bracket Revisited

The decomposition of the Kauffman bracket/Jones polynomial tangle representation into irreducible labels is usually done in the context of quantum groups (where it corresponds to the quantization of $SU(2)$). It was first done in [KR88], where every quantity you could dream of computing is calculated explicitly. A more combinatorial approach, which makes no reference to the structure of quantum groups, involves working with the Temperly-Lieb algebra, and occurs in many papers, such as [Lic91]. The chief difference between that work and what follows is that everything there is done in terms of abstract algebras which are not represented on explicit vector spaces. Such an approach works well for three manifold invariants, but does not appear to be strong enough to construct Topological Quantum Field Theories, as we will do in the next chapter.

We work with the Kauffman bracket, for ease of computation, but this all could just as well be done starting from the Jones polynomial.

2.4.1 The Kauffman and Jones Tangle Representations

Let $V_{1/2}$ be the vector space over C spanned by the vectors $v_{1/2}$ and $v_{-1/2}$. Assign to each product of generators of unoriented framed tangles which is equivalent to an (n, m) tangle a linear map from $V_{1/2}^{\otimes n}$ to $V_{1/2}^{\otimes m}$ as follows.

Let an ordered basis for $V_{1/2} \otimes V_{1/2}$ be $v_{+1/2} \otimes v_{+1/2}$, $v_{+1/2} \otimes v_{-1/2}$, $v_{-1/2} \otimes v_{+1/2}$, $v_{-1/2} \otimes v_{-1/2}$. Then

$$\begin{aligned}
\mathcal{F}(\mathcal{A}) &= \begin{bmatrix} A^{-1} & 0 & 0 & 0 \\ 0 & A^{-1} - A^{3} & A & 0 \\ 0 & A & 0 & 0 \\ 0 & 0 & 0 & A^{-1} \end{bmatrix} \\
\mathcal{F}(\mathcal{B}) &= \begin{bmatrix} A & 0 & 0 & 0 \\ 0 & 0 & A^{-1} & 0 \\ 0 & A^{-1} & A - A^{-3} & 0 \\ 0 & 0 & 0 & A \end{bmatrix} \\
\mathcal{F}(\mathcal{C}) &= \begin{bmatrix} 0 & -iA & iA^{-1} & 0 \end{bmatrix}
\end{aligned}$$

$$\mathcal{F}(\mathcal{D}) = \begin{bmatrix} 0 \\ -iA \\ iA^{-1} \\ 0 \end{bmatrix}$$

$$\mathcal{F}(\mathcal{E}) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Define $\mathcal{F}(1^{\otimes k} \otimes X \otimes 1^{\otimes j}) = 1_{V_{1/2}}^{\otimes k} \otimes \mathcal{F}(X) \otimes 1_{V_{1/2}}^{\otimes j}$ and $\mathcal{F}(X \circ Y) = \mathcal{F}(X)\mathcal{F}(Y)$.

Proposition 29 *For each nonzero $A \in \mathbf{C}$, the above association of products of generators to operators is invariant under the relations of Theorem (6a), and thus gives an unoriented framed tangle representation with a one element label set. Furthermore, the value of $\mathcal{F}$ on a $(\hat{0}, \hat{0})$ tangle is exactly the Kauffman bracket of the corresponding link.*

Pf: First note that if X, X', H, and V are the products of generators shown in Figure (24), then an easy computation yields

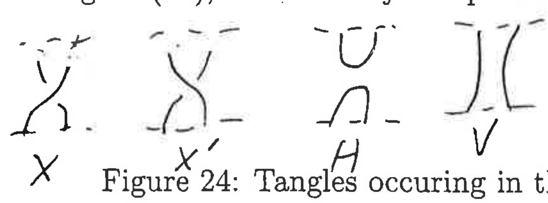


Figure 24: Tangles occuring in the skein relations

$$\begin{aligned} \mathcal{F}(X) &= A\mathcal{F}(H) + A^{-1}\mathcal{F}(V) \\ \mathcal{F}(X') &= A^{-1}\mathcal{F}(H) + A\mathcal{F}(V) \end{aligned} \qquad (13)$$

Also, if Y is a product of generators containing the product of two generators called O in Figure (24), and Y' is the same product with those two generators deleted, then

$$\mathcal{F}(Y) = (-A^2 - A^{-2})\mathcal{F}(Y'). \qquad (14)$$

From these equations it is clear that if F is a tangle representation, it is equal to the Kauffman bracket.

To discuss the question of whether it is a tangle representation, let the set of all length n tensor products of $v_{+1/2}$ and $v_{-1/2}$ be an ordered basis for $V_{1/2}^{\otimes n}$ by saying that one product precedes the other if it has $v_{1/2}$ in the

$$\mathcal{F}(\mathcal{D}) = \begin{bmatrix} 0 \\ -iA \\ iA^{-1} \\ 0 \end{bmatrix}$$

$$\mathcal{F}(\mathcal{E}) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Define $\mathcal{F}(1^{\otimes k} \otimes X \otimes 1^{\otimes j}) = 1_{V_{1/2}}^{\otimes k} \otimes \mathcal{F}(X) \otimes 1_{V_{1/2}}^{\otimes j}$ and $\mathcal{F}(X \circ Y) = \mathcal{F}(X)\mathcal{F}(Y)$.

Proposition 29 *For each nonzero $A \in \mathbf{C}$, the above association of products of generators to operators is invariant under the relations of Theorem (6a), and thus gives an unoriented framed tangle representation with a one element label set. Furthermore, the value of $\mathcal{F}$ on a $(\hat{0}, \hat{0})$ tangle is exactly the Kauffman bracket of the corresponding link.*

Pf: First note that if X, X', H, and V are the products of generators shown in Figure (24), then an easy computation yields

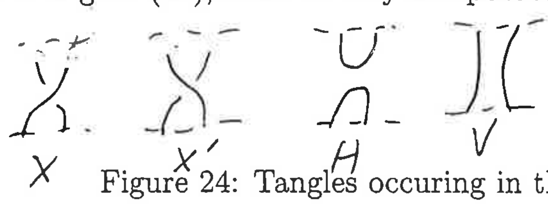


Figure 24: Tangles occuring in the skein relations

$$\begin{aligned} \mathcal{F}(X) &= A\mathcal{F}(H) + A^{-1}\mathcal{F}(V) \\ \mathcal{F}(X') &= A^{-1}\mathcal{F}(H) + A\mathcal{F}(V) \end{aligned} \qquad (13)$$

Also, if Y is a product of generators containing the product of two generators called O in Figure (24), and Y' is the same product with those two generators deleted, then

$$\mathcal{F}(Y) = (-A^2 - A^{-2})\mathcal{F}(Y'). \qquad (14)$$

From these equations it is clear that if F is a tangle representation, it is equal to the Kauffman bracket.

To discuss the question of whether it is a tangle representation, let the set of all length n tensor products of $v_{+1/2}$ and $v_{-1/2}$ be an ordered basis for $V_{1/2}^{\otimes n}$ by saying that one product precedes the other if it has $v_{1/2}$ in the

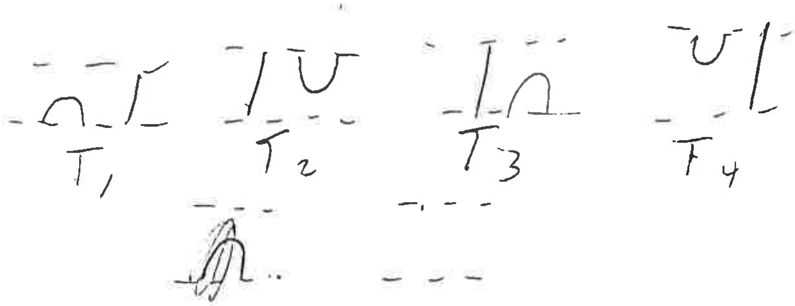


Figure 25: The tangles occurring in Relation V

first entry in which they differ (i.e. the lexicographic ordering). With this convention we have, for example

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \otimes A = \begin{bmatrix} aA & bA \\ cA & dA \end{bmatrix}.$$

Using this we see if T_1, T_2, T_3, T_4 are the tangles shown in Figure (25), then

$$\mathcal{F}(T_1) = \begin{bmatrix} 0 & 0 & -iA & 0 & iA^{-1} & 0 & 0 & 0 \\ 0 & 0 & 0 & -iA & 0 & iA^{-1} & 0 & 0 \end{bmatrix} \quad \text{and}$$

$$\mathcal{F}(T_2) = \begin{bmatrix} 0 & 0 \\ -iA & 0 \\ iA^{-1} & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & -iA \\ 0 & iA^{-1} \\ 0 & 0 \end{bmatrix} \quad \text{so}$$

$$\mathcal{F}(T_1)\mathcal{F}(T_2) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Likewise

$$\mathcal{F}(T_3)\mathcal{F}(T_4) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Thus $\mathcal{F}$ is invariant under Relation V of Theorem (6a). As usual it is trivially invariant under Relation IV. But now it is easy. Applying Equation (13) to the crossings on both sides of Relation VI, it reduces to Relation V. Likewise relations I, II, and III follow from Equations (13) and (14) and Relation V exactly as in the proof of the invariance of the Kauffman bracket. ■

Corollary 6 *The ϕ matrix is given by*

$$\phi_{1/2} = \begin{bmatrix} -A^{-2} & 0 \\ 0 & -A^2 \end{bmatrix}.$$

Pf: The permutation matrix P is given in this basis by

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

so one can check that $\mathcal{F}(\mathcal{C}) = \mathcal{F}(\mathcal{C})(\phi_{1/2}\otimes 1)P$, or $\mathcal{F}(\mathcal{C})(y\otimes x) = \mathcal{F}(\mathcal{C})(\phi_{1/2}(x)\otimes y)$, which is exactly the definition of $\phi_{1/2}$. ■

Proposition 30 *If V_{+x} and V_{-x} are both equal to $V_{1/2}$ defined above then the following identifications give a one label tangle representation whose value on links is the Jones polynomial.*

$$R_{x,x} = \begin{bmatrix} -t^{-1/2} & 0 & 0 & 0 \\ 0 & -t^{-1} & t^{-3/2} - t^{-1/2} & 0 \\ 0 & 0 & -t^{-1} & 0 \\ 0 & 0 & 0 & -t^{-1/2} \end{bmatrix}$$

$$\phi_x = \begin{bmatrix} -t^{-1/2} & 0 \\ 0 & -t^{1/2} \end{bmatrix}.$$

Pf: These are just the initial data of the Kauffman tangle representation, with A replaced by $t^{1/4}$ and the R matrix multiplied by $-t^{-3/4}$. This is easily seen to give another tangle representation, and a simple calculation shows that it satisfies the Jones skein relation. ■

Define the quantum integers

$$[n] \stackrel{\text{def}}{=} (A^{2n} - A^{-2n})/(A^2 - A^{-2}) \tag{15}$$

and notice that $[0] = 0$, $[1] = 1$, $[-n] = -[n]$ and $\lim_{A\to 1}[n] = n$. The following equation will come up frequently, and is easily checked:

$$A^{2k}[n] - A^{2n}[k] = [n-k].$$

2.4.2 The Generic Case

Call A *generic* if $A \neq 0$ and A is not a root of unity.

Proposition 31 *For generic A, There exist vector spaces V_j for each $j \in \frac{1}{2}\mathbf{Z}^{\geq 0}$, spanned by vectors $\{v_k^j : -j \leq k \leq j,\ j-k \in \mathbf{Z}\}$, and maps $f : V_j \otimes V_{1/2} \to V_{j-1/2}$ and $g : V_j \otimes V_{1/2} \to V_{j+1/2}$ (if $j = 0$, interpret this to mean $f \equiv 0$) given by*

$$\begin{aligned}
f_j(v_k^j \otimes v_{1/2}^{1/2}) &= (-1)^{j-k} A^{-j-k-1} \left(\frac{[j-k]^{1/2}}{[2j+1]^{1/2}} \right) v_{k+1/2}^{j-1/2} \\
f_j(v_k^j \otimes v_{-1/2}^{1/2}) &= A^{j-k+1} \left(\frac{[j+k]^{1/2}}{[2j+1]^{1/2}} \right) v_{k-1/2}^{j-1/2} \\
g_j(v_k^j \otimes v_{1/2}^{1/2}) &= A^{j-k} \left(\frac{[j+k+1]^{1/2}}{[2j+1]^{1/2}} \right) v_{k+1/2}^{j+1/2} \\
g_j(v_k^j \otimes v_{-1/2}^{1/2}) &= (-1)^{j-k} A^{-k-j} \left(\frac{[j-k+1]^{1/2}}{[2j+1]^{1/2}} \right) v_{k-1/2}^{j+1/2}
\end{aligned}$$

satisfying

(a) ff^t and gg^t are the identity on the spaces on which they act and gf^t and fg^t are zero, where f^t and g^t are the transposes of f and g in the given basis. $P \overset{\text{def}}{=} f^t f$ and $Q \overset{\text{def}}{=} g^t g$ are idempotents with $P + Q = 1$ and $PQ = 0$.

(b)

$$f_j = i \left(\frac{[2j]^{1/2}}{[2j+1]^{1/2}} \right) (1 \otimes \mathcal{F}(C))(g_{j-1/2}^t \otimes 1) \tag{16}$$

and

$$f_j^t = i \left(\frac{[2j]^{1/2}}{[2j+1]^{1/2}} \right) (g_{j-1/2} \otimes 1)(1 \otimes \mathcal{F}(\mathcal{D})). \tag{17}$$

(c) Each V_j can be identified with a label in the completion of $\mathcal{F}$ in such a way that the maps f, g, f^t, and g^t are all intertwiners.

Remark 2 *Since we are living in the complex numbers, the square roots occuring in the above formulas need some attention. In fact, since we are looking only at one value of A at a time, and are not concerned with continuity, it suffices to choose a square root for each $[n]$ once and for all.*

Pf: We will prove the first two points first by computation, and then prove that the each V_j corresponds to a label and that the four maps are intertwiners by induction.

By a simple calculation

$$\begin{aligned}
f_j f_j^t(v_{k-1/2}^{j-1/2}) &= f_j(A^{j-k+1}\frac{[j+k]^{1/2}}{[2j+1]^{1/2}} v_k \otimes v_{-1/2} \\
&\quad +(-1)^{j-k+1}A^{-j-k}\frac{[j-k+1]^{1/2}}{[2j+1]^{1/2}} v_{k-1} \otimes v_{1/2}) \\
&= (A^{2j-2k+2}\frac{[j+k]}{[2j+1]} + A^{-2j-2k}\frac{[j-k+1]}{[2j+1]}) v_{k-1/2} \\
&= v_{k-1/2}^{j-1/2}
\end{aligned}$$

and similarly

$$\begin{aligned}
g_j g_j^t(v_{k+1/2}^{j+1/2}) &= g_j(A^{j-k}\frac{[j+k+1]^{1/2}}{[2j+1]^{1/2}} v_k \otimes v_{1/2} \\
&\quad +(-1)^{j-k-1}A^{-j-k-1}\frac{[j-k]^{1/2}}{[2j+1]^{1/2}} v_{k+1} \otimes v_{-1/2}) \\
&= (A^{2j-2k}\frac{[j+k+1]}{[2j+1]} + A^{-2j-2k-2}\frac{[j-k]}{[2j+1]}) v_{k-1/2} \\
&= v_{k+1/2}^{j+1/2}.
\end{aligned}$$

Also notice

$$\begin{aligned}
g_j f_j^t(v_{k-1/2}) &= g_j(A^{j-k+1}\frac{[j+k]^{1/2}}{[2j+1]^{1/2}} v_k \otimes v_{-1/2} \\
&\quad +(-1)^{j-k+1}A^{-j-k}\frac{[j-k+1]^{1/2}}{[2j+1]^{1/2}} v_{k-1} \otimes v_{1/2}) \\
&= ((-1)^{j-k}A^{1-2k}\frac{[j+k]^{1/2}[j-k+1]^{1/2}}{[2j+1]} \\
&\quad +(-1)^{j-k+1}A^{1-2k}\frac{[j-k+1]^{1/2}[j+k]^{1/2}}{[2j+1]}) v_{k-1/2} \\
&= 0.
\end{aligned}$$

Thus $f_j f_j^t = 1$, $g_j g_j^t = 1$, $g_j f_j^t = (f_j g_j^t)^t = 0$, and therefore $P^2 = P$, $Q^2 = Q$, and $PQ = QP = 0$. This means that $P + Q$ is also an idempotent. We also have that f_j is a linear isomorphism from the range of P to $V_{j-1/2}$ with inverse f_j^t, and likewise g_j is a linear isomorphism from the range of Q to $V_{j+1/2}$, with g_j^t as its inverse. Therefore, the range of Q is $2j$ dimensional, and the range of P is $2j+2$ dimensional, so the range of $P+Q$ is $4j+2$ dimensional. Since all of $V_j \otimes V_{1/2}$ is $4j+2$ dimensional, $P+Q=1$.

The reader is left to verify that

$$\begin{aligned}
&(1 \otimes \mathcal{F}(C))g_{j-1/2}^t v_k^j \otimes v_{1/2} \\
&\quad = i(-1)^{j-k-1}A^{-j-k-1}\left(\frac{[j-k]^{1/2}}{[2j]^{1/2}}\right) v_{k+1/2}^{j-1/2} \quad \text{and} \\
&(1 \otimes \mathcal{F}(C))g_{j-1/2}^t v_k^j \otimes v_{-1/2} \\
&\quad = -iA^{j-k+1}\left(\frac{[j+k]^{1/2}}{[2j]^{1/2}}\right) v_{k-1/2}^{j-1/2}
\end{aligned}$$

from which Equation (16) follows immediately. Equation (17) is just the transpose of Equation (16).

We have only to show inductively that each j gives a label and all four maps are intertwiners. Assume that we have a label j, with vector space V_j for each $j \le j_0$, and assume f_j, f_j^t, g^j and g_j^t are all intertwiners for $j < j_0$. But then by Equations (16) and (17), f_{j_0} and $f_{j_0}^t$ are the products of intertwiners and hence intertwiners themselves. Thus P and therefore Q are in the commutant. Since Q is an idempotent, its range corresponds to a label. But identifying its range with $V_{j_0+1/2}$ via g_{j_0}, we can consider the new label to be $V_{j_0+1/2}$ and then g_{j_0} restricted to the range of Q is an isomorphism, so g_{j_0}, which is this isomorphism composed with Q, is an intertwiner. Then $g_{j_0}^t$ is the inverse of this isomorphism, and hence is also an intertwiner. ■

Lemma 5 *Let V be a vector space with a basis $v_1, \ldots, v_n$, and let A be an algebra of operators on V. Suppose A contains an operator ϕ for which each v_i is an eigenvector with a distinct eigenvalue. Suppose A also contains an operator x with the coefficient of v_{i+1} in xv_i nonzero for $i < n$, and an operator y with the coefficient of v_{i-1} in yv_i nonzero for $i > 1$. Then $A = End(V)$.*

Pf: If λ_i is the eigenvalue of ϕ associated to v_i, then the operator $\prod_{i \neq j}(\phi - \lambda_i)$ is zero on each v_i except v_j and multiplies v_j by some nonzero scalar.

Thus an appropriate multiple of it is an idempotent $u_{j,j} \in A$ such that $u_{j,j}v_i = \delta_{i,j}v_j$. Likewise $u_{j+1,j+1}xu_{j,j}$ will be zero on every v_i except v_j, which it will take to a nonzero multiple of v_{j+1}. Define $u_{j,j+1} \in A$ to be an appropriate multiple of this operator so that $u_{j,j+1}v_i = \delta_{i,j}v_{j+1}$. Also define a multiple $u_{j,j-1} \in A$ of $u_{i-1,i-1}yu_{i,i}$ so that $u_{j,j-1}v_i = \delta_{i,j}v_{j-1}$. Finally, define $u_{j,k} = \prod_{i=j}^{k-1} u_{i,i+1}$ for $k > j$ and $u_{j,k} = \prod_{i=0}^{j-k-1} u_{j-i,j-i-1}$. The operators $u_{j,k}$, which are all in A, written in this basis are the n^2 matrices with all zero entries but one 1. These span $\mathrm{End}(V)$.

■

Proposition 32 *For each j we have*

(a) $\phi_j(v_k^j) = (-1)^{2j}A^{-4k}v_k^j$

(b) $PR_{1/2,j}PR_{j,1/2} = A^{4j}g_j^tg_j + A^{-4j-4}f_j^tf_j$ *and*

(c) j *is irreducible.*

Pf:

(a) By induction. Assume that $\phi_j(v_k^j) = (-1)^{2j}A^{-4k}$. Then

$$(\phi_j \otimes \phi_{1/2})(v_kj \otimes v_{1/2}) = (-1)^{2j+1}A^{-4k-2}(v_k \otimes v_{1/2})$$

and

$$(\phi_j \otimes \phi_{1/2})(v_k^j \otimes v_{-1/2}) = (-1)^{2j+1}A^{-4k+2}(v_k \otimes v_{-1/2}).$$

But

$$\begin{aligned}
&\phi_{j+1/2}(v_{k+1/2}^{j+1/2}) \\
&= g_j(\phi_j \otimes \phi_{1/2})g_j^t(v_{k+1/2}) \\
&= g_j(\phi_j \otimes \phi_{1/2})(A^{j-k}\frac{[j-k+1]^{1/2}}{[2j+1]^{1/2}}v_k \otimes v_{1/2} \\
&\quad +(-1)^{j-k-1}A^{j+k+1}\frac{[j-k]^{1/2}}{[2j+1]^{1/2}}v_{k+1} \otimes v_{-1/2}) \\
&= (-1)^{2j+1}A^{-4k-2}g_j(A^{j-k}\frac{[j-k+1]^{1/2}}{[2j+1]^{1/2}}v_k \otimes v_{1/2} \\
&\quad +(-1)^{j-k-1}A^{j+k+1}\frac{[j-k]^{1/2}}{[2j+1]^{1/2}}v_{k+1} \otimes v_{-1/2}) \\
&= (-1)^{2j+1}A^{-4k-2}v_{k+1/2}.
\end{aligned}$$

(b) (and c.) By induction simultaneously.

First assuming that statements b. and c. are true of the label j, we will show that

$$\mathcal{F}(T) = -\frac{[4j+2]}{[2j+1]} 1_{V_j}, \tag{18}$$

where T is as in Figure (26).

Figure 26: The identity tangle encircled by an unknot

Of course, assuming j is irreducible, we know that $\mathcal{F}(T)$ is an intertwiner, so it is $\alpha \cdot 1_{V_j}$. Thus

$$\begin{aligned} \alpha \mathrm{qdim}_j &= \mathrm{qtr}(\mathcal{F}(T)) \\ &= \mathrm{qtr}(PR_{1/2,j} PR_{j,1/2}) \end{aligned}$$

Applying b. gives

$$\begin{aligned} \alpha \mathrm{qdim}_j &= A^{4j} \mathrm{qtr}(g_j^t g_j) + A^{-4j-4} \mathrm{qtr}(f_j^t f_j) \\ &= A^{4j} \mathrm{qtr}_{j+1/2}(g_j g_j^t) + A^{-4j-4} \mathrm{qtr}_{j-1/2}(f_j f_j^t) \\ &= A^{4j} \mathrm{qdim}_{j+1/2} + A^{-4j-4} \mathrm{qdim}_{j-1/2}. \end{aligned}$$

By a., $\mathrm{qdim}_j = (-1)^{2j}[2j+1]$, so

$$\begin{aligned} \alpha &= \frac{-A^{4j}[2j+2] - A^{-4j-4}[2j]}{[2j+1]} \\ &= -\frac{[4j+2]}{[2j+1]}. \end{aligned}$$

Now notice that $PR_{1/2,j+1/2} PR_{j+1/2,1/2}$ is $\mathcal{F}$ of the tangle pictured in the upper left of Figure (27). It is therefore

$$(g_j \otimes 1) \mathcal{F}(T) (g_j^t \otimes 1),$$

where T is the tangle between the horizontal lines in the upper right of Figure (27) .

Figure 27: The inductive computation of $\mathcal{F}$ of the full twist

Now

$$\mathcal{F}(T) = A^2\mathcal{F}(X) + \mathcal{F}(Y) + \mathcal{F}(Z) + A^{-2}\mathcal{F}(W)$$

by Equation (13), where X, Y, Z, and W are as pictured in Figure (27). Applying the inductive hypothesis to the first three tangles and recalling $f_j g_j^t = g_j f_j^t = 0$, $g_j g_j^t = 1$, and the value we computed for the tangle in Figure (26)

$$PR_{1/2,j+1/2}PR_{j+1/2,1/2} = A^2 A^{4j} 1_{j\otimes 1/2} + \left(2A^{4j} - A^{-2}\frac{[4j+2]}{[2j+1]}\right)(g_j \otimes 1)\mathcal{F}(S)(g_j^t \otimes 1)$$

where S is as pictured at the bottom of Figure (27). But by Equations (16) and (17),

$$(g_j \otimes 1)\mathcal{F}(S)(g_j^t \otimes 1) = -\frac{[2j+2]}{[2j+1]} f_{j+1/2}^t f_{j+1/2}.$$

So recalling that

$$1_{j\otimes 1/2} = g_{j+1/2}^t g_{j+1/2} + f_{j+1/2}^t f_{j+1/2},$$

we get

$$PR_{1/2,j+1/2}PR_{j+1/2,1/2} = A^{4j+2} g_{j+1/2}^t g_{j+1/2} - (\left(2A^{4j} - A^{-2}\frac{[4j+2]}{[2j+1]}\right)\frac{[2j+2]}{[2j+1]} + A^{4j+2}) f_{j+1/2}^t f_{j+1/2}.$$

And now a simple calculation gives

$$
\begin{aligned}
&(\left(2A^{4j} - A^{-2}\frac{[4j+2]}{[2j+1]}\right)\frac{[2j+2]}{[2j+1]} + A^{4j+2}) \\
&= \left(A^{4j+2} - A^{4j}\frac{[2j+2]}{[2j1]}\right) \\
&\quad + \frac{[2j+2]}{[2j+1]}\left(A^{-2}\frac{[4j+2]}{[2j+1]} - A^{4j}\right) \\
&= A^{-2}\left(A^{4j+4} - A^{4j+2}\frac{[2j+2]}{[2j+1]}\right) \\
&\quad + A^{-4j-4}\frac{[2j+2]}{[2j+1]}\left(A^{4j+2}\frac{[4j+2]}{[2j+1]} - A^{8j+4}\right) \\
&= A^{-2}\frac{[1]}{[2j+1]} + A^{-4j-4}\frac{[2j+2]}{[2j+1]} \\
&= A^{-4j-6}\frac{A^2[2j+2] - A^{4j+4}[1]}{2j+1]} \\
&= A^{-4j-6}
\end{aligned}
$$

and b. is proven.

But now for c., notice that if A is not a root of unity, each v_k is an eigenvector for ϕ_j with distinct eigenvalue. We have only to construct the operators x and y of Lemma (5) out of $PR_{1/2,j}PR_{j,1/2}$ and we are done.

Writing $PR_{1/2,j}PR_{j,1/2} = \sum_k f_k \otimes f^k$, we see that

$$x = \sum_k (v_{-1/2}, f^k v_{1/2}) f_k$$

and

$$y = \sum_k (v_{1/2}, f^k v_{-1/2}) f_k$$

are in the algebra, where $(v_k, v_j) = \delta_{k,j}$, so it suffices to show that (v_{k+1}, xv_k) and (v_{k-1}, yv_k) nonzero for each k.

$$(v_{k+1}, xv_k) = A^{4j}(v_{k+1}, Qv_k) + A^{-4j-4}(v_{k+1}, Pv_k)$$

$$\begin{aligned}
&= (-1)^{2j} A^{4j+2k+1} \frac{[j+k+1]^{1/2}[j-k]^{1/2}}{[2j+1]} \\
&\quad +(-1)^{2j+1} A^{-4j+2k-3} \frac{[j+k+1]^{1/2}[j-k]^{1/2}}{[2j+1]} \\
&= (-1)^{2j} \frac{[j+k+1]^{1/2}[j-k]^{1/2}}{[2j+1]} A^{2k-1}(A^{4j+2} - A^{-4j-2}).
\end{aligned}$$

Likewise

$$(v_{k-1}, y v_k) = (-1)^{2j} A^{2k-3} \frac{[j+k]^{1/2}[j-k+1]^{1/2}}{[2j+1]} (A^{4j+2} - A^{-4j-2}).$$

Since $A \neq 0$ and A is not a root of unity, the powers of A, differences of powers of A, and hence $[n]$ for all n, are all nonzero. Thus both quantities are nonzero.

■

Corollary 7 *For each j and k, $j \otimes k = \bigoplus_{i=|j-k|}^{j+k} i$, where the sum increases by integers.*

Pf: By induction on the lesser of the two: Assume it is k. Clearly it is true if $k = 0$, and we have proven it in Proposition (31) for $k = 1/2$. Assume it is true up to k. Then

$$\begin{aligned}
(j \otimes k) \otimes 1/2 &= \bigoplus_{i=j-k}^{j+k} i \otimes 1/2 \\
&= \bigoplus_{i=j-k}^{j+k} ((i - 1/2) \oplus (i + 1/2)) \\
&= \bigoplus_{i=j-k-1/2}^{j+k+1/2} i \oplus \bigoplus_{i=j-k+1/2}^{j+k-1/2} i.
\end{aligned}$$

On the other hand

$$\begin{aligned}
(j \otimes k) \otimes 1/2 &= j \otimes (k \otimes 1/2) \\
&= j \otimes (k - 1/2) \oplus j \otimes (k - 1/2) \\
&= j \otimes (k + 1/2) \oplus \bigoplus_{i=j-k+1/2}^{j+k-1/2} i.
\end{aligned}$$

But by the uniquness of the decomposition of labels, this gives

$$j \otimes (k + 1/2) = \bigoplus_{i=j-k-1/2}^{j+k+1/2} i.$$

∎

Theorem 9 *$\mathcal{F}$, with labels all direct sums of $j \in \mathbf{Z}^{\geq 0}$, forms a complete framed unoriented tangle representation with each j being irreducible and relevant, and with the structure constants given in the previous corollary.*

Pf: This all follows from the propositions of this section. ∎

2.4.3 Roots of Unity

Let A be a root of unity, and let J be the least half integer such that $A^{8J+4} = 1$.

Proposition 33 *For each $j < J$, V_j as defined in the previous subsection is irreducible, and the formulas for f_j, f_j^t, g_j, g_j^t, ϕ_j, and $PR_{1/2,j}PR_{j,1/2}$ are exactly the same as in Propositions (31) and (32).*

Pf: The arguments are exactly the same as in the previous subsection. One need only check that the only things we ever need to divide by are powers of A and $[2j+1]$, which are never zero, and that the nonzero quantities (v_{k+1}, xv_k) and (v_{k-1}, yv_k) are still nonzero, since $A^{4j+2} - A^{-4j-2} \neq 0$ for $j < J$. ■

Proposition 34 *The image of $g_{J-1/2}$, V_J, gives an irreducible irrelevant label.*

Pf: By the same reasoning as in Proposition (32),

$$\phi_J(v_k^J) = (-1)^{2J} A^{4k} v_k^J.$$

Since each A^{4k} is distinct for $-J \leq k \leq J$, ϕ_J still satisfies the conditions of Lemma (5). We just need to find an appropriate x and y.

The reasoning of Proposition (32) used to compute the value of a full twist applies here so far as to show that

$$PR_{1/2,J}PR_{J,1/2} = A^{4J} 1_{J\otimes 1/2}$$
$$+ (2A^{4J-2} - A^{-2}\frac{[4J]}{[2J]})(g_{J-1/2} \otimes 1)\mathcal{F}(g_{J-1/2}^t \otimes 1)$$

where S is as pictured at the bottom of Figure (27). Notice that since $A^{8J+4} = 1$, we have $[2J] = -A^{4J+2}$ and $[4J] = -[2]$, so $2A^{4J-2} - A^{-2}[4J]/[2J] = -A^{4J+2} \neq 0$. Thus we will take $x = (1 \otimes v_{-1/2}, (PR_{1/2,J}PR_{J,1/2} - A^{4J}1_{j\otimes 1/2})1 \otimes v_{1/2})$ divided by this quantity and $y = (1 \otimes v_{1/2}, (PR_{1/2,J}PR_{J,1/2} - A^{4J}1_{j\otimes 1/2})1 \otimes v_{-1/2})$ divided by this quantity.

$$(v_{k+1}, xv_k) = (v_{k+1} \otimes v_{-1/2}, (g_{J-1/2} \otimes 1)\mathcal{F}(S)(g_{J-1/2}^t \otimes 1)v_k \otimes v_{1/2})$$

$$\begin{aligned}
&= (v_{k+1} \otimes v_{-1/2}, (g_{J-1/2} \otimes 1)(A^{k-J}\frac{[J+k]^{1/2}}{[2J]^{1/2}} v_{k-1/2}^{J-1/2} \otimes v_{1/2} \otimes v_{1/2} \\
&\quad +(-1)^{2J-1}A^{-k-J}\frac{[J-k]^{1/2}}{[2J]^{1/2}} v_{k+1/2}^{J-1/2} \otimes v_{-1/2} \otimes v_{1/2})) \\
&= (v_{k+1} \otimes v_{-1/2}, \\
&\quad (g_{J-1/2} \otimes 1)((-1)^{2J-1}A^{-k-J+1}\frac{[J-k]^{1/2}}{[2J]^{1/2}} v_{k+1/2} \otimes v_{1/2} \otimes v_{-1/2} \\
&\quad +(-1)^{2J}A^{-k-J-1}\frac{[J-k]^{1/2}}{[2J]^{1/2}} v_{k+1/2}^{J-1/2} \otimes v_{-1/2} \otimes v_{1/2})) \\
&= (-1)^{2J}A^{-2J+2}\frac{[J-k]^{1/2}[J+k+1]^{1/2}}{[2J]}.
\end{aligned}$$

This is nonzero for all $-J \leq k < J$.

Similarly,

$$(v_{k-1}, yv_k) = (-1)^{2J-1}A^{-2J}\frac{[J+k]^{1/2}[J-k+1]^{1/2}}{[2J]}$$

which is again nonzero of all $-J < k \leq J$.

Thus by Lemma (5), J is irreducible. But

$$\begin{aligned}
\text{qdim}_J &= \text{tr}(\phi_J) \\
&= (-1)^{2J}\sum_{k=-J}^{J} A^{-4k} \\
&= (-1)^{2J}[2J+1] = 0.
\end{aligned}$$

■

Proposition 35 *If $j, k < J$, then*

$$j \otimes k = \bigoplus_{l \in S_{j,k,J}} l \oplus Z$$

where Z is a direct sum of irrelevant labels and $S_{j,k,J}$ is the set of all l such that $j+k+l \in \mathbf{Z}$, $l \leq j+k$, $l \geq |k-j|$, and $j+k+l < 2J$.

Pf: If $J = 0$ there is nothing to prove and if $J = 1/2$ it is trivially true. o assume $J > 1/2$. We will prove it by induction on the lesser of k and j, which without loss of generality we will assume to be k. So assume the claim is true for all $k \leq k_0$ and all $j \geq k$ and choose a $j > k_0$.

$$j \otimes k_0 = \bigoplus_{l=j-k_0}^{M} l \oplus Z$$

where $M = \min\{j + k_0, 2J - j - k_0 - 1\}$, and the sum goes by integers. Then

$$(j \otimes k_0) \otimes 1/2 = \bigoplus_{l=j-k_0}^{M} (l \otimes 1/2) \oplus (Z \otimes 1/2).$$

But $Z \otimes 1/2$ is a sum of irrelevant terms, by Proposition (26). On the other hand, since $M < J$, and $j - k_0 > 0$, Proposition (33) implies that

$$l \otimes 1/2 = (l - 1/2) \oplus (l + 1/2).$$

So

$$(j \otimes k_0) \otimes 1/2 = \bigoplus_{l=j-k_0}^{M} ((l - 1/2) \oplus (l + 1/2)) \oplus (\text{irrel. terms})$$

$$= \bigoplus_{l=j-k_0-1/2}^{M+1/2} l \oplus \bigoplus l = j - k_0 + 1/2^{M-1/2} l \oplus (\text{irrel.}).$$

There are now three cases. If $j + k_0 < J - 1/2$, then $M = j + k_0$ so write the sum as

$$\bigoplus_{l=j-k_0-1/2}^{M+1/2} l \oplus \bigoplus_{l=j-k_0+1/2}^{M-1/2} l \oplus (\text{irrel.})$$

$$= \bigoplus_{S_{j,k_0+1/2,J}} l \oplus \bigoplus_{S_{j,k_0-1/2,J}} l \oplus (\text{irrel.}).$$

If $j + k_0 > J - 1/2$, then M is $2J - j - k_0 - 1$, so write it as

$$\bigoplus_{l=j-k_0-1/2}^{M-1/2} l \oplus \bigoplus_{l=j-k_0+1/2}^{M+1/2} l \oplus (\text{irrel.})$$

$$= \bigoplus_{S_{j,k_0+1/2,J}} l \oplus \bigoplus_{S_{j,k_0-1/2,J}} l \oplus (\text{irrel.}).$$

Finally, if $j + k_0 = J - 1/2$, then $M = J - 1/2$, so write the sum as

$$\bigoplus_{l=j-k_0-1/2}^{M-1/2} l \oplus \bigoplus_{l=j-k_0+1/2}^{M-1/2} l \oplus J \oplus (\text{irrel.})$$

$$= \bigoplus_{S_{j,k_0+1/2,J}} l \oplus \bigoplus_{S_{j,k_0-1/2,J}} l \oplus (\text{irrel.}),$$

by the irrelevance of J. Thus the result is the same and by induction we get

$$(j \otimes k_0) \otimes 1/2 = j \otimes (k_0 - 1/2) \oplus \bigoplus_{S_{j,k_0+1/2,J}} l \oplus (\text{irrel.}).$$

But

$$\begin{aligned} (j \otimes k_0) \otimes 1/2 &= j \otimes (k_0 \otimes 1/2) = j \otimes ((k_0 - 1/2) \oplus (k_0 + 1/2)) \\ &= (j \otimes (k_0 - 1/2)) \oplus (j \otimes (k_0 + 1/2)). \end{aligned}$$

From this the result follows. ∎

Let $H_{\lambda,\gamma}$ refer to the positively oriented Hopf link, with one component labeled by λ and the other by γ, as shown in Figure (28).

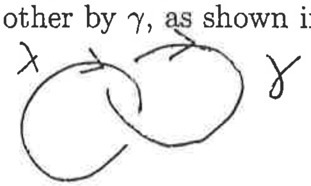


Figure 28: The Hopf link labeled by λ and γ

Proposition 36 *If A is generic, or if J is the least J such that $A^{8J+4} = 1$, and $i, j < J$, then*

$$\mathcal{F}(H_{i,j}) = (-1)^{2i+2j}[(2i+1)(2j+1)].$$

Pf: By induction on j. The result is clear if $j = 0$. If $j = 1/2$, recall Equation (18) says that if T is the identity tangle on i encircled by a component

Figure 29: Decomposing Hopf link labeled by a tensor product of labels

labeled by 1/2, then $\mathcal{F}(T) = -[4i+2]/[2i+1]$. But $\mathcal{F}(H_{i,1/2}) = \mathrm{qtr}(\mathcal{F}(T))$, which is

$$-\frac{[4i+2]}{[2i+1]}(-1)^{2i}[2i+1] = (-1)^{2i+1}[4i+2].$$

Now assume it is true for j_0, and consider $\mathcal{F}(H_{i,j_0\otimes 1/2})$. On the one hand we see in Figure (29) that it is just $\mathcal{F}(H_{i,j_0}\#H_{i,1/2})$, the connect sum being along the i component. By Proposition (27), this equals

$$\mathcal{F}(H_{i,j_0}) \cdot \mathcal{F}(H_{i,1/2})/\mathrm{qdim}_i,$$

which by induction is

$$(-1)^{2i+2j_0}\frac{[(2i+1)(2j_0+1)][2(2i+1)]}{[2i+1]}$$

which is easily seen to be

$$(-1)^{2i+2j_0-1}[(2i+1)2j_0] + (-1)^{2i+2j_0+1}[2i+1)(2j_0+2)].$$

On the other hand, $j_0 \otimes 1/2 = (j_0 - 1/2) \oplus (j_0 + 1/2)$, so

$$\begin{aligned} \mathcal{F}(H_{i,j_0\otimes 1/2}) &= \mathcal{F}(H_{i,j_0-1/2}) + \mathcal{F}(H_{i,j_0+1/2}) \\ &= (-1)^{2i+2j_0-1}[(2i+1)2j_0] + \mathcal{F}(H_{i,j_0+1/2}). \end{aligned}$$

Subtracting from the above formula, we get the desired result. ■

Addendum To Section 2.3

Lemma 6 *Let λ be nonreducible, and $x \in C_\lambda$. Then x is nilpotent if and only if it is singular if and only if its trace is zero. In particular, the set of nilpotent operators forms an ideal of C_λ. If λ is relevant, x satisfies either of these two conditions if and only if $qtr(x) = 0$.*

Pf: Write x as $\alpha 1_\lambda + \eta$, where α is scalar and η is a nilpotent operator. Writing η in its Jordan canonical form, we see $\det(x) = \alpha^n$ and $\mathrm{tr}(x) = n\alpha$, where n is the dimension of $V_{+\lambda}$. Thus all three conditions are equivalent to $\alpha = 0$.

For the ideal, notice that the nilpotents are a linear subspace because the set of things with tr equal to zero is, and the product of a nilpotent with any other operator (on either side) is again nilpotent because the same is true of nonsingular operators.

Finally, if λ is relevant, then $\mathrm{qtr}(x) = \alpha \mathrm{qdim}_\lambda$ is nonzero if and only if $\alpha = 0$. ■

Proposition 37 *Let f be an invertible intertwiner from $\bigoplus_i n_i\lambda_i$ to $\bigoplus_i m_i\lambda_i$, where each λ_i is nonreducible and n_i and m_i are multiplicities in the direct sum. Then $n_i = m_i$ for all i.*

Pf: Let $p_{i,j}$ for $j = 1$ to n_i be a resolution of the identity of the domain into minimal idenpotents in in the commutant, with range of $p_{i,j}$ isomorphic to λ_i, and let $q_{i,j}$, for $j = 1$ to m_i be the same for the range. Choose an i,and consider the intertwiner $p_{i,j'} f^{-1} q_{k,l} f p_{i,j}$. This is an intertwiner from the range of $p_{i,j}$ to that of $p_{i,j'}$, and thus can be identified with an element of C_{λ_i}. If it is invertible, then so is $q_{k,l} f p_{i,j}$, which means that $k = i$, for otherwise it would give an isomorphism between λ_i and λ_k. Thus if $Q = \sum_{k\neq i} q_{k,l}$, then $p_{i,j'} f^{-1}(1-Q) f p_{i,j} = \delta_{j,j'} - \eta_{j,j'}$ as an element of of C_{λ_i}, where $\eta_{j,j'}$ is nilpotent. Let $P = \sum_j p_{i,j}$, and consider $Pf^{-1}(1-Q)fP$ as an operator on the range of P. Writing it in block matrix form we see that its determinant is the determinant of the identity plus a sum of products of nilpotents in C_{λ_i}, and hence is 1. Thus it is invertible. But then $(1-Q)fP$ must be invertible as a map from the range of P to the range of $1-Q$, and thus they must have the same dimension. Since one is a direct sum of n_i copies of $V_{+\lambda_i}$ and the other of m_i copies of the same space, n_i must equal m_i. ■

Thus it makes sense to ask the multiplicity of a nonreducible label in a given label. In particular, define the structure constants $n^\delta_{\lambda,\gamma}$ by $\lambda \otimes \gamma = \bigoplus_\delta n^\delta_{\lambda,\gamma}\delta$, where λ, γ and δ are all nonreducible. These structure constants together with the permutation map σ on the nonreducibles given by $\sigma(\lambda) = \lambda^*$, completely determine the algebraic structure of λ.

Proposition 38 *If λ, γ, and δ are relevant, then $n^\delta_{\lambda,\gamma} = n^{\gamma^*}_{\gamma,\delta^*}$.*

Pf: We will actually show that $n^\delta_{\lambda,\gamma} \leq n^{\lambda^*}_{\gamma,\delta^*}$. By the mod 3 symmetry of the statement, this suffices.

Choose $n^\delta_{\lambda,\gamma}$ minimal idempotents p_i in $\mathcal{C}_{\lambda\otimes\gamma}$ with $p_i p_j = \delta_{i,j} p_i$ and each p_i equivalent as a label to δ. Also choose isomorphisms from $V_{+\delta}$ to the range of each p_i, so that we have intertwiners $\chi_i : V_{+\delta} \to V_{+\lambda} \otimes V_{+\gamma}$ and $\psi_i : V_{+\lambda} \otimes V_{+\gamma} \to V_{+\delta}$ such that $\psi_i \chi_j = \delta_{i,j}$ and $p_i = \chi_i \psi_i$. Define

$$\begin{aligned} \tilde{\chi}_i : V_{+\lambda^*} &\to V_{+\gamma} \otimes V_{+\delta^*} \\ \tilde{\psi}_i : V_{+\gamma} \otimes V_{+\delta^*} &\to V_{+\lambda^*} \end{aligned}$$

by

$$\begin{aligned} \tilde{\chi}_i &= (\mathcal{F}(D) \otimes 1 \otimes 1)(1 \otimes \chi_i \otimes 1)(1 \otimes \mathcal{F}(F)) \\ \tilde{\psi}_i &= (1 \otimes \mathcal{F}(C))(1 \otimes \psi_i \otimes 1)(\mathcal{F}(E) \otimes 1 \otimes 1), \end{aligned}$$

as illustrated pictorially in Figure (30).

Figure 30: Rotating idempotents

Notice that $\tilde{\psi}_i \tilde{\chi}_j = [(1 \otimes \mathrm{iqtr})(\chi_j \psi_i)]^\dagger$, and thus is an element of $\mathcal{C}_{\lambda^*}$. Thus $\mathrm{qtr}_{\lambda^*}(\tilde{\psi}_i \tilde{\chi}_i) = \mathrm{iqtr}_{\lambda\otimes\gamma}(\chi_j \psi_i) = \mathrm{qtr}_{\lambda\otimes\gamma}(\chi_j \psi_i)$. But each p_i has nonzero quantum trace (since λ, γ, δ are relevant), and each $\chi_j \psi_i$ for $i \neq j$ has range disjoint from its domain, and thus has quantum trace 0. So $\tilde{\psi}_i \tilde{\chi}_j = \alpha_{i,j} 1_{\lambda^*} + \eta_{i,j}$, where $\alpha_{i,j}$ are constants which are nonzero if and only if $i = j$ and $\eta_{i,j}$ are nilpotent. We will adjust $\tilde{\psi}_i$ and $\tilde{\chi}_j$ so that they satisfy the same assumptions as ψ_i and χ_j, and thus give a resolution of the identity into $n^{\delta}_{\lambda,\gamma}$ idempotents in $C_{\gamma\otimes\delta^*}$ with range isomorphic to λ^*. This shows that $n^{\lambda^*}_{\gamma,\delta^*} \geq n^{\delta}_{\lambda,\gamma}$.

First, since $\tilde{\psi}_i\tilde{\chi}_i$ is invertible, define $\tilde{\psi}'_i \stackrel{\text{def}}{=} (\tilde{\psi}_i\tilde{\chi}_i)^{-1}\tilde{\psi}_i$. Now $\tilde{\psi}'_i\tilde{\chi}_i = 1$, and $\tilde{\psi}'_i\tilde{\chi}_j$ for $i \neq j$ is an operator times a nilpotent and thus is another nilpotent by Lemma (6). Now recursively define $\tilde{\chi}'_i$, $\tilde{\psi}''_i$, so that $\tilde{\psi}''_i\tilde{\chi}'_j = \delta_{i,j}$. Assume this is true for $i, j < j_0$ and that $\tilde{\psi}'_j\tilde{\chi}'_i$ and $\tilde{\psi}''_i\tilde{\chi}_j$ are nilpotent for $i < j_0$ and $j \geq j_0$. Let $p = \sum_{i<j_0} \tilde{\chi}'_i\tilde{\psi}''_i$, an idempotent in $C_{\gamma\otimes\delta^*}$. Notice $\tilde{\psi}'_{j_0}(1-p)\tilde{\chi}_{j_0} = \tilde{\psi}'_{j_0}\tilde{\chi}_{j_0} - \sum_{i<j_0}(\tilde{\psi}'_{j_0}\tilde{\chi}'_i)(\tilde{\psi}''_i\tilde{\chi}_{j_0}) \stackrel{\text{def}}{=} X$. X is a nonnilpotent element of C_{λ^*} minus a product of nilpotent operators, so it is nonnilpotent and thus is invertible. Define $\tilde{\psi}''_{j_0} = \tilde{\psi}'_{j_0}(1-p)$ and $\tilde{\chi}'_{j_0} = (1-p)\tilde{\chi}_{j_0}X^{-1}$. Then $\tilde{\psi}''_{j_0}\tilde{\chi}'_{j_0} = 1$, $\tilde{\psi}''_i\tilde{\chi}'_{j_0} = \tilde{\psi}'_i(1-p)\tilde{\chi}_{j_0}X^{-1} = 0$ for $i < j_0$ because $\tilde{\psi}''_i(1-p) = 0$, $\tilde{\psi}''_{j_0}\tilde{\chi}'_i = \tilde{\psi}'_{j_0}(1-p)\tilde{\chi}'_i = 0$ for $i < j_0$, because $(1-p)\tilde{\chi}'_i = 0$. Furthermore, $\tilde{\psi}'_j\tilde{\chi}'_{j_0} = \tilde{\psi}'_j(1-p)\tilde{\chi}_{j_0}X^{-1} = \tilde{\psi}'_j\tilde{\chi}_{j_0}X^{-1} - \sum_i(\tilde{\psi}'_j\tilde{\chi}'_i)(\tilde{\psi}''_i\tilde{\chi}_{j_0})X^{-1}$ and $\tilde{\psi}''_{j_0}\tilde{\chi}'_j = \tilde{\psi}'_{j_0}(1-p)\tilde{\chi}'j = \tilde{\psi}'_{j_0}\tilde{\chi}'_j - \sum_i(\tilde{\psi}'_{j_0}\tilde{\chi}'_i)(\tilde{\psi}''_i\tilde{\chi}'_j)$ for $j > j_0$, which are both nilpotent since every term contains a nilpotent factor.

Thus we have constructed $\tilde{\psi}''_i$ and $\tilde{\chi}'_i$ with $\tilde{\psi}''_i\tilde{\chi}'_j = \delta_{i,j}$. So defining $q_i = \tilde{\chi}'_i\tilde{\psi}''_i$, we see that the q_i satisfy $q_iq_j = \delta_{i,j}q_i$ so q_i are minimal idempotents in $C_{\gamma\otimes\delta^*}$ onto disjoint subspaces isomorphic to $V_{+\lambda^*}$ (via $\tilde{\chi}'_i$). Decomposing $1 - \sum_i q_i$ into minimal idempotents, we have written $\gamma \otimes \delta^*$ as a sum of nonreducible labels, with at least $n^{\delta}_{\lambda,\gamma}$ copies of λ^*. Thus $n^{\lambda^*}_{\gamma,\delta^*} \geq n^{\delta}_{\lambda,\gamma}$.

∎

Corollary 8 *If λ, γ are relevant, then $n^1_{\lambda,\gamma} = 0$ unless $\gamma = \lambda^*$, and $n^1_{\lambda,\lambda^*} = 1$.*

3 Three Manifolds

3.1 Three Manifold Invariants

The basis for everything in this chapter is [Wit88], which in fact gives an outline of a rigorous proof of all the claims made in it (though Witten actually only claims these as conjectures). In the following years an avalanch of rigorous presentations followed. Primacy in this regard goes to [RT91] who show a three manifold invariant arises from any "modular Hopf algebra" (our definition of modular tangle representation is meant to correspond, though it is in fact somewhat weaker), and show that such an algebra arises from the Jones polynomial (namely the quantum group $U_q(sl_2)$). A nonexhaustive list of other versions includes [KM91], [Lic91], [CBV92], [Koh92], [Mor92], [TW93], [Wen93], and [Wal]. Our approach is a mixture of all of these. [KM91] and [Lic91] are recommended as the simplest, [Wal] is recommended as the most powerful and complete.

3.1.1 Link Labels and Satellites

Let $\mathcal{F}$ be a complete tangle representation with label set Λ. We return to considering links, and make two modifications on Λ. First we allow each component of a link to be labeled by an arbitrary (finite) linear combination of elements of Λ with coefficients in the ground field. If $L(\sum_i c_i\lambda_i)$ is L with one component labeled by $\sum_i c_i\lambda_i$, and $L(\lambda_i)$ is the same link with that component labeled by λ_i, we define

$$\mathcal{F}(L(\sum c_i\lambda_i)) \stackrel{\text{def}}{=} \sum c_i\mathcal{F}(L(\lambda_i)).$$

We say that two linear combinations l and l' are equivalent, $l \equiv l'$, if given any link L and any component, $\mathcal{F}(L(l)) = \mathcal{F}(L(l'))$. Define the set of link labels $\mathcal{L}$, to be the set of such linear combinations modulo equivalence.

Proposition 39

(a) If $\lambda, \lambda' \in \Lambda$ and $\lambda \equiv \lambda'$ as elements of $\mathcal{L}$, then $\lambda^* \equiv \lambda'^*$, and $\lambda \otimes \gamma \equiv \lambda' \otimes \gamma$.

(b) If $n_i \in \mathbf{Z}$, then $\sum_i n_i\lambda_i \equiv \bigoplus n_i\lambda_i$, where in the right hand side $n_i\lambda_i$ refers to the direct sum of λ_i with itself n times.

(c) $\mathcal{L}$ is spanned by equivalence classes of relevent labels.

(d) $\mathcal{L}$ is a commutative algebra, with multiplication $\cdot$ the image of $\otimes$ and with an involution * the image of *.

Pf:

(a) Clearly $\mathcal{F}(L(\lambda^*)) = \mathcal{F}(L(\lambda'^*))$ and $\mathcal{F}(\lambda \otimes \gamma) = \mathcal{F}(\lambda' \otimes \gamma)$ because in each case the left and right hand side can be written as a link labeled by λ and λ' respectively.

(b) By the definition of direct sum of labels, $\mathcal{F}(L(\bigoplus n_i\lambda_i)) = \sum_i n_i\mathcal{F}(L(\lambda_i)) = \mathcal{F}(L(\sum_i n_i\lambda_i))$.

(c) $\mathcal{L}$ is certainly spanned by equivalence classes of nonreducible labels. But by Proposition (25), irrelevent labels are equivalent to 0 as link labels.

(d) Immediate from Corollary (3).

■

Let $\mathbf{T}$ be a solid torus with a distinguished longitude. Of course, we mean $\mathbf{T}$ to be a PL manifold. A link in $\mathbf{T}$ is just a one dimensional submanifold, with equivalence given by ambient isotopy. Framed and oriented links are defined by analogy with ordinary links. Given a framed oriented link L in S^3 with a distinguished component c, we can imbed $\mathbf{T}$ into S^3 by removing a tubular neighborhood of c, and gluing in $\mathbf{T}$ by an orientation preserving homeomophism of the boundary such that the longitude gets taken to the image of $S^1 \times 0$ in c (thus $\mathbf{T}$ "twists the same amount as c"). The reader may check that such an imbedding is uniquely determined up to isotopy.

If $\mathbf{T}$ contains a link t, then the imbedding yields a new link L_c^t, called the t satellite of L along c.

Proposition 40 *Let t be a link in $\mathbf{T}$ with components labeled by elements of $\mathcal{L}$. Then t is equivalent to some $l \in \mathcal{L}$, in the sense that $\mathcal{F}(L_c^t) = \mathcal{F}(L(l))$ for every link L in S^3 and every component c.*

Pf: Assume first that t is labeled by elements of Λ. Identify $\mathbf{T}$ with $D \times I$, with $D \times \{0\}$ and $D \times \{1\}$ identified in such a way that $\{1\} \times I$ corresponds to the longitude. Isotope t so that it intersects $D \times \{0\}$ transversely at evenly spaced points along the x-axis. Call the resulting tangle T. Write L as the closure of a $(+1, +1)$ tangle S by cutting along c. If T is an $(\hat{n}, \hat{n})$ tangle, $\hat{n} = (n_1, n_2, \ldots, n_n)$ and $\lambda = \otimes\lambda_i$ where $\lambda_i = \delta$ if $n_i = \delta$ and $\lambda_i = \delta^*$ if $n_i = -\delta$, let the open component of S be labeled by λ. Then $\mathcal{F}(L_c^t) = \text{qtr}(\mathcal{F}(S)\mathcal{F}(T))$. Let p_i be a resolution of the identity of λ into minimal idempotents. So

$$\begin{aligned}\mathcal{F}(L_c^t) &= \text{qtr}_\lambda(\mathcal{F}(S)\mathcal{F}(T)\sum p_i) \\ &= \sum_i \text{qtr}_\lambda(\mathcal{F}(S)\mathcal{F}(T)p_i)\end{aligned}$$

$$
\begin{aligned}
&= \sum_i \mathrm{qtr}_\lambda(\mathcal{F}(S)\mathcal{F}(T)p_i^3) \\
&= \sum_i \mathrm{qtr}_\lambda(p_i^2\mathcal{F}(S)\mathcal{F}(T)p_i) \\
&= \sum_i \mathrm{qtr}_\lambda(p_i\mathcal{F}(S)p_i\mathcal{F}(T)p_i).
\end{aligned}
$$

The last two steps because p_i, being in the commutant, commutes with ϕ and $\mathcal{F}(S)$. But $p_i\mathcal{F}(S)p_i$ is by definition $\mathcal{F}(S_i)$, where S_i is S labeled by λ_i. Also $\mathrm{qtr}_\lambda(p_i x) = \mathrm{qtr}_{\lambda_i}(x)$, and $p_i\mathcal{F}(T)p_i = \alpha_i p_i + \eta_i$ for some constant α_i and nilpotent η_i. So the above quantity equals

$$
\begin{aligned}
\sum_i \mathrm{qtr}_{\lambda_i}(\mathcal{F}(S_i)(\alpha_i 1_{\lambda_i} + \eta_i)) &= \sum_i \alpha_i \mathrm{qtr}_{\lambda_i}\mathcal{F}(S_i) + \mathrm{qtr}_{\lambda_i}(\mathcal{F}(S_i)\eta_i) \\
&= \sum_i \alpha_i \mathrm{qtr}_{\lambda_i}\mathcal{F}(S_i)
\end{aligned}
$$

by the Lemma of the Addendum. But now this is exactly $\mathcal{F}(L)$ when L is labeled by $\sum_i \alpha_i\lambda_i$.

If t is labeled by linear combinations of labels, then by linearity $\mathcal{F}(L_c^t) = \sum_i c_i\mathcal{F}(L_c^{t_i})$, where the sum is over various labelings t_i of t by labels in Λ and c_i are constants. But then since the claim is true for the t_i, it is true for t. ■

3.2 Modular Tangle Representations

Let $\mathcal{F}$ be a complete tangle representation with label set Λ. Let $\lambda_1, \ldots, \lambda_n$ be a finite collection of relevant labels, including 1. For simplicity of notation let $\lambda_1 = 1$. Suppose that every λ_i^* and every $\lambda_i \otimes \lambda_j$ is equivalent as a link label to a linear combination of $\lambda_1, \ldots, \lambda_n$. This would happen, for example, if there were only finitely many relevant labels, as is the case for the Kauffman bracket tangle representation at roots of unity. Now consider the Hopf link, H_{λ_i,λ_j}, labeled by λ_i and λ_j, as pictured in Figure (28). Suppose the λ_i's also satisfy that the matrix of values $\mathcal{F}(H_{\lambda_i,\lambda_j})_{ij\leq n}$ is nonsingular. If $\mathcal{F}$ admits a set $\lambda_1, \ldots, \lambda_n$ with these properties, it is called a modular tangle representation.

Now suppose $\mathcal{F}$ is a modular tangle representation, and restrict the algebra of link labels to the span of the image of $\lambda_1, \ldots, \lambda_n$. Notice this is a subalgebra closed under the involution, and if t is a link in the solid torus labeled by elements of this subalgebra, then t as a satellite is equivalent by

Proposition (40) to an element of this subalgebra. Thus we will restrict attention to this subalgebra, and when we say the algebra of link labels we will mean only this subalgebra.

The value $\mathcal{F}(H_{l_1,l_2})$ of the Hopf link labeled by labels l_1 and l_2 gives a bilinear form on the algebra of link labels, which is nondegenerate if $\mathcal{F}$ is modular. Let w_{λ_i} be a dual basis to λ_i. That is, w_{λ_i} is a link label such that $\mathcal{F}(H_{\lambda_j,w_{\lambda_i}}) = \delta_{ij}$.

Proposition 41

(a) $w_{\lambda_i^*} = w_{\lambda_i}^*$. *In particular,* $w_1^* = w_1$.

(b) $w_{\lambda_i} \cdot w_{\lambda_j} = \delta_{ij} w_{\lambda_i}/\text{qdim}_{\lambda_i}$.

(c) $w_1 \cdot l = \mathcal{O}(l) w_1$, *where* $\mathcal{O}(l)$ *is the value of* $\mathcal{F}$ *on the unknot labeled by* l.

(d) $w_1 = \sum_i \text{qdim}_{\lambda_i} \lambda_i / \sum (\text{qdim}_{\lambda_i})^2$.

Pf:

(a) Notice if you reverse the orientations of both components of the Hopf link, you get the Hopf link again. Thus $H_{l_1,l_2} = H_{l_1^*,l_2^*}$, so $\mathcal{F}(H_{\lambda_j^*,w_{\lambda_i}^*}) = \mathcal{F}(H_{\lambda_j,w_{\lambda_i}}) = \delta_{ij}$ so $w_{\lambda_i}^* - w_{\lambda_i^*}$ is in the null space of the matrix. Since the matrix is assumed to be nondegenerate, it is zero.

(b) Consider $H_{\lambda_k,w_{\lambda_i}\cdot w_{\lambda_j}}$. This has the same value as the connect sum $H_{\lambda_k,w_{\lambda_i}} \# H_{\lambda_i,w_{\lambda_j}}$ along λ_k, so

$$\begin{aligned} \mathcal{F}(H_{\lambda_k,w_{\lambda_i}\cdot w_{\lambda_j}}) &= \mathcal{F}(H_{\lambda_k,w_{\lambda_i}}) \cdot \mathcal{F}(H_{\lambda_k,w_{\lambda_j}})/\text{qdim}\lambda_k \\ &= \delta_{ik}\delta_{jk}/\text{qdim}_{\lambda_i}. \end{aligned}$$

Thus $w_{\lambda_i} \cdot w_{\lambda_j} - \delta_{ij} w_{\lambda_i}/\text{qdim}\lambda_i$ is in the null space of the matrix, and therefore is zero.

(c) Notice that $\mathcal{O}(l) = \mathcal{F}(H_{1,l})$ so $\mathcal{O}(w_{\lambda_i}) = \delta_{1,\lambda_i}$. Also, $\{w_{\lambda_i}\}$ forms a basis of the link labels, so $l = \sum \alpha_{\lambda_i} w_{\lambda_i}$, and therefore $\mathcal{O}(l) = \alpha_1$. Finally, by b),

$$\begin{aligned} w_1 \cdot l &= \sum \alpha_{\lambda_i} w_1 \cdot w_{\lambda_i} \\ &= \alpha_1 w_1/\text{qdim}_1 \\ &= \alpha_1 w_1 = \mathcal{O}(l) w_1. \end{aligned}$$

(d) First we will show that $w = \sum_i \text{qdim}_{\lambda_i}\lambda_i$ has the property that $w \cdot l = \mathcal{O}(l)w$. It suffices to show this for $l = \lambda_j$. Computing

$$\begin{aligned} w \cdot \lambda_j &= \sum_i \text{qdim}_{\lambda_i}\lambda_i \cdot \lambda_j \\ &= \sum_{i,k} \text{qdim}_{\lambda_i} n^{\lambda_k}_{\lambda_i\lambda_j}\lambda_k \\ &= \sum_k (\sum_i \text{qdim}_{\lambda_i} n^{\lambda_i^*}_{\lambda_j\lambda_k^*})\lambda_k. \end{aligned}$$

But clearly $\text{qdim}_{\lambda_i} = \text{qdim}_{\lambda_i^*}$, and

$$\begin{aligned} \sum_i n^{\lambda_i^*}_{\lambda_j\lambda_k^*}\text{qdim}_{\lambda_i^*} &= \text{qdim}_{\lambda_j \otimes \lambda_k^*} \\ &= \text{qdim}_{\lambda_j}\text{qdim}_{\lambda_k^*} \\ &= \text{qdim}_{\lambda_j}\text{qdim}_{\lambda_k}, \end{aligned}$$

and so

$$\begin{aligned} w \cdot \lambda_j &= \sum_k \text{qdim}_{\lambda_j}\text{qdim}_{\lambda_k}\lambda_k \\ &= \text{qdim}_{\lambda_j}\sum_k \text{qdim}_{\lambda_k}\lambda_k \\ &= w \cdot \text{qdim}_{\lambda_j}. \end{aligned}$$

But now this means

$$w \cdot w_1 = w\mathcal{O}(w_1) = w.$$

On the other hand,

$$w \cdot w_1 = w_1\mathcal{O}(w) = w_1 \cdot \sum_i (\text{qdim}_{\lambda_i})^2.$$

Now w is a nonzero linear combination of the x_i's, and hence cannot be equivalent to the 0 link label, because $\mathcal{F}$ is modular. Hence

$$\sum_i (\text{qdim}_{\lambda_i})^2 \neq 0$$

and

$$w_1 = w / \sum_i (\text{qdim}_{\lambda_i})^2.$$

■

Proposition 42 *Let L be the distant union of the $+1$ framed unknot with the -1 framed unknot, both labeled by w_1. Then $\mathcal{F}(L) = K$, where*

$$K = 1/\sum_i (\text{qdim}_{\lambda_i})^2.$$

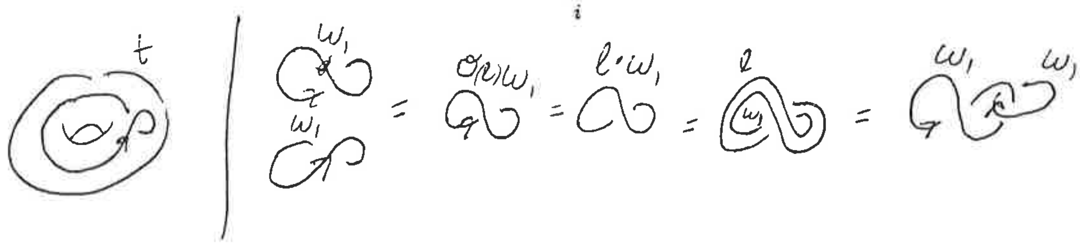


Figure 31: Computations with framed unknots

Pf: Of course $\mathcal{F}(L) = \mathcal{F}(O_{+1})\mathcal{F}(O_{-1})$, where O_{+1} and O_{-1} are the ±1 framed unknots labeled by w_1 respectively. Write $O_{+1} = O(l)$, where l is the label equivalent to the satellite t shown on the left of Figure (31). So $\mathcal{F}(L) = \mathcal{F}(O'_-)$ where O'_- is O_- labeled by $\mathcal{O}(l)w_1$. But by Proposition (41c), $\mathcal{O}(l)w_1 = l \cdot w_1$. Labeling O_- by $l \cdot w_1$ is the same (as far as $\mathcal{F}$ is concerned) as cabeling O_- and labeling one strand by w_1 and replacing the other by a t satellite. This gives the link L' shown on the right hand side of Figure (31), which is just the Hopf link with one component given a -1 framing and both labeled by w_1. By Proposition (41d), $w_1 = K\sum_i \text{qdim}_{\lambda_i}\lambda_i$, so $\mathcal{F}(L')$ is $K\sum \text{qdim}_{\lambda_i}\mathcal{F}(L'_{\lambda_i,w_1})$ where $L'_{\lambda_i w_1}$ is L' with the -1 framed component labeled by λ_i. Finally $L'_{\lambda_i w_1} = H_{\lambda_i w_1}\#O_-^{\lambda_i}$, the second link being the -1 framed unknot labeled by λ_i, so

$$\begin{aligned}\mathcal{F}(L) &= K\sum \text{qdim}_{\lambda_i}\mathcal{F}(L'_{\lambda_i w}) \\ &= K\sum_i \mathcal{F}(H_{\lambda_i w_1}) \cdot \mathcal{F}(O_-^{\lambda_i}).\end{aligned}$$

But $\mathcal{F}(H_{\lambda_i w_1}) = \delta_{\lambda_i 1}$, and $\mathcal{F}(O_-^{\lambda_i}) = 1$, so we are done. ■

3.3 Surgery and Framed 3-Manifolds

Let L be a framed unoriented link in S^3. For each component of L, consider the image of an imbedding of the solid torus $\mathbf{T}^2$ which takes its center to the center of the component and its preferred longitude to one edge of the component, as described in Section 3.1.1 for satellites. One can clearly do this so that none of the images intersect. The complement of the union of these images is a 3-manifold with boundary a union of tori, each boundary component having a homeomorphism h_i from boundary of $\mathbf{T}^2$ to it such that gluing a copy of $\mathbf{T}^2$ to each component of the complement via h_i gives back the original S^3.

Instead, let g be an automorphism of the boundary of $\mathbf{T}^2$ which takes the preferred longitude to a meridian and that meridian to the longitude with the opposite orientation. If we glue a copy of $\mathbf{T}$ to each boundary component of the complement by identifying each point in $\mathbf{T}$ with $h_i g$ of that point, we get a new 3-manifold, $M(L)$, which is potentially different from S^3. The reader should check that $M(L)$ does not depend up to homeomophism ambiguities on the imbeddings of $\mathbf{T}$ or the choice of g and thus the notation $M(L)$ is appropriate.

For example, the reader may check that if O_+, O_-, O_0 are the $+1$, -1, and 0 framed unknots, then $M(O_\pm) = S^3$ and $M(O_0) = S^2 \times S^1$.

We shall not prove the following two remarkable facts, as their proofs take us deep into otherwise unnecessary topology. Their proofs can be found in [Lic62] and [FR79], where they were first proven. The second theorem with a "nonlocal" set of moves was first proven in [Kir78], and Fenn and Rouke's theorem follows without much trouble. With some effort the arguments here could by modified to use the Kirby moves, perhaps more naturally.

Theorem 10 *Every compact oriented 3-manifold is $M(L)$ for some L.*

Theorem 11 *If $M(L_1)$ is homeomophic to $M(L_2)$, then L_1 and L_2 can be related by a sequence of the two Fenn-Rourke moves shown in Figure (32) and their mirror images, where in move II there can be any number of strands linking with the unknot.*

Define a framed 3-manifold to be an equivalence class of framed unoriented links, equivalent under the framed Fenn-Rourke moves of Figure (33).

Let L have n components. Choose an orientation for each and number them 1 to n. Let l_{ij} be the linking number of component i and j for $i \neq j$. Let l_{ii} be the self linking number of component i (linking and self linking number are defined in the exercises). The symmetric matrix $\{l_i j\}^{i,j \leq n}$ is

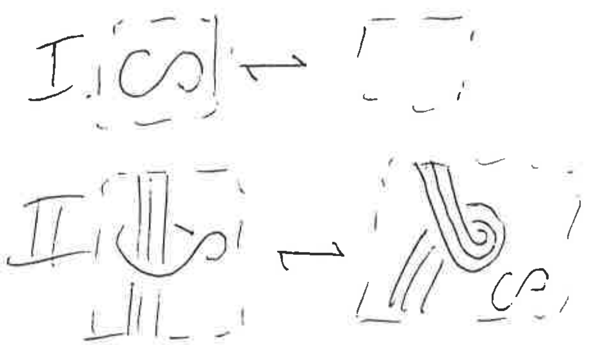


Figure 32: The Fenn-Rourke moves

Figure 33: The framed Fenn-Rourke moves

called the linking matrix of L. Since it is symmetric, it is diagonalizable over $\mathbf{R}$. Let $\sigma(L)$ be the signature of the linking matrix, i.e., the number of positive eigenvalues (counting multiplicities) minus the number of negative ones.

Proposition 43 *$\sigma(L)$ is a framed unoriented link invariant, and in fact is a framed 3-manifold invariant. What's more, if two links represent the same 3-manifold and have the same signature of their linking matrix, they represent the same framed 3-manifold.*

Pf: The signature of a matrix does not change with a change of basis, so reordering the components (reordering the basis) or changing the orientation of some components, (changing the signs of some basis elements) does not change the signature. Since these were the only choices to he made, it is a framed unoriented link invariant.

Invariance under the first framed Fenn-Rourke move is more easy. Let A be the linking matrix of the right hand link. Then the linking matrix of the right hand link is

$$\begin{bmatrix} A & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix},$$

assuming the $+1$ framed unknot is numbered $n+1$ and the -1 is labeled $n+2$. Then if D is A diagonalized, then the larger matrix diagonalized is just

$$\begin{bmatrix} D & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix},$$

which has the same signature as D.

For the second, number the components which go through the $+1$ framed unknot by 1 through k, and number the unknot n. Each $l_{i,n}$ for $i \leq k$, is just the number of times component i occurs in the lower left side of Figure (32), with signs for the direction. On the right hand side, $l_{k,n}$ gets replaced by 0, and since every strand gets wrapped around every other, the linking number of i and j, for $i, j \leq k$ gets $l_{i,n}l_{j,n}$ subtracted from it. This is exactly the effect of replacing each basis vector v_j for $j \leq k$ with $v_j - l_{k,n}v_n$. Thus the signature is unchanged. For the final claim, let L_1 and L_2 have the same signature, and be connected by a sequence of ordinary Fenn Rourke moves. We will adjust this sequence to a sequence of framed Fenn Rourke moves, by an approach analogous the the proof of the framed Reidermeister moves.

To do this replace each instance of move I which creates a framed unknot with the framed move I which creates two framed unknots. Move the extraneous one far away from the rest of the link. Don't do any instances of move I which delete a framed unknot, but instead move it far away from the rest of the link. When you are finished, you have shown L_1 gives the same framed manifold as L_2 distant unioned with a collection of ± 1 framed unknots. But all three links have the same signature. Since distant unioning a ± 1 framed unknot with a link add ± 1 to the signature. It follows the new link is L_2 distant unioned with an equal number of $+1$ and -1 framed unknots. Then a sequence of framed Fenn Rourke moves takes this link to L_2. ∎

We cannot end this subsection without some remark about framed 3-manifolds. Of course there is a geometric defintion. As expected, a framed 3-manifold is something like a 3-manifold with a smooth choice of framing of each tangent plane (here we should be working in the smooth category). Two framings are the same if they are isotopic when imbedded in $TM \oplus TM$ diagonally. Atiyah has shown ([Ati90]), that if the 3-manifold bounds a 4-manifold, then exactly one isotopy class of framings extends to that 4-manifold, and it will extend to every 4-manifold, with the same boundary and the same signature (the signature of the cup product of H^2). Remarkably, the process of surgery on links which we used to construct 3-manifolds quite naturally determines a 4-manifold which it bounds, and the signature of that manifold is exactly the signature of the linking matrix. Thus the signature of the linking matrix corresponds to a unique framing on the 3-manifold. This and Proposition (43) justify our senseless but very practical definition.

3.3.1 The Three Manifold Invariant

Recall that the connect sum of two connected 3-manifolds, $M_1 \# M_2$, is the 3-manifold formed by removing a 3-ball from each and then gluing the two together along the S^2 boundaries.

Theorem 12 *Let $\mathcal{F}$ be a modular tangle representation, and L be a link labeled by $\Omega = w_1/\sqrt{K}$. Then $\mathcal{F}(L)$ is an invariant of the framed 3-manifold determined by surgery on L with orientations removed. The quantity $I(M(L)) = C^{\sigma(L)}\mathcal{F}(L)$ is an invariant of $M(L)$, where $\sigma(L)$ is the signature of L and*

$$C = \mathcal{F}(O_-^{\Omega})$$

where O_-^{Ω} is the -1 framed unknot labeled by Ω. Furthermore, I sends the connect sum of two manifolds to the product of their invariants.

Pf: Since $\omega_1^* = \omega_1$, $\Omega^* = \Omega$ and $\mathcal{F}(L)$ is independent of the orientation of each component. For framed move I, notice $\mathcal{F}(L \amalg O_+^{\omega_1} \amalg O_-^{\omega_1}) = \mathcal{F}(L)\mathcal{F}(O_-^{\Omega} \amalg O_+^{\Omega})/K = \mathcal{F}(L)$ by Proposition (42). For move II, let the left hand side be as pictured on the left of Figure (34), and think of it as a Hopf link, with both components given a $+1$ framing and one component labeled by Ω, the other given the satellite t, where t is pictured on the right of Figure (34). But the Hopf link with both components given $+1$ framing is just O_+ cabled twice. So $\mathcal{F}$ of the left hand side is $\mathcal{F}(O_+^{t \cdot \Omega})$, where by t we mean the link label equivalent to t as described in Proposition (40). By Proposition (41c), this is just $\mathcal{O}(t)\mathcal{F}(O_+^{\Omega})$. This is exactly $\mathcal{F}$ of the right hand side. Thus $\mathcal{F}$ is a framed 3-manifold invariant.

To see the invariance of $I(M(L))$, notice both $\mathcal{F}(L)$ and $\sigma(L)$ are invariant under Fenn-Rourke move II, so $I(M(L))$ is. For move I, notice that $\mathcal{F}(O_+^{\Omega}) = \mathcal{F}(O_-^{\Omega})^{-1}$ by Proposition (41d). Thus if $L_+ = L \amalg O_+^{\Omega}$, $\mathcal{F}(L_+) = \mathcal{F}(L) \cdot \mathcal{F}(O_+^{\Omega}) = \mathcal{F}(L)/C$, and if $L_- = L \amalg O_-^{\Omega}$, $\mathcal{F}(L_-) = \mathcal{F}(L) \cdot C$. On the other hand clearly $\sigma(L_+) = \sigma(L) + 1$ and $\sigma(L_-) = \sigma(L) - 1$, so I has the same value on L_+, L_- and L, and thus is invariant under move I.

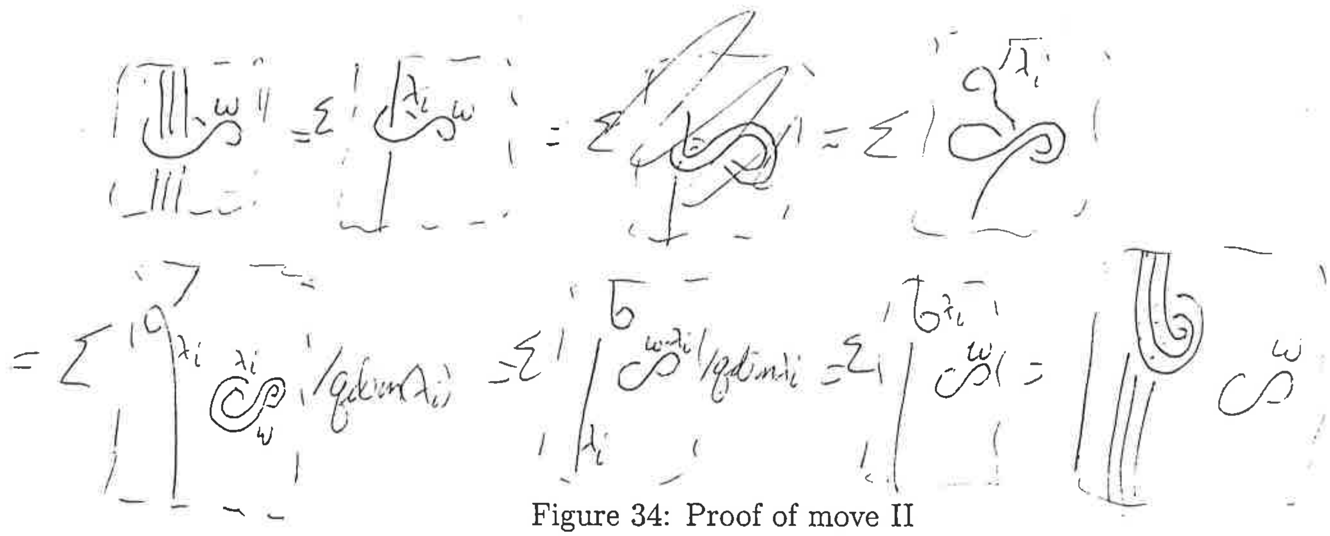


Figure 34: Proof of move II

For multiplicativity, notice that $M(L_1)\#M(L_2) = M(L_1 \amalg L_2)$ (since $S^3\#S^3 = S^3$). But $\mathcal{F}(L_1 \amalg L_2) = \mathcal{F}(L_1)\mathcal{F}(L_2)$, and $\omega(L_1 \amalg L_2) = \omega(L_1) + \omega(L_2)$. Thus

$$I(M(L_1)\#M(L_2)) = I(M(L_1 \amalg L_2)) = I(M(L_1)) \cdot I(M(L_2))$$

. ∎

As an example, we see

$$I(S^3) = \mathcal{F}(\text{unlink}) = 1,$$

and

$$I(S^1 \times S^2) = \mathcal{F}(O_0^{\Omega}) = \mathcal{O}(\Omega) = 1/\sqrt{K} = (\sum_i \text{qdim}^2_{\lambda_i})^{1/2}.$$

3.4 The Kauffman Bracket Again

Theorem 13 *Let $\mathcal{F}$ be a tangle representation, and let $\lambda_1, \ldots, \lambda_n$ be relevant labels such that each λ_i^* is λ_j for some j, and $\lambda_i \otimes \lambda_j = \oplus n_{ij}^k \lambda_k \oplus$ (irrel.). Then $\mathcal{F}$ is a modular tangle representation if and only if the quantity*

$$\sum_{j=1}^{n} \text{qdim}_{\lambda_j} \mathcal{F}(H_{\lambda_i,\lambda_j})$$

is zero for all i except $i = 1$, for which it is nonzero.

Pf: First suppose $\mathcal{F}$ is a modular tangle representation. Then we know

$$\omega = \sum_{i=1}^{n} \text{qdim}_{\lambda_i} \lambda_i$$

is a multiple of ω_1. Thus the matrix $\{\mathcal{F}(H_{\lambda_i,\lambda_j})\}_{i,j\le n}$ times ω should be a multiple of the vector $\sum_{i=1}^{n}\delta_{1,i}\lambda_i$. But in fact it is

$$\sum_{ij}\text{qdim}_{\lambda_i}\mathcal{F}(H_{\lambda_i,\lambda_j})\lambda_j,$$

which gives the desired results.

On the other hand, suppose these quantities have the value given. Let

$$\Omega_i=\sum_{j=1}^{n}\mathcal{F}(H_{\lambda_i,\lambda_j^*})\lambda_j.$$

Then

$$\begin{aligned}
\mathcal{F}(H_{\lambda_k,\Omega_i}) &= \sum_{j=1}^{n}\mathcal{F}(H_{\lambda_k,\lambda_j})\mathcal{F}(H_{\lambda_i,\lambda_j^*})\\
&= \sum_{j=1}^{n}\mathcal{F}(H_{\lambda_i,\lambda_j})\mathcal{F}(H_{\lambda_i^*,\lambda_j})\\
&= \sum_{j=1}^{n}\mathcal{F}(H_{\lambda_k\cdot\lambda_i^*,\lambda_j})\text{qdim}_{\lambda_j}\\
&= \sum_{j=1}^{n}\sum_{l=1}^{n}n_{\lambda_k\lambda_i^*}^{\lambda_l}\mathcal{F}(H_{\lambda_l,\lambda_j})\text{qdim}_{\lambda_j}\\
&= \sum_{l=1}^{n}\sum_{j=1}^{n}n_{\lambda_k\lambda_i^*}^{\lambda_l}\mathcal{F}(H_{\lambda_l^*,\lambda_j^*})\text{qdim}_{\lambda_j}\\
&= \alpha\sum_{l=1}^{n}n_{\lambda_k\lambda_i^*}^{\lambda_l}\delta_{l,1}\\
&= \alpha n_{\lambda_k\lambda_i^*}^{1}\\
&= \alpha\delta_{k,i}
\end{aligned}$$

where $\alpha\neq 0$.

Thus $\{\Omega_i\}$ is a multiple of a dual basis to $\{\lambda_i\}$ under this pairing, so the pairing is nondegenerate. ∎

Theorem 14 *The tangle representation coming from the Kauffman bracket at A a root of unity is modular.*

Pf:Let J be the least half integer with $A^{8J+4}=1$. By Proposition (36)

$$\mathcal{F}(H_{i,j}) = (-1)^{2i+2j}[(2i+1)(2j+1)]$$

so

$$\begin{aligned}
\sum_{j=0}^{J-1/2} \mathcal{F}(H_{ij})\mathrm{qdim}_j &= \sum_{j=0}^{J-1/2} (-1)^{2i}[(2i+1)(2j+1)][2j+1] \\
&= (-1)^{2i} \sum_{j=0}^{J-1/2} \frac{A^{(4i+4)(2j+1)} - A^{4i(2j+1)} - A^{-4i(2j+1)} + A^{-(4i+4)(2j+1)}}{(A^2-A^{-2})^2} \\
&= \frac{(-1)^{2i}}{(A^2-A^{-2})^2}(1+\sum_{j=0}^{J-1/2}(A^{(4i+4)(2j+1)}+A^{-(4i+4)(2j+1)})) \\
&\quad -\frac{(-1)^{2i}}{(A^2-A^{-2})^2}(1+\sum_{j=0}^{J-1/2}(A^{4i(2j+1)}+A^{-4i(2j+1)})) \\
&= \frac{(-1)^{2i}}{(A^2-A^{-2})^2}(\sum_{n=-2J}^{2J} A^{(4i-4)n} - \sum_{n=-2J}^{2J} A^{4in}) \\
&= \frac{(-1)^{2i}}{(A^2-A^{-2})^2}((A^{4i+4})^{-2J}(\frac{1-(A^{4i+4})^{4J+1}}{1-A^{4i+4}}) \\
&\quad -(A^{4i})^{-2J}(\frac{1-(A^{4i})^{-4J+1}}{1-A^{4i}})
\end{aligned}$$

if $i \neq 0$, where the denominators are nonzero since $4i, 4i+4 < 8J+4$. Since A^{2n} raised to the $4J+2$ is 1 for any integer n, we have

$$\begin{aligned}
&\sum_{j-0}^{J-1/2} \mathcal{F}(H_{i,j})\mathrm{qdim}_j \\
&= \frac{(-1)^{2i}}{(A^2-A^{-2})^2}((A^{4i+4})^{-2J}(\frac{1-A^{-4i-4}}{1-A^{4i+4}}) - (A^{4i})^{-2J}(\frac{1-A^{-4i}}{1-A^{4i}})) \\
&= \frac{(-1)^{2i}}{A^2-A^{-2}}(-(A^{4i+4})^{-2J-1}+(A^{4i})^{-2J-1}) \\
&= \frac{(-1)^{2i}}{A^2-A^{-2}}(A^{4i})^{-2J-1}(1-A^{-8J-4}) \\
&= 0
\end{aligned}$$

if $i \neq 0$.

On the other hand, if $i = 0$, then

$$\sum_{j=0}^{J-1/2} \mathcal{F}(H_{ij})\text{qdim}_j = \sum_{j=0}^{J-1/2} [2j+1]^2 > 0$$

since $[2j+1]$ is real for all $j \leq J - 1/2$. ∎

As an example, we compute

$$\begin{aligned}
1/K &= \sum_{j=0}^{J-1/2} \text{qdim}_j^2 \\
&= \sum_{j=0}^{J-1/2} [2j+1]^2 \\
&= \sum_{j=0}^{J-1/2} \frac{A^{4(2j+1)} + A^{-4(2j+1)} - 2}{(A^2 - A^{-2})^2} \\
&= \frac{1}{(A^2 - A^{-2})^2}\left(\frac{A^4 - A^{4(2J+1)}}{1 - A^4} + \frac{A^{-4} - A{-4(2J+1)}}{1 - A^{-4}} - (2J-1)\right) \\
&= \frac{1}{(A^2 - A^{-2})^2}(-1 + -1 - (2J-1)) \\
&= -\frac{2J+1}{(A^2 - A^{-2})^2}.
\end{aligned}$$

In particular, we have

$$I(S^1 \times S^2) = \frac{1}{\sqrt{K}} = \frac{\sqrt{2J+1}}{\sqrt{-(A^2 - A^{-2})^2}}.$$

3.5 Topological Quantum Field Theory

3.5.1 ATQFT's

Choose a dimension d, and choose a representative Σ of each homeomorphism class of connected, compact oriented d dimension manifolds. We shall find it notationally convenient to add the empty manifold $\emptyset$ to our set of representatives. A *parameterized* d surface is a compact oriented d dimensional manifold together with a homeomorphism from each component to a representative Σ. Parameterized sufaces are homeomorphic if they are homeomorphic as manifolds by a map which preserves the parameterization. A component of a parameterized d surface is called positive or negative according to whether its parameterization is an orientation preserving or reversing homeomophism. A *parameterized* $d+1$ *manifold* is a $d+1$ dimensional compact, oriented manifold with boundary together with a parameterization of the boundary and an ordering of the boundary components. Thus if the i^{th} boundary component of M is identified with the representative Σ_i, we have identified ∂M with the sequence $\Sigma_1 \cup \ldots \cup \Sigma_n$. A $d+1$ dimensional manifold without boundary is considered to be parameterized with boundary $\emptyset$.

The following defintion is an attempt to capture the concept of a Topological Quantum Field Theory arising in physics. It is usually referred to in the literature as a TQFT. However, since it misses quite a bit of what a physicist means by a TQFT (e.g. fields!), we have chosen to call it an Axiomatic Topological Quantum Field Theory.

A $d+1$ *dimensional Axiomatic Quantum Field Theory*, or *ATQFT*, over a field $\mathbf{F}$, is a map $\mathcal{Z}$, which sends:

- each representative Σ to a finite dimensional vector space $\mathcal{Z}(\Sigma)$. In particular it identifies $\mathcal{Z}(\emptyset)$ explicitly with $\mathbf{F}$,
- each parameterized $d+1$ manifold M whose boundary has been identified with $\Sigma_1 \cup \ldots \cup \Sigma_n$ with signs $\varepsilon_1, \ldots, \varepsilon_n$ to an element

  $$\mathcal{Z}(M) \in \bigotimes_{i=1}^{n} \mathcal{Z}(\Sigma_i)^{\varepsilon_i}$$

  where

  $$\mathcal{Z}(\Sigma_i)^{\varepsilon_i} = \begin{cases} \mathcal{Z}(\Sigma_i) & \text{if } \varepsilon_i = +, \\ \mathcal{Z}(\Sigma_i)^* & \text{if } \varepsilon_i = -, \end{cases}$$

subject to the following axioms:

(a) *Reordering*: if M_1 is simply M_2 with the boundary components reordered by a permutation map σ, then $\mathcal{Z}(M_1) = \sigma(\mathcal{Z}(M_2))$, where σ is the map

$$\sigma : \bigotimes_{i=1}^{n} \mathcal{Z}(\Sigma_i)^{\varepsilon_i} \longrightarrow \bigotimes_{i=1}^{n} \mathcal{Z}(\Sigma_{\sigma(i)})^{\varepsilon_{\sigma(i)}}$$

given by

$$\sigma(\bigotimes_{i=1}^{n} v_i) = \bigotimes_{i=1}^{n} v_{\sigma(i)};$$

(b) *Nontriviality*: If I_Σ is the $d+1$ manifold $\Sigma \times [0,1]$, with the obvious parameterization on the boundary and with the negative component coming first,

$$\mathcal{Z}(I_\Sigma) = \sum v_i^* \otimes v_i \in \mathcal{Z}(\Sigma)^* \otimes \mathcal{Z}(\Sigma)$$

where v_i is a basis of $\mathcal{Z}(\Sigma)$ and v_i^* is its dual basis in $\mathcal{Z}(\Sigma)^*$;

(c) *Tensor Product*: Suppose M_1, M_2 are parameterized $d+1$ manifolds, and $M_1 \cup M_2$ inherits their parameterization and ordering; putting the boundary of M_1 before that of M_2, then

$$\mathcal{Z}(M_1 \cup M_2) = \mathcal{Z}(M_1) \otimes \mathcal{Z}(M_2).$$

If one of these is closed, we interpret this equality with the canonical identification of vector spaces

$$\mathbf{F} \otimes H = H = H \otimes \mathbf{F};$$

(d) *Gluing*: Suppose M is a parameterized $d+1$ manifold, with the first and second boundary components homeomorphic to Σ, via f_1 and f_2 respectively, the first being positive and the second being negative. Let M' be M with the points of the first and second boundary components identified via the map $f_2^{-1} \circ f_1$. Then if

$$\mathcal{Z}(M) = \sum_{i,j} v_i \otimes v_j^* \otimes w_{ij},$$

where v_i is a basis of $\mathcal{Z}(\Sigma)$, v_i^* is its dual basis and w_{ij} are elements of the vector space associated to the boundary of M', then

$$\mathcal{Z}(M') = \sum_i w_{ii}.$$

Of course, an ATQFT gives a numerical invariant of d-manifolds without boundary, since it assigns to each an element of $\mathcal{Z}(\emptyset) = \mathbf{F}$. On the other hand, it gives much more, including a representation of the mapping class group of every $d-1$ dimensional manifold, as follows.

Suppose M is a parameterized manifold with two boundary components homeomorphic to Σ, the first negative and the second positive. Then $\mathcal{Z}(M) \in \mathcal{Z}(\Sigma)^* \otimes \mathcal{Z}(\Sigma)$ can be interpreted as an operator on $\mathcal{Z}(\Sigma)$. Specifically, if v_i is a basis for $\mathcal{Z}(\Sigma)$ and v_i^* is its dual basis, then for

$$\mathcal{Z}(M) = \sum a_{ji} v_i^* \otimes v_j$$

identify $\mathcal{Z}(M)$ with the operator which sends v_i to $\sum_j a_{ji} v_j$.

Now suppose M_1 and M_2 are both of this form, and consider the manifold M formed by taking their union and then gluing the first component of M_2 to the second of M_1. Then if

$$\begin{aligned} \mathcal{Z}(M_1) &= \Sigma a_{ji} v_i^* \otimes v_j \\ \mathcal{Z}(M_2) &= \Sigma b_{ji} v_i^* \otimes v_j \end{aligned}$$

we have

$$\mathcal{Z}(M_1 \cup M_2) = \sum_{i,j,k,l} a_{ji} b_{lk} v_i^* \otimes v_j \otimes v_k^* \otimes v_l$$

and

$$\mathcal{Z}(M) = \sum_{i,j,l} b_{lj} a_{ji} v_i^* \otimes v_l.$$

Thus $\mathcal{Z}(M)$ as an operator is the product of $\mathcal{Z}(M_2)$ times $\mathcal{Z}(M_1)$ as operators.

Given an automorphism f of Σ, let I_f be the manifold $\Sigma \times [0,1]$, with the first (negative) boundary component identified with Σ via the identity and the second (positive) component identified with Σ via f. Notice gluing the second component of I_f to the first of I_g results in the parameterized manifold $I_{g \circ f}$. Thus if we define $\mathcal{Z}(f)$ to be the operator associated to $\mathcal{Z}(I_f)$, we have $\mathcal{Z}(g \circ f) = \mathcal{Z}(g)\mathcal{Z}(f)$. Thus $\mathcal{Z}$ represents the group of automorphisms of Σ on $\mathcal{Z}(\Sigma)$. Also, if f is isotopic to the identity, then I_f is homeomorphic as a parameterized manifold to I_Σ, so $\mathcal{Z}(f)$ is the identity. So we really have a representation of the mapping class group of Σ.

The following result is not strictly necessary to the course, but justifies the term nontriviality.

Proposition 44 *Suppose $\mathcal{Z}$ is a map which satisfies all the assumptions of an ATQFT except nontriviality. Then $\mathcal{Z}(I_\Sigma)$ as an operator is idempotent, and if we define $\mathcal{Z}'(\Sigma) = Range(\mathcal{Z}(I_\Sigma))$ and $\mathcal{Z}'(M) = \mathcal{Z}(M)$ considered as a vector in $\bigotimes_{i=1}^n \mathcal{Z}'(\Sigma_i)^{\varepsilon_i}$, then $\mathcal{Z}'$ is an honest ATQFT.*

Pf: Notice the above argument representing the mapping class group of Σ applies to $\mathcal{Z}$ here. Thus if f is the identity on Σ then $Z(f) = Z(f \circ f) = Z(f)Z(f)$,so $\mathcal{Z}(f)$ is an idempotent. Let $\mathcal{Z}'(\Sigma)$ be its range. If M is a parameterized manifold with boundary identified with $\Sigma_1 \cup \ldots \cup \Sigma_n$, and $\mathcal{Z}(M) \in \otimes_{i=1}^n \mathcal{Z}(\Sigma_i)^{\varepsilon_i}$, then gluing a copy of I_{Σ_i} to each component gives M again. But it is easy to check that this means

$$\mathcal{Z}(M) = (\bigotimes_{i=1}^{n} Z(f))\mathcal{Z}(M),$$

so $\mathcal{Z}(M)$ is in the tensor product of the range of $\mathcal{Z}(f)$ (on negative components, we should use the adjoint $\mathcal{Z}(f)^*$, but the range of $\mathcal{Z}(f)^*$ can be canonically identified with the dual of the range of $\mathcal{Z}(f)$). Thus $\mathcal{Z}(M) \in \otimes \mathcal{Z}'(M)^{\varepsilon_i}$. Defining $\mathcal{Z}'(M) = \mathcal{Z}(M)$, it is easy to check that $\mathcal{Z}'$ now satisfies all the axioms of an ATQFT. ■

Suppose $\mathcal{Z}_1$ and $\mathcal{Z}_2$ are ATQFT's. We can form the direct sum $\mathcal{Z}_1 \oplus \mathcal{Z}_2$ as follows. Define $(\mathcal{Z}_1 \oplus \mathcal{Z}_2)(\Sigma) = \mathcal{Z}_1(\Sigma) \oplus \mathcal{Z}_2(\Sigma)$. Thus we can identify $\otimes_{i=1}^n (\mathcal{Z}_1 \oplus \mathcal{Z}_2)(\Sigma_i)^{\varepsilon_i}$ with

$$\bigoplus_{j_1,\ldots,j_n \in \{1,2\}} \bigotimes_{i=1}^{n} \mathcal{Z}_{j_i}(\Sigma_i)^{\varepsilon_i}.$$

Now let M have boudary identified with $\Sigma_1 \cup \ldots \cup \Sigma_n$, and let the connected components of M be $M_1, \ldots, M_k$, and notice $\mathcal{Z}_{j_1}(M_1) \otimes \ldots \otimes \mathcal{Z}_{j_k}(M_k)$ is an element of the above direct sum for each sequence $j_1, \ldots, j_k$ of 1's and 2's. Define

$$(\mathcal{Z}_1 \oplus \mathcal{Z}_2)(M) = \sum_{j_1 \ldots j_k} \bigotimes_{i=1}^{k} \mathcal{Z}_{j_i}(M_i)$$

and notice it is an element of $(\mathcal{Z}_1 \oplus \mathcal{Z}_2)(\Sigma_1 \cup \ldots \cup \Sigma_n)$. The reordering, nontriviality and gluing axioms all follow from these axioms for $\mathcal{Z}_1$ and $\mathcal{Z}_2$, and the tensor product axiom is true by construction.

Let S^d be the d dimensional sphere. We say that $\mathcal{Z}$ is *simple* if $\mathcal{Z}(S^d)$ is one dimensional.

Proposition 45 *Every ATQFT can be written as a direct sum of simple theories. A simple ATQFT cannot be written as a direct sum of two nontrivial theories.*

Pf: Notice that between two manifolds homeomorphic to S^d there is exactly one orientation preserving homeomorphism and one orientation reversing homeomorphism up to isotopy. Thus to specify a union of copies of S^d as a parameterized surface, you just have to specify an ordering and signs on the components.

Let K be S^{d+1} with three B^{d+1} balls removed. Make the first two boundary components negative and the third positive. Thus $\mathcal{Z}(K) = \sum_i a_i \otimes b_i \otimes c_i \in \mathcal{Z}(S^d)^* \otimes \mathcal{Z}(S^d)^* \otimes \mathcal{Z}(S^d)$, which can be thought of as a multiplication on $\mathcal{Z}(S^d)$ which defines xy to be $\sum_i a_i(x)b_i(y)c_i$. It is commutative by the reordering axiom and associative by the gluing axiom. Likewise if N is S^{d+1} with two B^{d+1} balls removed, both parameterized negatively then $\mathcal{Z}(N)$ is a bilinear pairing of $\mathcal{Z}(S^d)$. On the other hand $\mathcal{Z}(N^*)$, where N^* is N with the opposite orientation, is some element of $\mathcal{Z}(S^d) \otimes \mathcal{Z}(S^d)$, say $\sum_i x_i \otimes y_i$. But gluing N to N' along one component gives I_{S^d}, so $\sum \langle x, x_i \rangle y_i = \mathcal{Z}(I_{S^d})(x) = x$, and thus $\langle \cdot, \cdot \rangle$ is nondegenerate (and x_i and y_i are dual bases).

Again by the gluing axiom, we have $\langle xy, z \rangle = \langle y, xz \rangle$. Thus $\mathcal{Z}(S^d)$ forms a commutative algebra of symmetric operators on $\mathcal{Z}(S^d)$, with respect to a nondegenerate pairing. They therefore can be simultaneously diagonalized. Let $p_1, \ldots, p_n$ be the eigenvectors. Since $p_i p_j$ is a multiple of p_i and p_j, $p_i p_j = 0$ if $i \neq j$. Normalizing, we also have $p_i^2 = p_i$ (if $p_i^2 = 0$, p_i could not be represented as a symmetric matrix). Thus the p_i's are a basis of idempotents. Of course $(\sum_i p_i)p_j = p_j$, so $\sum p_i$ is the identity. But $\mathcal{Z}(B^{d+1})$ is the identity for this multiplication, so $\mathcal{Z}(B^{d+1}) = \sum_i p_i$.

Now let I'_Σ be I_Σ with a copy of B^{d+1} removed, with that boundary component put first and parameterized negatively. Notice I'_Σ glued to I'_Σ is I_Σ with two holes removed, which can be thought of as I'_Σ glued to a copy of K (see Figure (35)). Putting the S^d boundary components first in each case, we have, if $\mathcal{Z}(I'_\Sigma) : \mathcal{Z}(S^d) \otimes \mathcal{Z}(\Sigma) \to \mathcal{Z}(\Sigma)$ is given by $F(x \otimes y)$, then $F(p_i \otimes \cdot)$ is an operator A_i on $\mathcal{Z}(\Sigma)$ such that

$$A_i A_i = F(p_i p_i \otimes \cdot) = F(p_i \otimes \cdot) = A_i.$$

Similarly

$$A_i A_j = F(p_i p_j \otimes \cdot) = 0.$$

Also $F(\sum p_i \otimes \cdot)$ is $\mathcal{Z}$ of I'_Σ glued to B_{d+1}, and hence is $\mathcal{Z}(I_\Sigma) = 1$. Thus the A_i's are a resolution of the identity, and decompose $\mathcal{Z}(\Sigma)$ into a direct sum of subspaces. Define $\mathcal{Z}_i(\Sigma) = \text{Range}(A_i)$, so that

$$\mathcal{Z}(\Sigma) = \bigoplus_{i=1}^{n} \mathcal{Z}_i(\Sigma).$$

Likewise $A_i^* A_j^* = \delta_{i,j} A_i^*$, and so $\mathcal{Z}(\Sigma)^* = \bigoplus_1^n \text{Range}(A_i^*)$ and $\text{Range}(A_i^*)$ can be identified with $\text{Range}(A_i)^* = \mathcal{Z}_i(\Sigma)^*$.

Figure 35: The operator A_i is an idempotent

Now for a parameterized manifold M, with k components, let M' be M with a B^{d+1} ball removed from each component, with all the new S^d boundary components parameterized negatively and put at the beginning. Define $\mathcal{Z}_i(M) = \mathcal{Z}(M')(p_i^{\otimes k} \otimes \cdot)$. Notice if we act A_i on any tensor factor of $\mathcal{Z}_i(M)$, we get $\mathcal{Z}$ of M' glued to I'_Σ, with p_i put in each tensor factor corresponding to an S^d boundary component. This manifold is M' with one extra B^{d+1} removed from one component. This can be written as $\mathcal{Z}$ of M' glued to K. Since p_i is put in the two factors corresponding to K, this is the same as not gluing in K but putting p_i in the tensor factor corresponding to the S^d along which K is glued. But this is just $\mathcal{Z}(M')(p_i^{\otimes k} \otimes \cdot) = \mathcal{Z}_i(M)$. Thus $A_i \mathcal{Z}_i(M) = \mathcal{Z}_i(M)$. Similarly, A_i^* on a tensor factor of $\mathcal{Z}_i(M)$ in $\mathcal{Z}(\Sigma)^*$ is again $\mathcal{Z}_i(M)$. Therefore

$$\mathcal{Z}_i(M) \in \bigotimes_{j=1}^{n} A_i^{\varepsilon_i} \mathcal{Z}(\Sigma_j)^{\varepsilon_j} = \bigotimes_{i=1}^{n} \mathcal{Z}_i(\Sigma_j)^{\varepsilon_j}.$$

$\mathcal{Z}_i$ clearly satisfies reordering, nontriviality, and the tensor product axiom. For gluing notice if you are gluing two boundary components of the same component of M, then M' glued is M glued with a B^{d+1} removed. Thus the gluing axiom for $\mathcal{Z}_i$ follows from the gluing axiom for $\mathcal{Z}$. If two different components of M are being glued, then the result of gluing M' has one component with two S^d boundary components. As above, these can be combined into one by separating off K. Thus each $\mathcal{Z}_i$ is an ATQFT. Since the range of p_i was one dimensional, they are simple.

Finally, suppose a simple theory could be written as $\mathcal{Z}_1 \oplus \mathcal{Z}_2$. Then either $\mathcal{Z}_1(S^d) = \{0\}$ or $\mathcal{Z}_2(S^d) = \{0\}$, lets assume the former. Then for any manifold M, $\mathcal{Z}_1(M)$ can be written as $\mathcal{Z}_1(M')(\mathcal{Z}_1(B^{d+1}))$, but since $\mathcal{Z}_1(B^{d+1}) = 0$, we get $\mathcal{Z}_1(M) = 0$ for all M. In particular every $\mathcal{Z}_1(\Sigma)$ would be zero dimensional. Neglecting this trivial theory, no simple theory admits a direct sum decomposition. ■

3.5.2 Manifolds With Boundary and Framing

We can define framed, oriented links in a parameterized 3-manifold in exactly the same way as for S^3. That is, the image of a PL imbedding of copies of $S^1 \times I$ in the 3-manifold, considered equivalent if there is an isotopy of the parameterized manifold taking the image of one in an orientation preserving fashion to the image of the other. Surgery on framed links also works the same way. That is, remove a tubular neighborhood of each component and glue it back in by a map which sends the longitude (determined by the framing) to the meridian and the meridian to the reverse of the longitude.

Choose for each genus g a representative genus g surface Σ_g sitting standardly in S^3 so that its interior H_g^+ and exterior H_g^- are both genus g handlebodies (with opposite orientations).

Now consider an orientation preserving imbedding N of $H_{g_1}^{\varepsilon_1} \cup \ldots \cup H_{g_n}^{\varepsilon_n}$ into S^3, where each ε_i is plus or minus, and let L be a framed, unoriented link in the complement of the image of N. We construct a parameterized manifold (N, L) with boundary identified with $\Sigma_{g_1} \cup \ldots \cup \Sigma_{g_n}$ with signs $-\varepsilon_1, \ldots, -\varepsilon_n$ as follows. Remove the image of the interior of $H_{g_1}^{\varepsilon_1} \cup \ldots \cup H_{g_n}^{\varepsilon_n}$ under N from S^3. Parameterize the boundary by the inverse of N restricted to $\Sigma_{g_1} \cup \ldots \cup \Sigma_{g_n}$. (N, L) is then the result of surgery on L in the parameterized manifold. The following is a natural generalization of the theorems of Lickorish and Kirby to parameterized 3-manifolds. ~~The proof can be found in [Saw].~~ It follows from results of Justin Roberts.

Theorem 15 *Every connected, oriented parameterized 3-manifold M is homeomorphic as a parameterized manifold to some (N, L). Two manifolds (N, L)*

and (N', L') are parametrically homeomorphic if and only if they can be connected by a sequence of the extended Kirby moves, which are those pictured in Figure (36), and their mirror images. Move II is explained below.

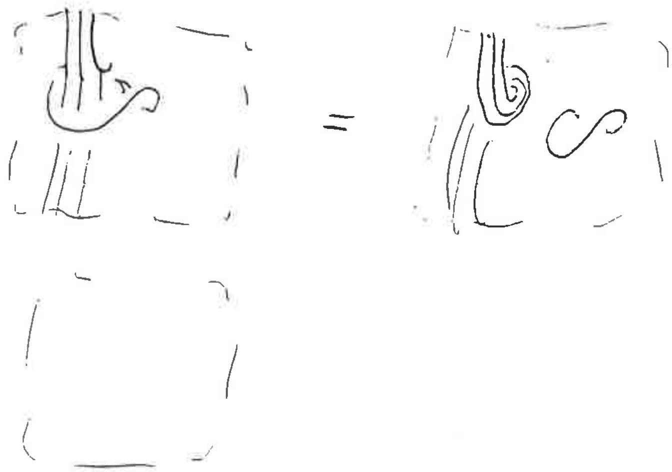


Figure 36: The extended Kirby or Fenn-Rourke moves

The left hand side of move II contains a component which is a ± 1 framed unknot as a knot in S^3, but the disk it bounds may intersect the link and the boundary of N transversely. The right hand side is obtained by removing the unknot, thickening the disk to a cylinder, removing the cylinder and gluing it back in with a clockwise (resp. counterclockwise) full twist, and adding a distant union of a ± 1 framed unknot. Notice this gluing gives S^3 back again.

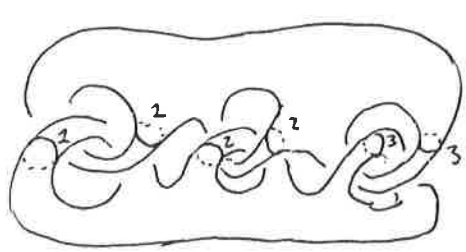


Figure 37: The manifold I_{Σ_g}

This gives a combinatorial presentation of a parameterized $2+1$ manifold as a surgery on a disjoint union of copies of S^3 with imbedded handle bodies removed. As an example, Figure (37) shows a presentation of I_{Σ_g}. The parameterization is specified by marking meridians of H_g^+ on both boundary components.

It will be useful to have a presentation in which some or all of the handlebodies are imbedded in standard position. That is, a presentation in which the imbedding of the given component extends to a homeomorphism of $S^3 = H_g^+ \cup H_g^-$ to S^3.

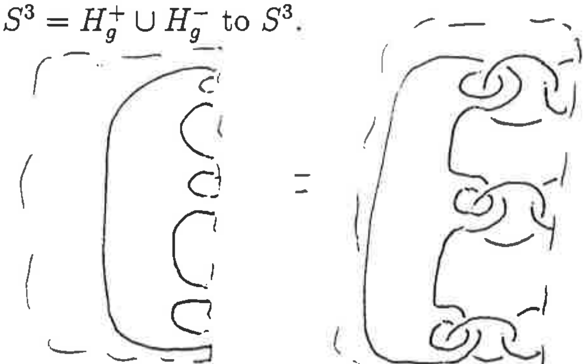


Figure 38: Putting handlebodies in standard position

Proposition 46 *The parameterized* 2+1 *manifolds presented in Figure (38) are homeomorphic, where in the second picture the cylinders of* Σ_g *not shown are sent to a tubular neighorhood of the pieces of the framed link not shown in the first picture, with the longitudes following the framing.*

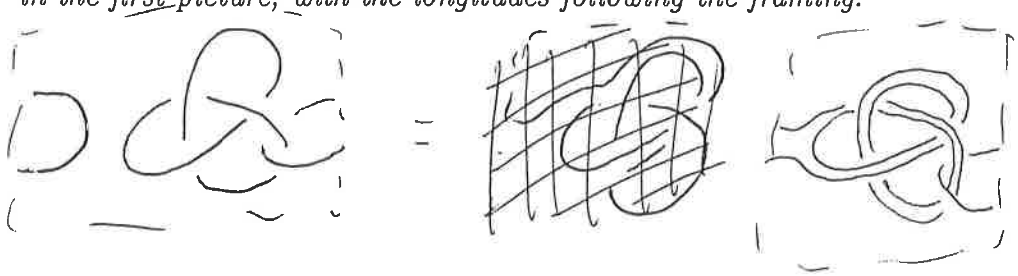
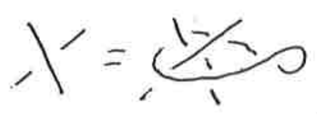


Figure 39: ~~Pushing tubes past an unknot~~ Handle slide

The above handle-slide move does not change the manifold, where the open component to the left can be any collection of strands and handlebody tubes, the trefoil represents any closed link component and the open strand is made to parallel it.

Pf: ~~We~~ first observe the equality in Figure (39), where the open strands are meant to stand for ~~a tube~~ containing an arbitrary ~~number~~ of link components and handles of the handlebodies. ~~The~~ bulk of the proof is given in Figure (40). This proves ~~the equality~~ in Figure ~~(39) except~~ the enclosed tube ends up with ~~a full~~ twist. But applying the same argument ~~with~~ the other tube

To prove handlesliding, apply Move I to create many ±1 framed unknots, the apply move II to attach these unknots to the closed component, thus changing certain crossings by



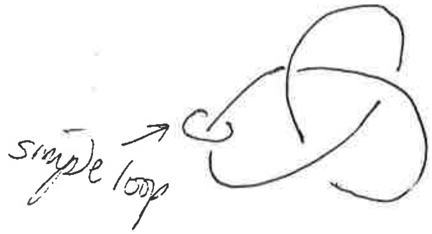
and making it an +1 framed unknot. It then suffices to show the result for a +1 framed unknot linking some # of strands. Using Move II on that unknot we can cast off the strands, then re attach them with the addition handle-sliding strand.

Using the handleslide, note that any surgery component with simple loop around as in Figure 41, can be erased, together with its loop. This is because we can change the sign of any crossing by handlesliding it across the loop.

Figure 40: Proof of ~~above equation~~

In this manner we can make the component a ±1 framed unknot unlinked with anything but the loop.

empty, we see that the full twist can be removed by a sequence of extended Kirby moves.



~~Figure 41: Layering a component by extended Kirby moves~~

Notice also that any component of the surgery link can be made into an unframed unknot wrapped around some collection of handlebody handles and link components by a sequence of extended Kirby moves as shown in Figure (41).

With this in hand, the proof of the proposition is contained in Figure (42). Specifically, for each component shown partially in Figure (38), apply the moves in Figure (41) to make it have zero framing and bound a disk which may intersect handlebodies and the link, do the moves in Figure (42), and then do the inverse of the moves used to make it the unknot, which will put the given handle into the position occupied by the component. The last step of Figure (42), removing the chain links, is shown in Figure (43). ■

To continue with our example, I_{Σ_g} can be presented as in Figure (44).

Using this, the proof of the proposition is easy. Handle slide the handle over the component shown partially. The component shown partially now has a simple loop around it, and can be erased

Figure 42: Proof of the proposition

Figure 43: Removing a chain link

We will also need a description of gluing in terms of this presentation. There are two cases: If the two components of ∂M to be identified lie in different components of M, and if they lie in the same component.

Suppose they lie in different components of M, M_1 and M_2. A typical M_1 and M_2 are pictured in Figure (45). Here the boundary component of each which is to glued is shown, and is understood to be positive for M_1 and negative for M_2. They are assumed to have been put in standard position, as shown in Proposition (46). The parameterization is specified by marking meridians of H_g^+ on the boundaries.

Figure (46) shows M_1 presented as surgery on H_g^+ with some imbedded

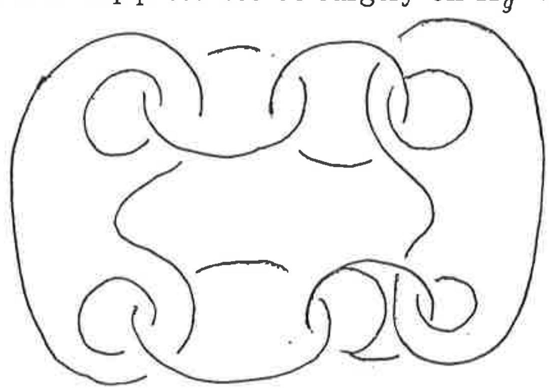


Figure 44: Presentation of I_{Σ_g}

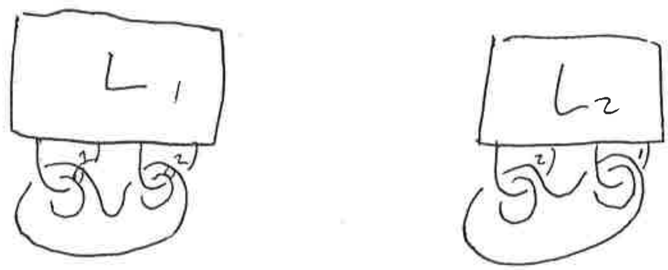


Figure 45: Two boundary components of a disconnected manifold

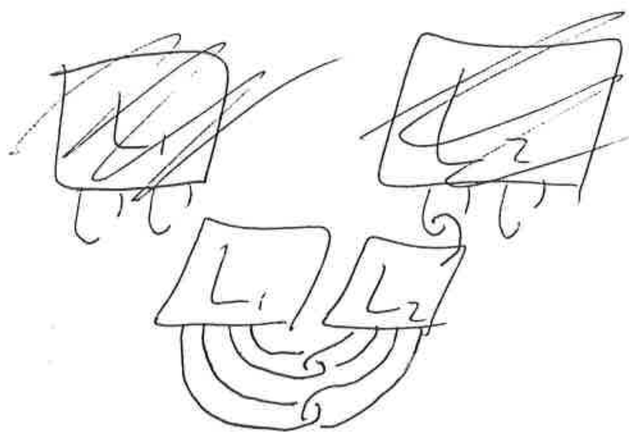


Figure 46: Gluing different components together

handlebodies removed, and then shows the result of gluing this in to M_2.

The case of a single component is a bit subtler. A typical presentation of M is shown in Figure (47), with as before the handlebodies in standard form and with meridians marked.

The top of Figure (48) shows a presentation of M as surgery on H_g^+ with handlebodies removed. The dashed line shows a sphere separating the other boundary component to be glued from the rest of the handlebody. The second line shows the result of cutting out this ball and gluing it back in along the two Σ_g boundary components, which gives $S^2 \times I$. The gluing of M is gotten by gluing the two S^2 boundaries together. Treating $S^2 \times I$ as two cylinders glued together, one empty and one containing the link and the other imbedded handlebodies, we see that M glued is the result of gluing an empty solid torus to a solid torus containing the links and handlebodies by

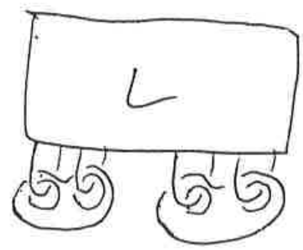


Figure 47: Two boundary components of a connected manifold

a map sending longitude to longitude and meridian to minus the meridian. This is simply a surgery with one more component, as shown in the last line of Figure (48).

To return to our example, $\Sigma_g \times S^1$ is obtained by gluing the boundary components of I_{Σ_g}, and thus can be presented as in Figure (49).

3.5.3 Intertwiner Tangles

A ribbon tangle is the image in $D \times I$ of a map of a collection of copies of $S^1 \times I$ and $I \times I$, called *ribbons*, and of $I \times I$, called *coupons*, subject to the following condition. The map is one-to-one on each component, and is one-to-one on the union minus every copy of $I \times \{0\}$ and $I \times \{1\}$. The image of the boundary of coupons only intersect the image of the boundary of ribbons, and the image of $I \times \{0\}$ and $I \times \{1\}$ of each open ribbon gets mapped one-to-one into either the boundary of a coupon or $[-1, 1] \times \{0\}$ or $[-1, 1] \times \{1\}$ in $D \times I$, the map being increasing with respect to the ordering on the intervals. Two ribbon graphs are equivalent if there is an ambient isotopy of $D \times I$ fixing the boundary which takes each ribbon or coupon one to one to a ribbon or coupon respectively, and so that the image of the sequence of the points $((0,0),(0,1),(1,0),(1,1))$ of any $I \times I$ gets sent to the sequence $((0,0),(0,1),(1,0),(1,1))$ of some $I \times I$. An example of a ribbon tangle is given in Figure (50). Here ribbons are drawn as lines, with an arrow indicating the positive direction and the line understood to be thickened parallel to the page. Coupons are drawn as squares, with arrows indicating the positive direction of each I.

Ribbon tangles have the same sense of composition and tensor product,

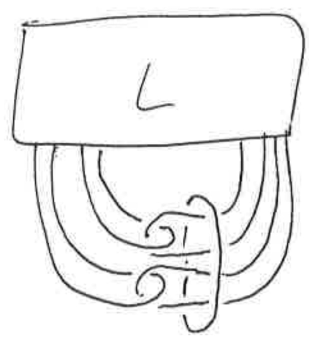


Figure 48: Gluing a connected manifold to itself

and form a strict monoidal category which extends the category of framed oriented tangles.

Theorem 16 *The category of ribbon tangles is generated by those shown in Figure (51), with the relations shown in Figure (52).*

Pf: The argument for this works by very precise analogy to the presentation of the tangle category given in Corollary (2). ■

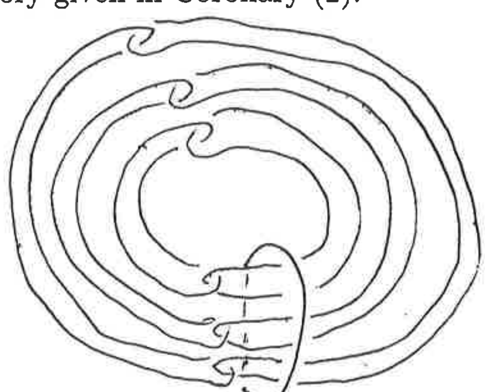


Figure 49: The manifold $\Sigma_g \times S^1$

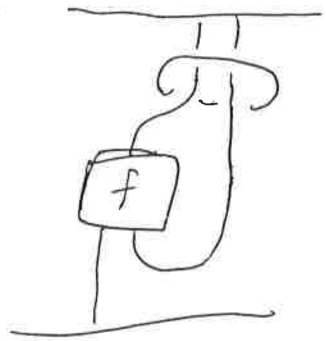


Figure 50: A typical ribbon tangle

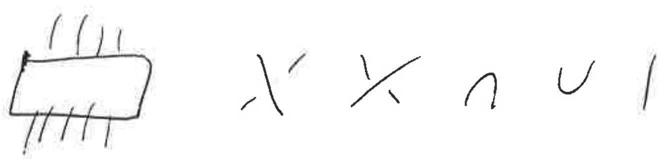


Figure 51: Generators of the ribbon tangle category

Given a tangle representation $\mathcal{F}$ with label set Λ, consider the strict monoidal category of ribbon tangles, with each ribbon labeled by an element of Λ, and each coupon is labeled by a (λ, γ) intertwiner, where λ and γ are defined as follows. If $I \times \{0\}$ of the coupon intersects ribbons labeled by λ_i, and if ε_i is plus if the i^{th} ribbon intersects along its $I \times \{1\}$ boundary minus if the i^{th} ribbon intersects along its $I \times \{0\}$ boundary, then $\lambda = \otimes_{i=1}^{n} \lambda_i^{\varepsilon_i}$, where $\lambda_i^+ = \lambda_i$ and $\lambda_i^- = \lambda_i^*$. Similarly, if $I \times \{1\}$ of the coupon intersects ribbons γ_1 through γ_n, and δ_i is plus or minus according to whether the intersection is along $I \times \{0\}$ or $I \times \{1\}$, then $\gamma = \otimes_{i=1}^{n} \gamma_i^{\delta_i}$.

Theorem 17 *$\mathcal{F}$ extends to a functor from the category of labeled ribbon tangles to the category of intertwiners, which takes tensor product to tensor product and composition to composition, and sends the ribbon tangle I labeled by an intertwiner f to f.*

Pf: Assign to I the values given in the theorem, and assign to A through H the values $\mathcal{F}(A)$ through $\mathcal{F}(H)$, as tangles. We have only to check moves I-IX of Theorem (16). Relations I-VI are satisfied because $\mathcal{F}$ is a tangle representation. For Relations VII-IX, assume the coupon shown is labeled by f, an intertwiner from λ to γ. Then the four versions of Relation VII follow from the fact that $R_{\delta,\gamma}(1 \otimes f) = (1 \otimes f)R_{\delta,\lambda}$ and $R_{\gamma,\delta}(f \otimes 1) = (f \otimes 1)R_{\lambda,\delta}$. Relation VIII says that $\phi_\gamma f \phi_\lambda^{-1} = f$, which follows from the fact that f commutes with ϕ. Finally, relation IX says that $g_\gamma f g_\lambda^{-1} = f = g_\gamma^{-1} f g_\lambda$, A ribbon graph is just a $(\hat{0}, \hat{0})$ ribbon tangle which we may consider to be sitting inside S^3.

3.5.4 Constructing the ATQFT

Let Σ_g, H_g^+, and H_g^- be as in Section 3.2. Define a ribbon graph in the handlebody H_g^+ or H_g^- exactly as for an ordinary ribbon graph, except the imbeddings are into $H_g^\pm$ rather than S^3. Let V_g^+, V_g^- be the vector space of formal linear combinations of labeled ribbon graphs in H_g^+, H_g^- respectively. If h is a labeled ribbon graph in H_g^+, and g is a labeled ribbon graph in H_g^-, then since $H_g^+ \cup H_g^- = S^3$, we can consider the union of h and g as a labeled ribbon graph in S^3. Define $\langle h, g\rangle$ to be $\sqrt{K}\mathcal{F}$ of this ribbon graph, where recall $K = 1/\sum \mathrm{qdim}^2\lambda_j$. This extends to a bilinear pairing $\langle\cdot,\cdot\rangle : V_g^+ \otimes V_g^- \to \mathbf{F}$. Let N_g^+ be the subspace

$$\{v \in V_g^+ : \langle v, w\rangle = 0 \;\forall w \in V_g^-\}$$

and N_g^- be the subspace

$$\{v \in V_g^- : \langle w, v\rangle = 0 \;\forall w \in V_g^+\}.$$

Then the pairing descends to the quotients, $\langle\cdot,\cdot\rangle : V_g^+/N_g^+ \otimes V_g^-/N_g^- \to \mathbf{F}$, and here is nondegenerate. Thus if we define $\mathcal{Z}(\Sigma_g) = V_g^+/N_g^+$, it is natural to identify $\mathcal{Z}(\Sigma_g)^*$ with V_g^-/N_g^-.

Proposition 48 *Let g be a labeled ribbon graph in S^3, with closed ribbon components labeled by link labels. Then $\mathcal{F}$ of g is the same as $\mathcal{F}$ of the result of one of the Kirby-like moves in Figure (53), where move II is interpreted as in Section 3.2.*

Pf: By exactly the same argument as in Theorem (12). Notice for move II that Proposition (40) still applies to the more general situation where the solid torus may contain ribbon graphs. ∎

Now consider a connected, framed parameterized 3-manifold M, presented by (N, L). Suppose the boundary of M is homeomorphic to $\Sigma_{g_1} \cup \ldots \cup \Sigma_{g_n}$ with signs $\varepsilon_1, \ldots, \varepsilon_n$, so that N is an imbedding of $H_{g_1}^{-\varepsilon_1} \cup \ldots \cup H_{g_n}^{-\varepsilon_n}$. Suppose h_i is a labeled ribbon graph in $H_{g_i}^{-\varepsilon}$ for $1 \leq i \leq n$. Consider the ribbon graph in S^3 formed by imbedding each h_i in S^3 via N, and choosing an orientation for each component of L and labeling it by $\Omega = \omega_1/\sqrt{K}$.

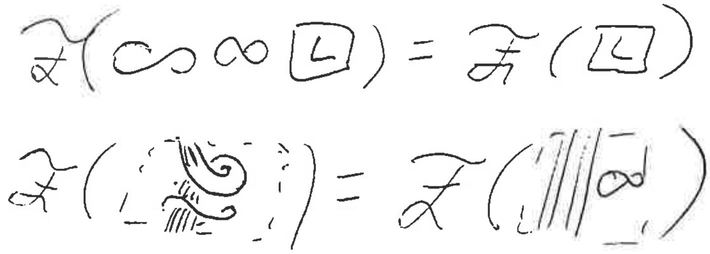


Figure 53: Kirby-like invariance of $\mathcal{F}$

Figure 54: Proof of invariance

The value $\sqrt{K}\mathcal{F}$ of this ribbon graph assigns a number to each sequence $h_1, \ldots, h_n$, which extends to a linear functional on $V_{g_1}^{-\varepsilon_1} \otimes \ldots \otimes V_{g_n}^{-\varepsilon_n}$.

Proposition 49 *This functional depends only on M, and not on the particular presentation (N, L). Thus we may call this functional $\mathcal{Z}'(M)$.*

Pf: Of course, the functional does not depend on the choice of orientation of the components of L because $\Omega^* = \Omega$. By Theorem (15) we only need check that this quantity does not change under the extended framed Kirby moves. It also suffices to check this fact for any sequence of handlebodies $h_1, \ldots h_n$. But this is exactly the statement of Proposition (48). ∎

In particular, we can choose an (N, L) so that the image of $H_{g_j}^{-\varepsilon_j}$ is in standard position, by Proposition (46). Choose labeled handlebodies h_i in

$H_{g_i}^{-\varepsilon_i}$, and consider the functional on $V_{g_j}^{-\varepsilon_j}$ defined by

$$\mathcal{Z}'(M)(h_1 \otimes \cdots \otimes h_{j-1} \otimes \cdot \otimes h_{j+1} \otimes \cdots \otimes h_n).$$

We may compute this functional by imbedding $h_1, \ldots \hat{h_j}, \ldots h_n$ into S^3 via N to get a labeled handlebody g in $S^3 - H_{g_j}^{-\varepsilon_j}$, (including the labeled components of L) which we may think of as sitting in $H_{g_j}^{\varepsilon_j}$, since $H_{g_1}^{-\varepsilon_j}$ was imbedded in standard position. The functional on any labeled handlebody $h_j \in H_{g_j}^{-\varepsilon_j}$ is then $\sqrt{K}\mathcal{F}$ of the union of g and h_j in S^3. In other words the functional is computed by taking the bracket with an element of $V_{g_j}^{\varepsilon_j}$. In particular it is zero on all of $N_{g_j}^{-\varepsilon_j}$. So $\mathcal{Z}(M)$ is zero on

$$V_{g_1}^{-\varepsilon_1} \otimes \cdots \otimes V_{g_{j-1}}^{-\varepsilon_{j-1}} \otimes N_{g_j}^{-\varepsilon_j} \otimes V_{g_{j+1}}^{-\varepsilon_{j+1}} \otimes \cdots \otimes V_{g_n}^{-\varepsilon_n}$$

for each $j \leq n$. Therefore $\mathcal{Z}'(M)$ descends to a functional on $\mathcal{Z}(\Sigma_{g_1})^{-\varepsilon_1} \otimes \cdots \mathcal{Z}(\Sigma_{g_n})^{-\varepsilon_n}$, or equivalently an element of $\mathcal{Z}(\Sigma_{g_1})^{\varepsilon_1} \otimes \cdots \otimes \mathcal{Z}(\Sigma_{g_n})^{\varepsilon_n}$, which we call $\mathcal{Z}(M)$. Here as usual $\mathcal{Z}(\Sigma_g)^+$ means $\mathcal{Z}(\Sigma_g)$ and $\mathcal{Z}(\Sigma_g)^-$ means $\mathcal{Z}(\Sigma_g)^*$.

If M in not connected, write it as a union of connected pieces $M = M_1 \cup \cdots \cup M_k$, and order the boundary components so that ∂M_i precedes ∂M_j if $i < j$. Then we define

$$\mathcal{Z}(M) = \mathcal{Z}(M_1) \otimes \ldots \otimes \mathcal{Z}(M_k).$$

3.5.5 Verifying the Axioms

We are almost ready to verify that the map $\mathcal{Z}$ we construct in the previous section is indeed an ATQFT. We will need the following two lemmas, which may seem quite innocuous, but in fact represent the only place in the construction of the ATQFT where the structure of a tangle representation is used in an essential way.

Lemma 7 *For each $\hat{n}$, there exist $(\hat{0}, \hat{n})$ intertwiners T_i, and $(\hat{n}, \hat{0})$ intertwiners T^i, for $i = 1, \ldots, k$, such that for any $(\hat{n}, \hat{0})$ intertwiner S and $(\hat{0}, \hat{n})$ intertwiner R,*

$$\sum_{i=1}^{k} S T_i T^i R = SR$$

Pf: Consider a resolution of the identity of $\mathcal{C}(\hat{n})$ into minimal idempotents corresponding to the nonreducible label λ. If SqR is nonzero, then $qRSq$ is nonzero and nonnilpotent. But $qRSq$ is in $\mathcal{C}(\lambda)$. Since λ is nonreducible, $qRSq$ is invertible, so Sq is an invertible intertwiner from the range of q to the trivial label. Therefore λ is isomorphic to the trivial label whenever SqR is nonzero.

Thus if $p_1, \ldots, p_k$ are the minimal idempotents in this decomposition with range isomorphic to the trivial label, then

$$SR = \sum_{i=1}^{k} Sp_iR.$$

But if $f_i : \text{Range}(q) \to \mathbf{F}$ is this isomorphism, then $T_i = f_i$, $T^i = f_i^{-1}q$ satisfy the statement of the lemma. ∎

Lemma 8 *If $\mathcal{F}$ is a modular tangle representation, let S be the $(\hat{m}, \hat{m})$ intertwiner*

$$S = \sum_{j=1}^{n} \sqrt{K}\,\text{qdim}_{\lambda_j} S_j,$$

where S_j is $\mathcal{F}$ of $1_{\hat{m}}$ with a zero framed unknot labeled by λ_j around it positively, as in Figure (26). Then if R is any $(\hat{m}, \hat{m})$ intertwiner

$$\text{qtr}(RS) = \frac{1}{\sqrt{K}} \sum_{i=1}^{k} T^i R T_i.$$

Pf: Consider the same resolution of the identity as in the proof of the previous lemma, and let q be a minimal idempotent corresponding to λ. Then $\text{qtr}(RSq) = \text{qtr}(qRqS)$ is $\sqrt{K}$ times $\mathcal{F}$ of the Hopf link with one component labeled by Ω and the other labeled by the label correponding to the satellite qRq, as discussed in Proposition (40). This label is a multiple of λ, so the whole quantity is zero unless λ is isomorphic to the trivial label. Arguing as in the previous lemma, we have

$$\text{qtr}(RS) = \sum_{i=1}^{k} \text{qtr}(RST_iT^i).$$

But now

$$ST_i = T_i\mathcal{O}(\Omega) = T_i/\sqrt{K},$$

so

$$\mathrm{qtr}(RS) = \frac{1}{\sqrt{K}} \sum_{i=1}^{k} \mathrm{qtr}(RT_i T^i) = \frac{1}{\sqrt{K}} T^i R T_i.$$

■

Theorem 18 *The map $\mathcal{Z}$ defined in the previous section is a simple ATQFT.*

Pf: Reordering is trivial, and tensor product is by definition. For nontriviality, present I_{Σ_g} as in Figure (37). As a map on $\mathcal{Z}(\Sigma_g) \otimes \mathcal{Z}(\Sigma_g)^*$, it sends $h \in H_g^+$, $h' \in H_g^-$ to $\sqrt{K}\mathcal{F}$ of the ribbon graph formed by their union in S^3. This is exactly $\langle h, g \rangle$. Thus as an operator A on $\mathcal{Z}(\Sigma_g)$, $\mathcal{Z}(I_{\Sigma_g})$ acts by

$$\langle Ah, g \rangle = \langle h, g \rangle$$

so

$$A = \text{identity}.$$

To show gluing, suppose ∂M is identified with $\Sigma_{g_1} \cup \ldots \cup \Sigma_{g_n}$ with signs $\varepsilon_1, \ldots, \varepsilon_n$, where $\Sigma_{g_1} = \Sigma_{g_2}$ and $\varepsilon_1 = +$, $\varepsilon_2 = -$. Let M' be M with the first two components glued. If

$$\mathcal{Z}(M) = \sum_{i,j} v_i \otimes v_j^* \otimes w_{i,j}$$

where v_i is a basis of $\mathcal{Z}(\Sigma_{g_1})$, v_j^* is the dual basis of $\mathcal{Z}(\Sigma_{g_1})^*$, and $w_{i,j}$ is an element of $\mathcal{Z}(\Sigma_{g_3})^{\varepsilon_3} \otimes \cdots \otimes \mathcal{Z}(\Sigma_{g_n})^{\varepsilon_n}$, then we wish to show

$$\mathcal{Z}(M') = \sum_i w_{i,i}.$$

We may clearly assume M has only those components bounded by the first two boundary components. There are two cases.

First assume the first boundary component of M lies in a different component of M from the second. Write $M = M_1 \cup M_2$ and present M_1 and M_2 by (N_1, L_1) and (N_2, L_2) respectively, with the first two boundary components presented in standard form. Let h_i be a labeled ribbon graph in $H_{g_i}^{-\varepsilon_i}$ for $3 \leq i \leq n$, and let N_1', N_2' be N_1 and N_2 respectively with h_i glued in by the imbeddings N_1 and N_2. Thus N_1' is now an imbedding of $H_{g_1}^-$ in standard position with a ribbon graph and the link L_1 in the complement, and N_2' is and imbedding of $H^+_{g_1}$ in standard position with a ribbon graph and the link L_2 in the complement. Thus, orienting the components of L_1 and L_2 and labeling them by Ω, we may think of N_1' as a labeled ribbon graph h in $H^+_{g_1}$, and N_2' as a labeled ribbon graph g in H^-_g.

By the definition of $\mathcal{Z}(M)$, if $\mathcal{Z}(M) = \sum_{i,j} v_i \otimes v_j^* \otimes w_{i,j}$, then

$$h \otimes g = \sum_{i,j} v_i \otimes v_j^* \otimes \langle w_{i,j}, h_3 \otimes \cdots \otimes h_n \rangle,$$

identifying $h, g, h_3, \ldots, h_n$ with their equivalence classes in $\mathcal{Z}(\Sigma_g)^\epsilon$. In particular

$$\langle h, g \rangle = \sum_i \langle h, v_i^* \rangle \langle v_i, g \rangle = \sum_i \langle w_{i,i}, h_3 \otimes \cdots \otimes h_n \rangle.$$

But $\langle h, g \rangle$ is just $1/\sqrt{K}\mathcal{F}$ of M' with $h_3, \ldots, h_n$ glued in along the boundary, which is

$$\langle \mathcal{Z}(M'), h_3 \otimes \cdots \otimes h_n \rangle.$$

This proves the gluing axiom in this case.

Now suppose M consists of one component and is presented by (N, L) with the first two handlebodies imbedded in standard form in N. Again glue in $h_3, \ldots h_n$ and orient each component of L and label it by Ω. The result is a ribbon graph in $H^+_g \# H^-_g$. Writing Ω as a linear combination of tangle labels we can write the ribbon graph as a linear combination $\sum \alpha_j g_j$ with each ribbon graph g_j having every component of L labeled by a tangle label. Now choose $h_\pm$ labeled ribbon graphs in $H^\pm_{g_1}$ respectively, and glue them in via N. The result is a linear combination $\sum \alpha_j g_j'$ of ribbon graphs in S^3, $1/\sqrt{K}\mathcal{F}$ of which is

$$\langle \mathcal{Z}(M), h_- \otimes h_+ \otimes h_3 \otimes \cdots \otimes h_n \rangle.$$

Now choose an imbedding of S^2 into S^3 which separates the imbedding of $H^+_{g_1}$ and $H^-_{g_1}$, intersects L transversely and doesn't intersect the image of N at all. This cuts each g_j' into a composition of a $(\hat{n}, \hat{0})$ ribbon tangle S_j and an $(\hat{0}, \hat{n})$ tangle R_j. But by Lemma (7), this has the same $\mathcal{F}$ value as $\sum_{j,i} S_j T_i T^i R_j$, for some $(\hat{0}, \hat{n})$ and $(\hat{n}, \hat{0})$ tangles T_i and T^i (they will depend on j too, but we'll suppress that).

But $S_jT_iT^iR_j$ is a distant union of ribbon graphs, so

$$\begin{aligned}
&\langle \mathcal{Z}(M), h_- \otimes h_+ \otimes h_1 \otimes \cdots \otimes h_n \rangle \\
&= \sqrt{K} \sum_{j,i} \mathcal{F}(S_jT_i)\mathcal{F}(T^iR_j) \\
&= \frac{1}{\sqrt{K}} \sum_{j,i} \sqrt{K}\mathcal{F}(S_jT_i)\sqrt{K}\mathcal{F}(T^iR_j).
\end{aligned}$$

But $\sqrt{K}\mathcal{F}(S_jT_i)$ is just $\langle S_j'T_i, h_- \rangle$, where S_j' is just S_j with a neighborhood of h_- removed, and similarly $\sqrt{K}\mathcal{F}(T^iR_j)$ is $\langle h_+, T^iR_j' \rangle$, where R_j' is R_j with a neighborhood of h_+ removed. Therefore

$$\begin{aligned}
\sum_k \langle w_{kk}, h_3 \otimes \cdots \otimes h_n \rangle &= \sum_k \langle \mathcal{Z}(M), v_k^* \otimes v_k \otimes h_3 \otimes \cdots \otimes h_n \rangle \\
&= \frac{1}{\sqrt{K}} \sum_{k,j,i} \langle S_j'T_i, v_k^* \rangle \langle v_k, T^iR_j' \rangle \\
&= \frac{1}{\sqrt{K}} \sum_{j,i} \langle S_j'T_i, T^iR_j' \rangle
\end{aligned}$$

which is

$$\mathcal{F}(\sum_{j,i} T^iU_jT_i),$$

where U_j is the $(\hat{n}, \hat{n})$ tangle gotten by gluing S_j to R_j.

Now by Lemma (8),

$$\sum_{j,i} \mathcal{F}(T^iU_jT_i) = \sum_j \sqrt{K}\mathrm{qtr}(U_jS) = \sqrt{K}\mathrm{qtr}(\sum_j U_jS).$$

But comparing to Figure (48), we see this is exactly $\langle \mathcal{Z}(M'), h_3 \otimes \cdots \otimes h_n \rangle$, where M' is M glued. ■

References

[Ati90] M. F. Atiyah. On framings of 3-manifolds. *Topology*, 29:1–8, 1990.

[BZ85] Gerhard Burde and Heiner Zieschang. *Knots*. Walter de Gruyter, Berlin – New York, 1985.

[CBV92] G. Massbaum C. Blanchet, N. Habegger and P. Vogel. Three-manifold invariants derived from the Kauffman bracket. *Topology*, 1992.

[FR79] R. Fenn and C. Rourke. On Kirby's calculus of framed links. *Topology*, 18:1–15, 1979.

[FY89] P. Freyd and D. Yetter. Braided compact closed categories with applications to low-dimensional topology. *Adv. in Math.*, 77:156–182, 1989.

[Hum72] James E. Humphreys. *Introduction to Lie Algebras and Representation Theory.* Springer-Verlag, New York-Heidelberg-Berlin, 1972.

[Jon85] Vaughan F. R. Jones. A polynomial invariant for knots via von Neumann algebras. *Bulletin AMS*, 12:103–111, 1985.

[Kau] L. H. Kauffman. An invariant of regualr isotopy. Announcement–1985, preprint–1986.

[Kau88] L. H. Kauffman. Statistical mechanics and the Jones polynomial. *Contemporary Mathematics*, 1988.

[Kir78] Robion Kirby. A calculus of framed links in s^3. *Invent. Math.*, 45:35–56, 1978.

[KM91] R. Kirby and P. Melvin. On the 3-manifold invariants of Witten and Reshetikhin-Turaev for $sl(2, \mathbf{c})$. *Invent. Math.*, 105:473–545, 1991.

[Koh92] T. Kohno. Topological invariants of three manifolds using representations of mapping class groups. *Topology*, 31:203–230, 1992.

[KR88] A. N. Kirillov and N. Yu. Reshetikhin. Representations of the algebra $U_q(sl_2)$, q-orthogonal polynomials and invariants of links. *Infinite Dimensional Lie Algebras and Groups*, 7:285–339, 1988.

[Lan71] Saunders Mac Lane. *Categories For the Working Mathematician.* Graduate Texts in Mathematics. Springer-Verlag, New York-Heidelberg-Berlin, 1971.

[Lic62] W. B. R. Lickorish. A representation of orientable combinatorial 3-manifolds. *Ann. Math.*, 1962.

[Lic91] W. B. R. Lickorish. Invariants for 3-manifolds from the combinatorics of the Jones polynomial. *Pacific. J. Math.*, 1991.

[Mor92] H. R. Morton. Invariants of links and 3-manifolds from skein theory and from quantum groups. In *Topics in Knot Theory*. Kluwer, 1992.

[RT90] N. Yu Reshetikhin and V. G. Turaev. Ribbon graphs and their invariants derived from Quantum groups. *Comm. Math. Phys.*, 127:1–26, 1990.

[RT91] N. Yu Reshetikhin and V. G. Turaev. Invariants of three manifolds via link polynomials and quantum groups. *Invent. Math.*, 103:547–597, 1991.

Justin Roberts

~~[Saw]~~ S~~teve Sawin. Extending the kirby calculus to three manifolds~~ with bo~~undary. preprint.~~

[Tur] V. G. Turaev. Operator invariants of tangles and R-matrices. To appear in Izv. ANN SSSR.

[TW93] V. Turaev and H. Wenzl. Quantum invariants of 3-manifolds associated with classical simple Lie algebras. *Int. J. Math.*, 4:323–358, 1993.

[Wal] Kevin Walker. On Witten's 3-manifold invariants. preprint.

[Wen90] H. Wenzl. Representations of braid groups and the quantum Yang-Baxter equation. *Pacific J. Math.*, 145:153–180, 1990.

[Wen93] H. Wenzl. Braids and invariants of 3-manifolds. *Invent. Math.*, 114:235–273, 1993.

[Wit88] Edward Witten. Quantum field theory and the Jones polynomial. *Institute for Advanced Study*, IASSNS-HEP-88/33, 1988.