South Pointing Chariot: An Invitation to Geometry

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Fairfield University

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The South-Pointing Chariot was a two-wheeled vehicle in ancient China with a moveable pointer that always pointed south, no matter how the chariot turned.

Dubious legends place its origins as far back as 2635 BCE, but most believe one was built by Ma Jun around 250 CE, and that it probably involved gears.

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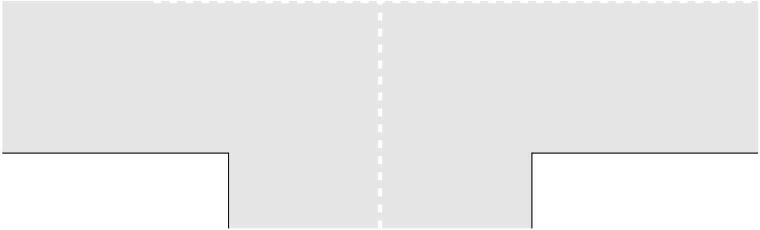
How Does This Work? Geometry

The left wheel and right wheel travel different distances around a turn



How much more does the left wheel travel than the right wheel? Call the width of the axle w. In a turn of radius r and angle θ (in radians) the left wheel travels $(r + w)\theta$ and the right wheel travels $r\theta$, so the difference is $w\theta$. When the chariot rotates θ degrees right, the left wheel travels $w\theta$ more than the right wheel.

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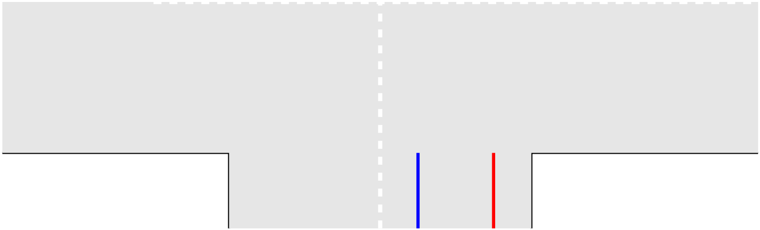

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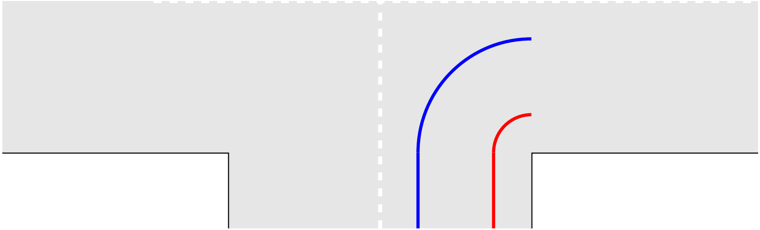

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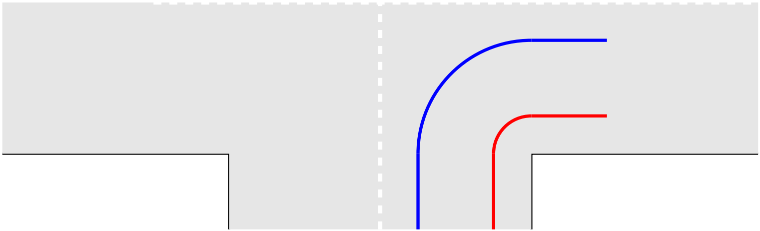

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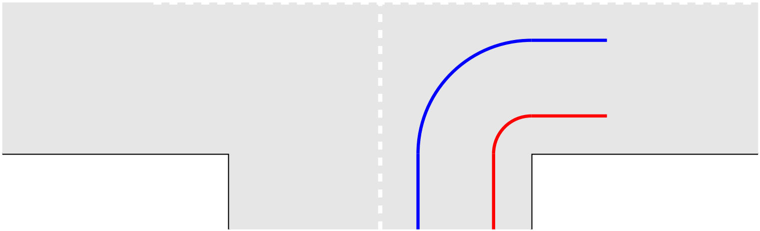

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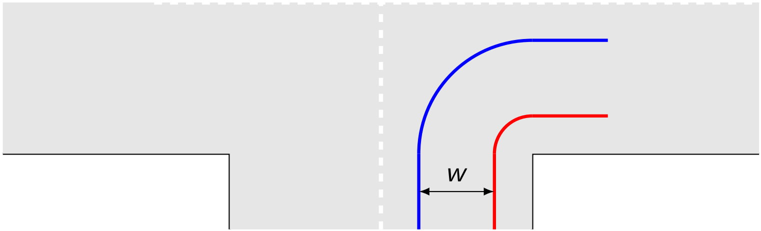

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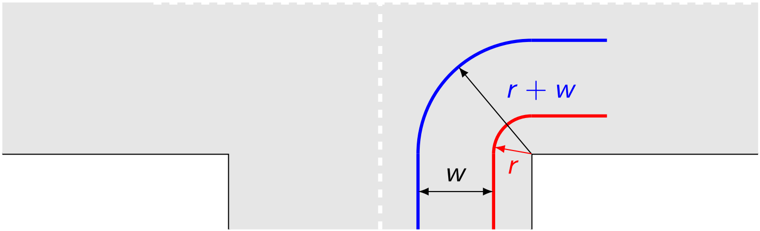

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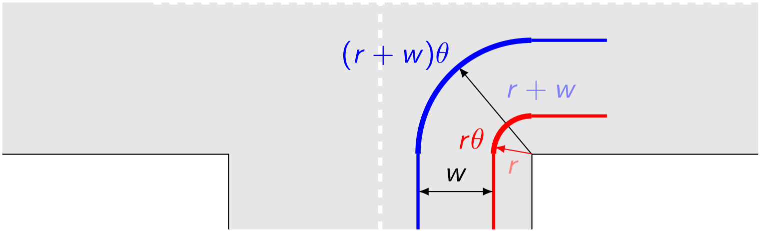

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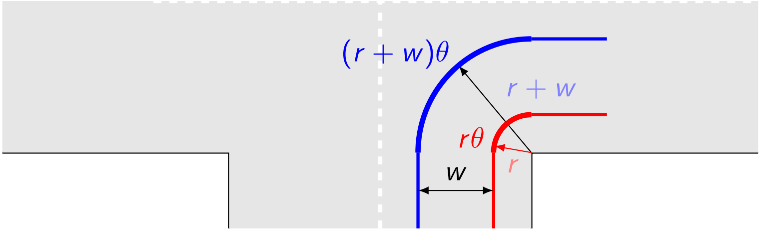

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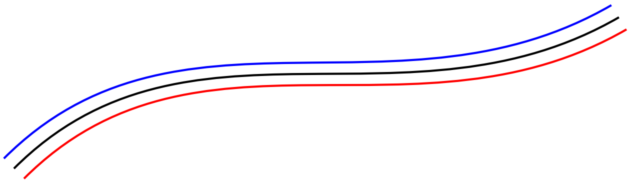
Any path can be approximated by straight lines and arcs of circles. So in any path the left wheel travels $w\theta_{\text{tot}}$ more than the right wheel, where θ_{tot} is the sum of all the rotations.



$$\theta_{\text{tot}} = \frac{d_l - d_r}{w}$$

where

- θ_{tot} is the total rotation clockwise undergone by the chariot
- d_l is the distance traveled by the left wheel
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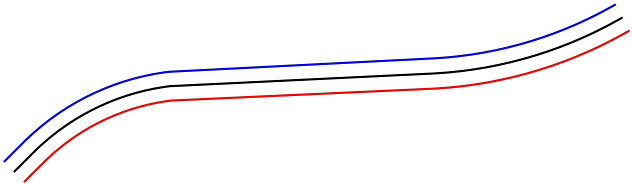

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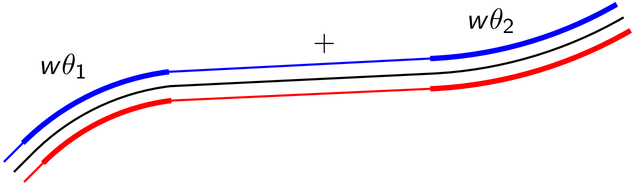

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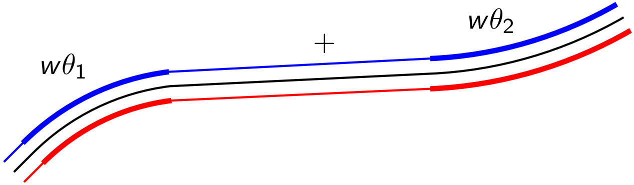

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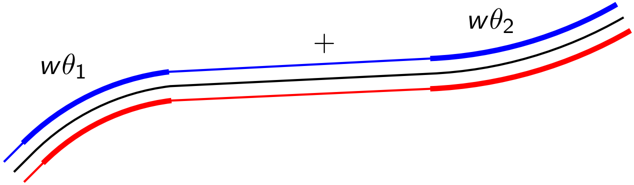

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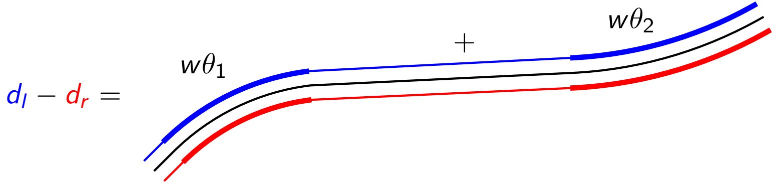

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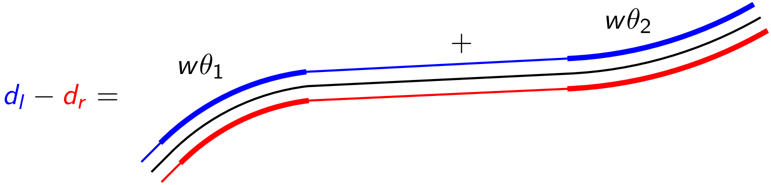

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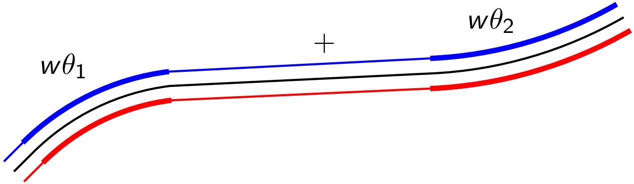

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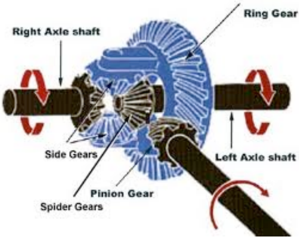

the middle axle rotates at a rate that is the average of the left and right axles' rotations so

$$\frac{d\theta}{dt} = \frac{1}{2}\left(\frac{d\theta_l}{dt} + \frac{d\theta_r}{dt}\right).$$

The middle axle is connected to the pointer. The left axle by an odd number of gears to the left wheel, so $d\theta_l/dt \propto v_l$ the velocity of left wheel and right axle is connected by even number of gears to right wheel, so $d\theta_r/dt \propto -v_r$ the velocity of the right wheel.

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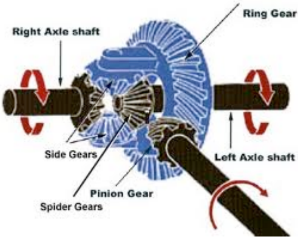

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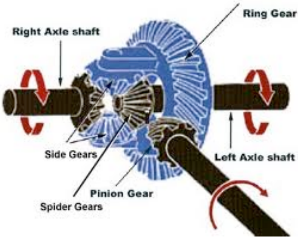

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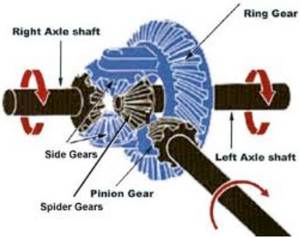

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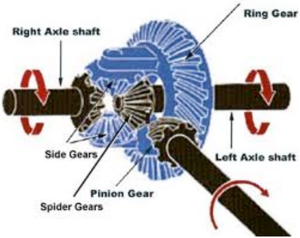

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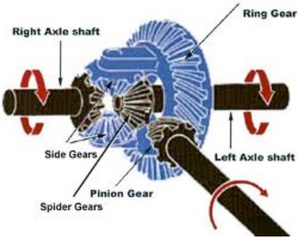

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$$\frac{d\theta_{\text{point}}}{dt} \propto v_l - v_r$$

Integrating over the time of travel yields

$$\theta_{\text{point}} \propto d_l - d_r = \frac{d_l - d_r}{w}$$

if we size the gears right, where θ_{point} is the total angle of rotation of the pointer counterclockwise (relative to the chariot) during the journey.

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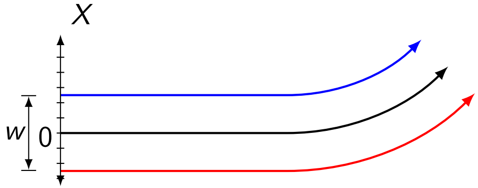

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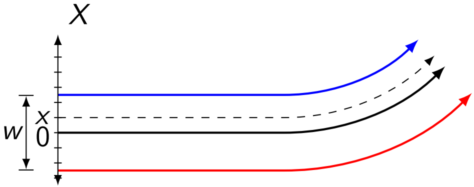

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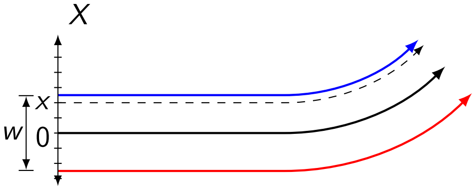

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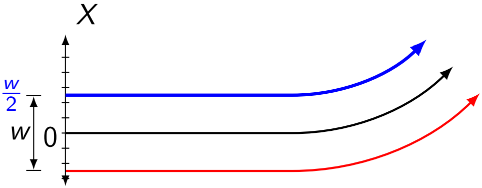

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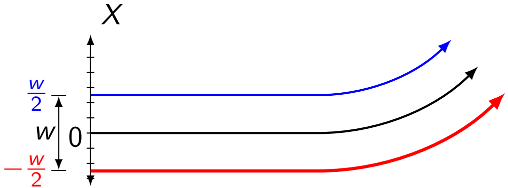

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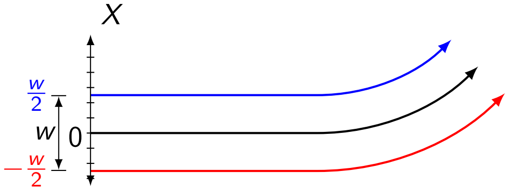

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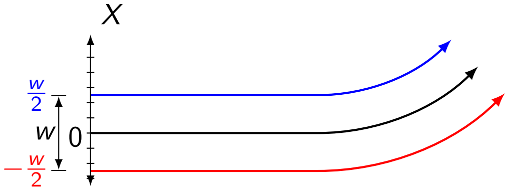

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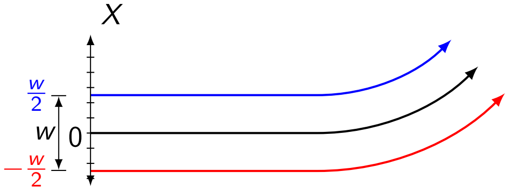

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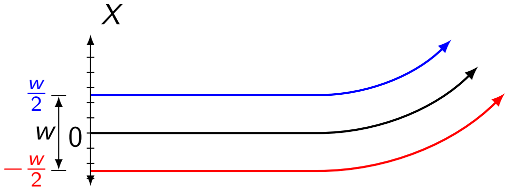

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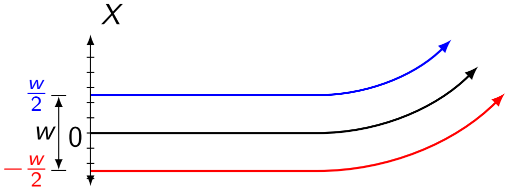

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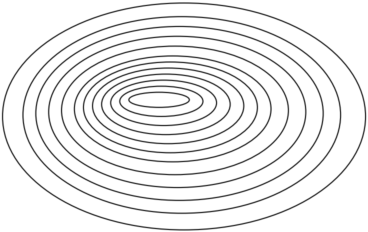

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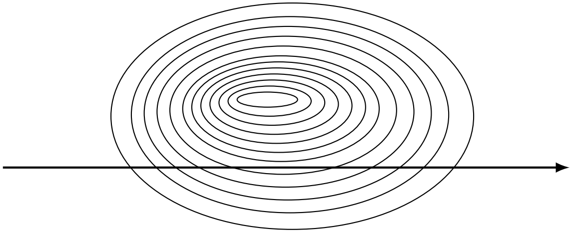

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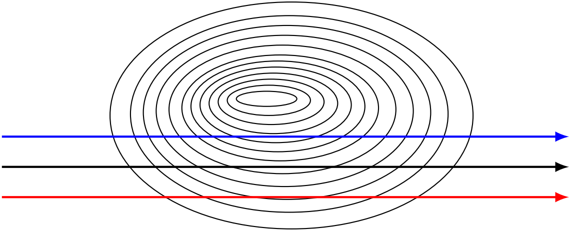

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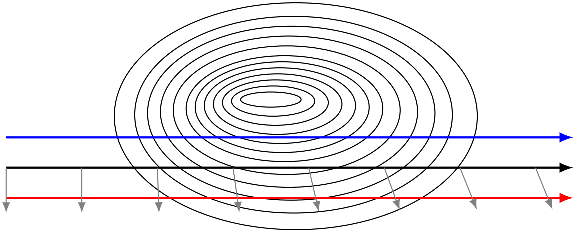

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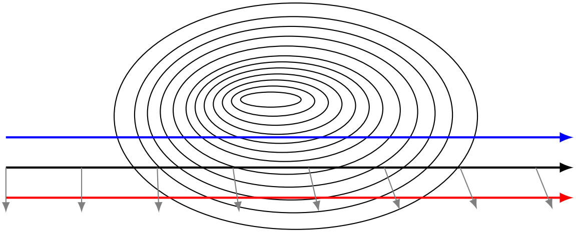
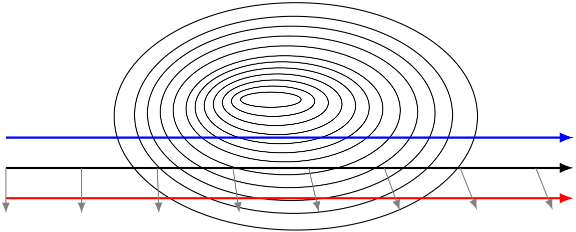


A bird thinks the chariot is going straight, but the pointer thinks it is turning right!

Who Is Right: Bird or Chariot?

We need a neutral referee.



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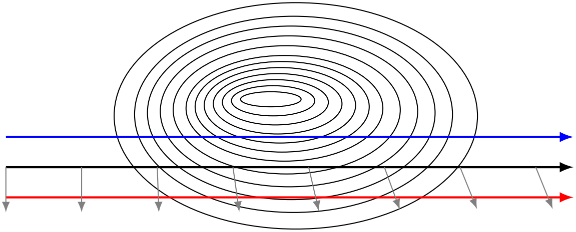

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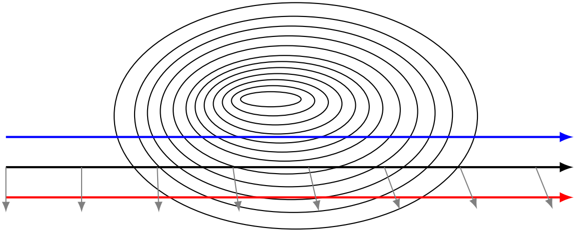

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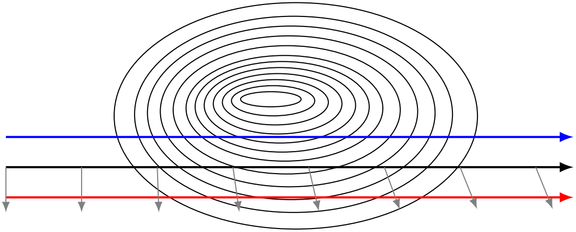

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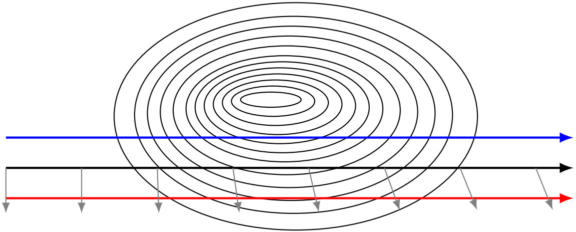

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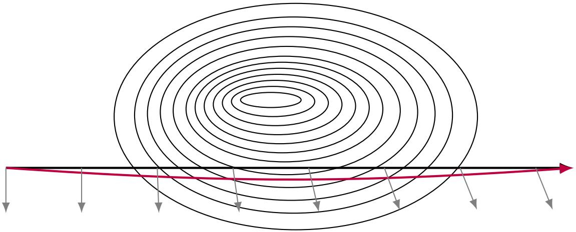

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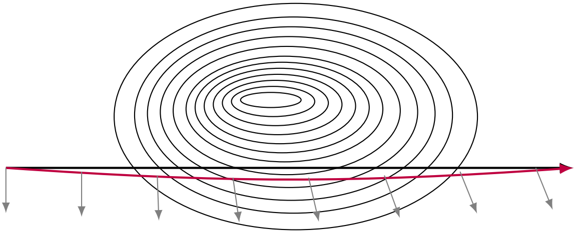

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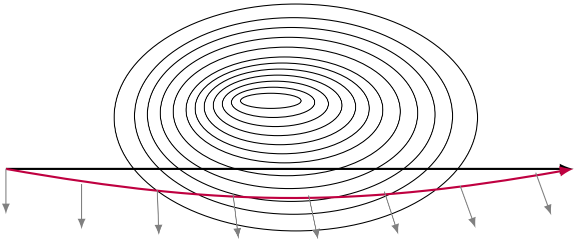

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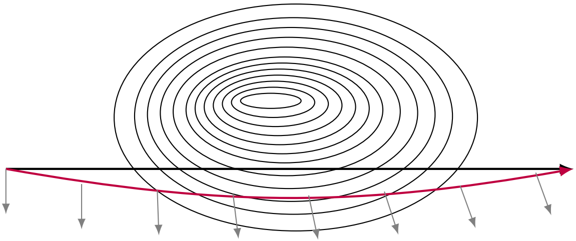

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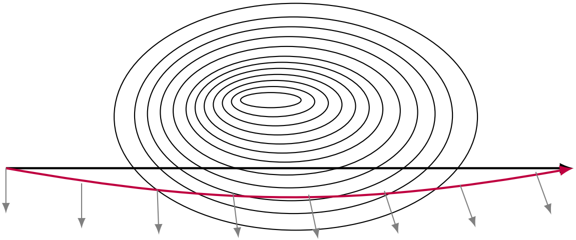

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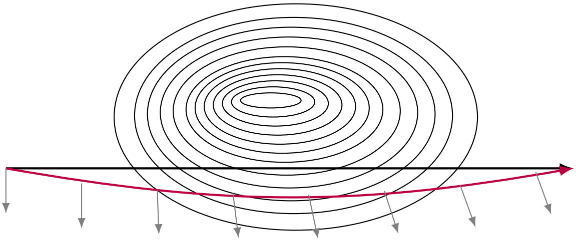

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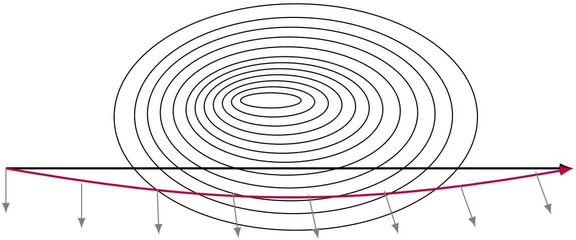

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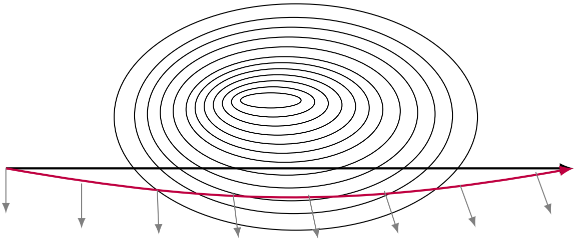

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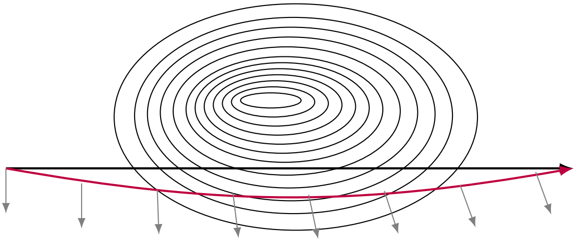

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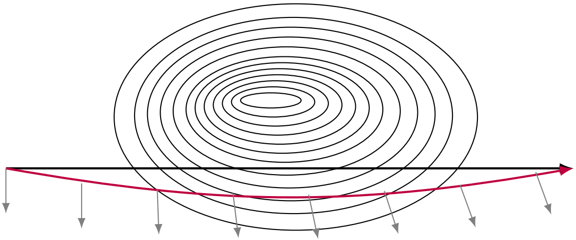

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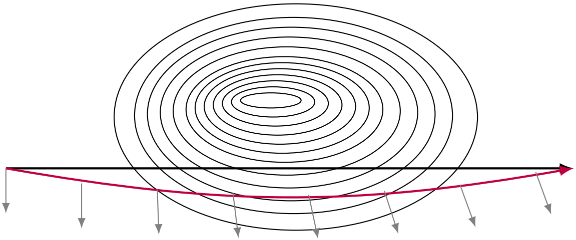

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W-W-Wait. The minimum length happens when the derivative is zero? Where have I heard that before? We can think of the set of all possible paths between two points as a (infinite dimensional!) space. Length is a continuous function in it. A (local) minimum should be a critical point. A critical point is typically where the derivative is zero, i.e. where any small perturbation of the path causes no first order change in length. That is what $\delta d/\delta x = 0$ tells us. Of course you have to trust multivariable calculus on infinite dimensional spaces.

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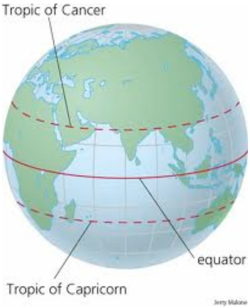
An Example

What does this look like on a sphere? If SPC traveled along equator, its pointer would not turn. Everything is symmetric about plane the equator lies on, so left wheel and right wheel travel same distance. Equator is a geodesic. Any "great circle," on plane through through origin is a geodesic.



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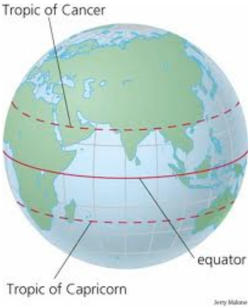

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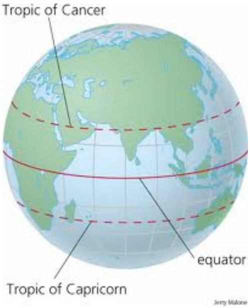

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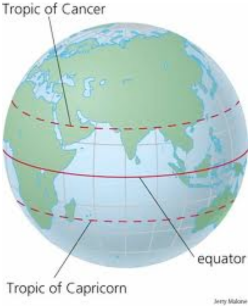

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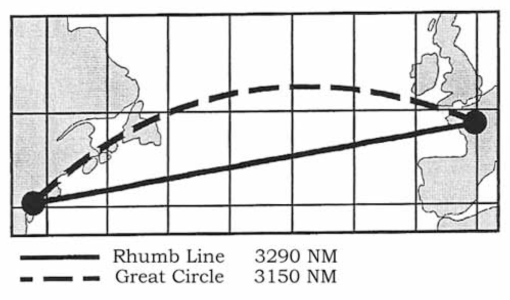

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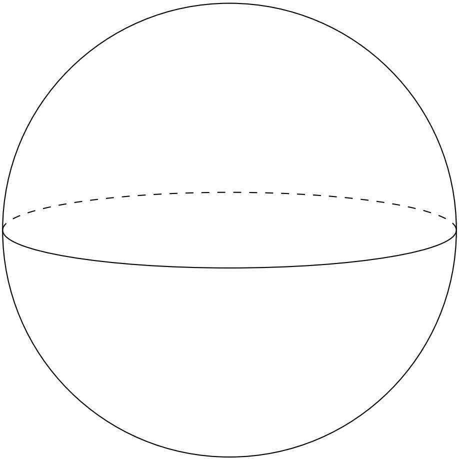
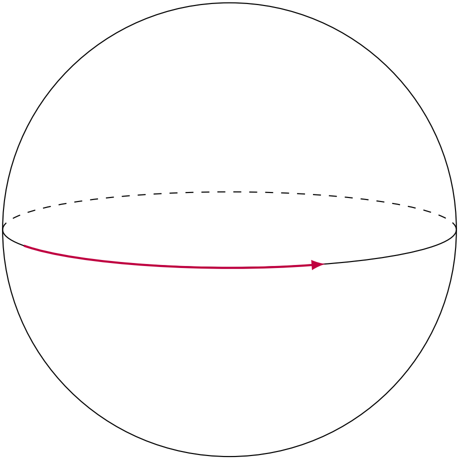


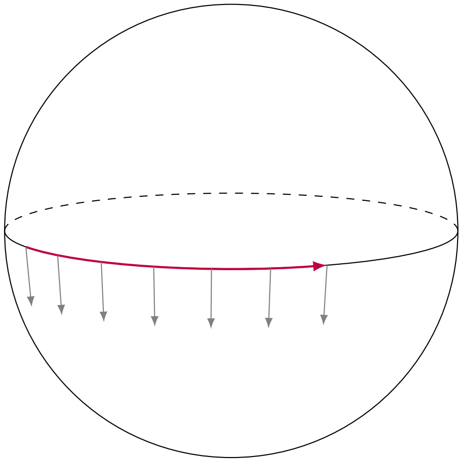
Now let's think about a loop.

More Example



Now let's think about a loop. Head east 1/4 way round equator. It's a geodesic, so pointer stays pointing south.

More Example



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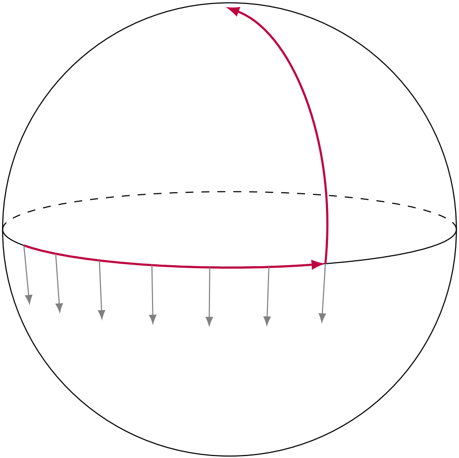

Now let's think about a loop. Turn 90° left and head to north pole.
Again a geodesic, so pointer stays point to the south.

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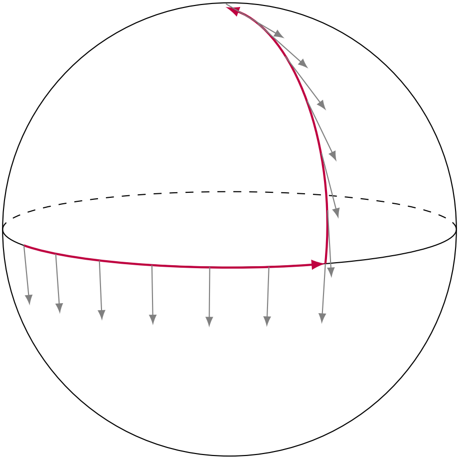

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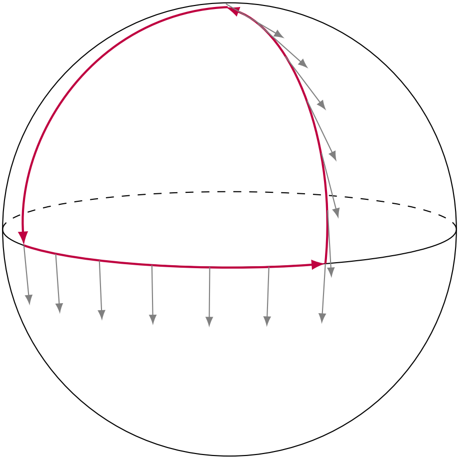
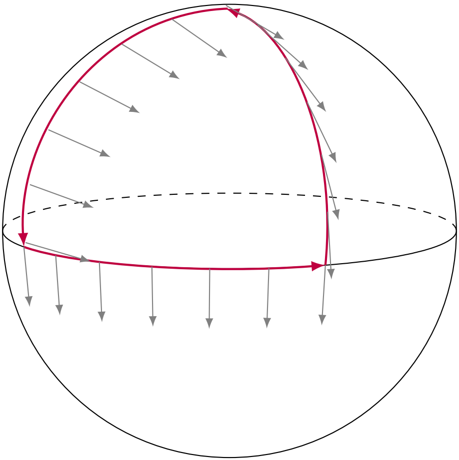


Now let's think about a loop. Turn 90° left again and head back south. pointer remains pointing east.

More Example



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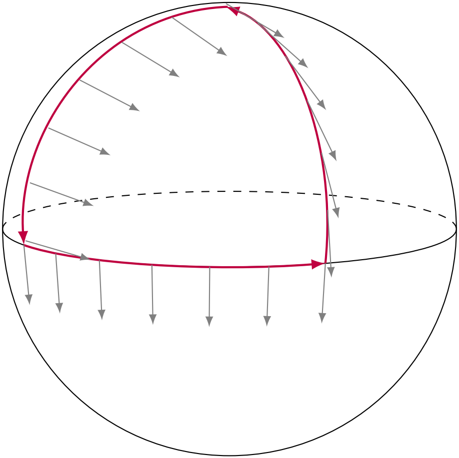

Now let's think about a loop. We are back where we started, but the "south pointer" has turned counterclockwise 90°! Not only doesn't it agree with the south, it doesn't even agree with itself!

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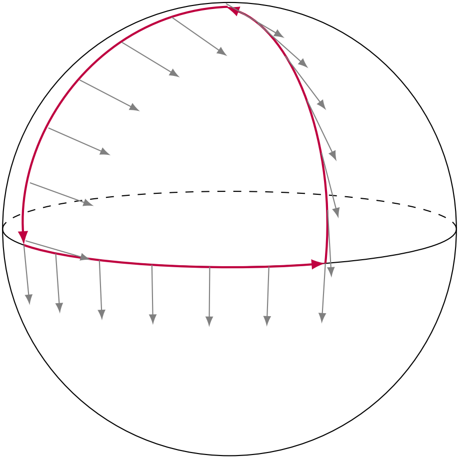

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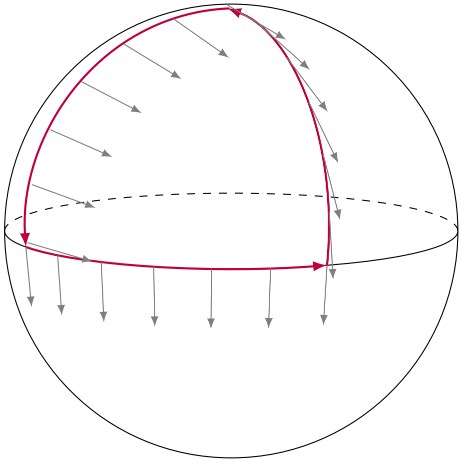

Now let's think about a loop. Another way to look at it is we drew a "triangle" with three right angles. Interior angles add up to 270°, which is 90° too much!

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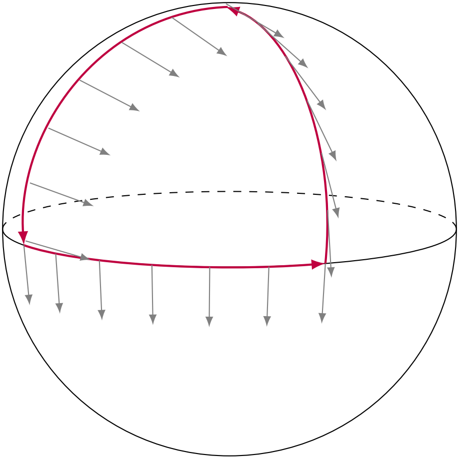

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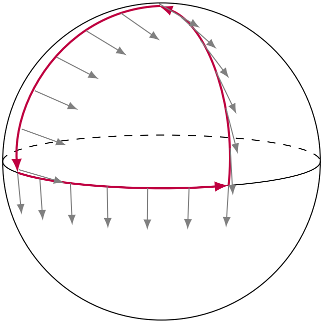
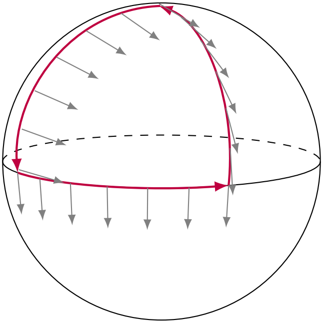


The 90° rotation SPC underwent through that loop happens because the surface is curved. Let's use as tool to explore and measure a surface's curvature. To each loop L on the surface associate a number, the *holonomy* $H(L)$ of the loop, the amount the pointer on SPC rotates from its starting position as it traverses the loop. To understand what it tells us about curvature, need to understand its properties.

Holonomy



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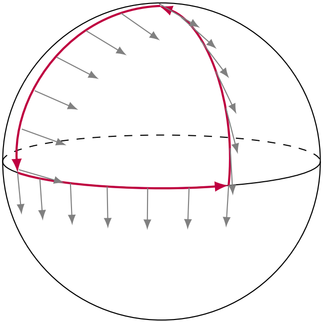

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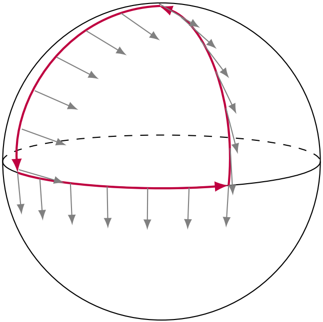

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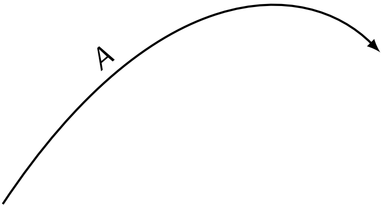
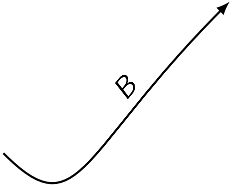
Holonomy - Concatenation



The concatenation AB of two paths A and B is the path AB that traverses one then the other. You can compose two loops as well.

$$H(LK) =$$

Holonomy - Concatenation



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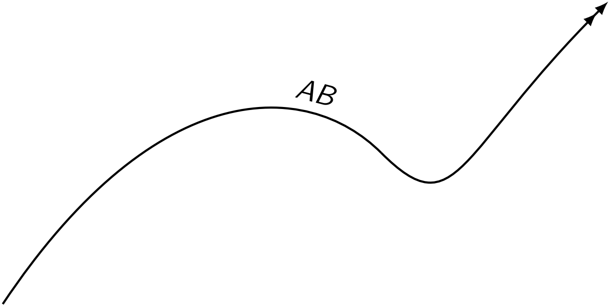

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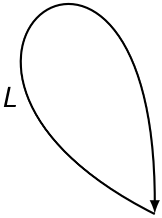

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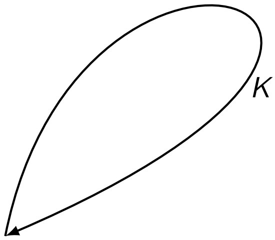

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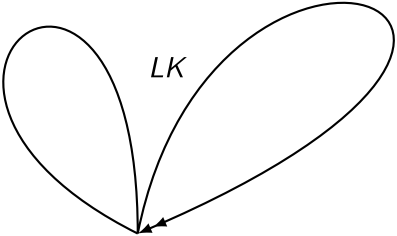

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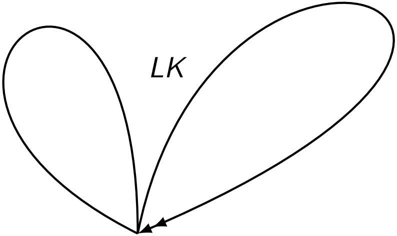

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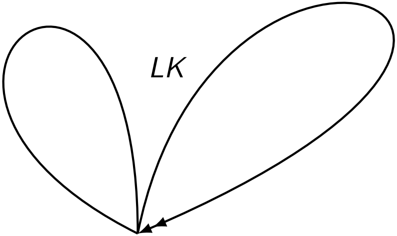

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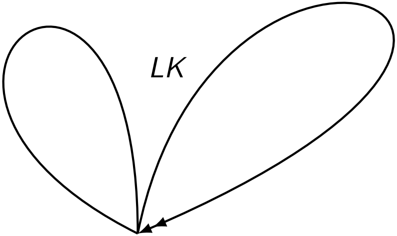

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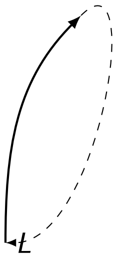
Holonomy - Starbucks Move



Suppose SPC is traversing loop L when it remembers coffee... it traverses over then back then finishes the loop to make a new loop L'.

$$H(L') = \quad .$$

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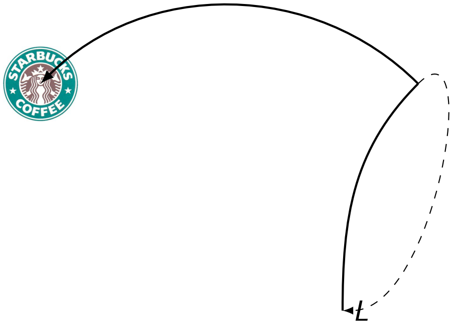

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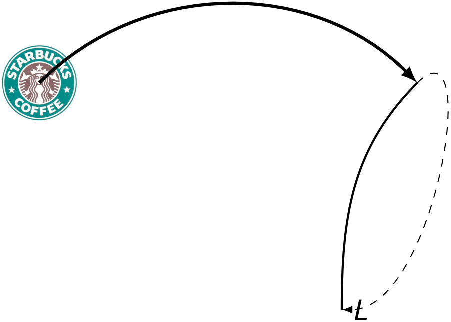

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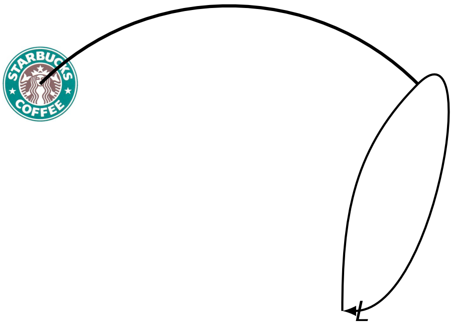

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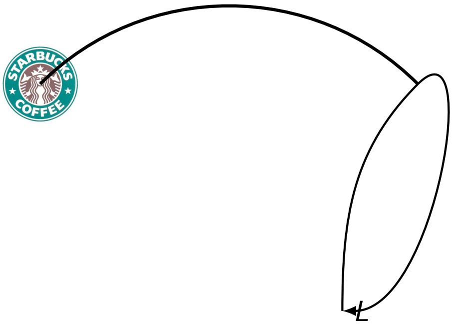

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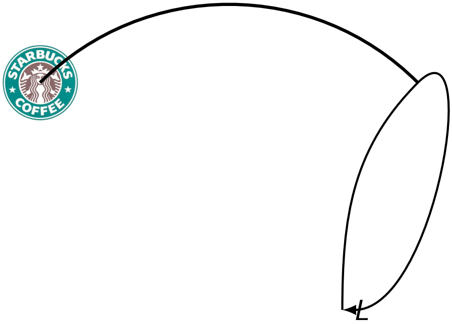

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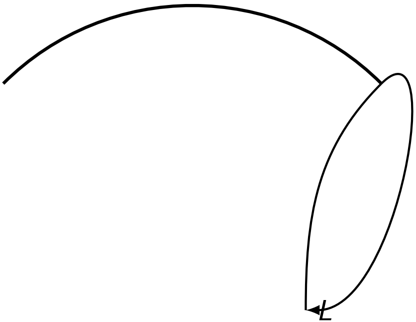
Holonomy - Starbucks Move - Why?



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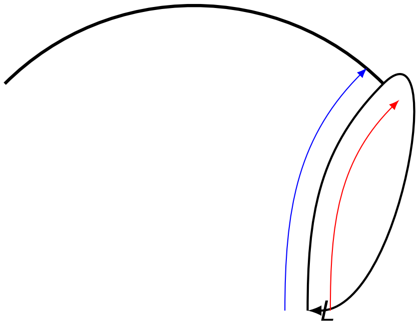

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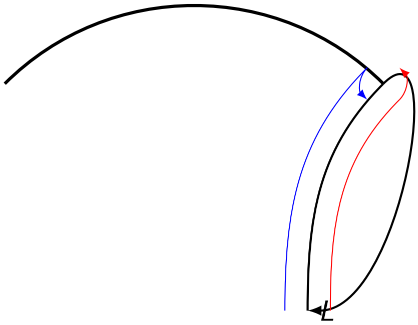

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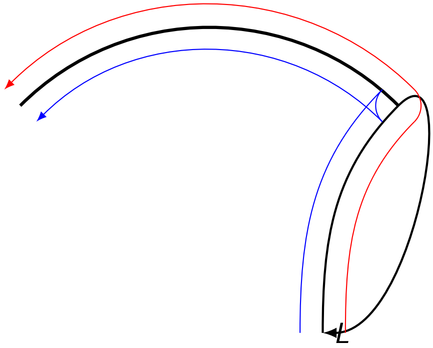

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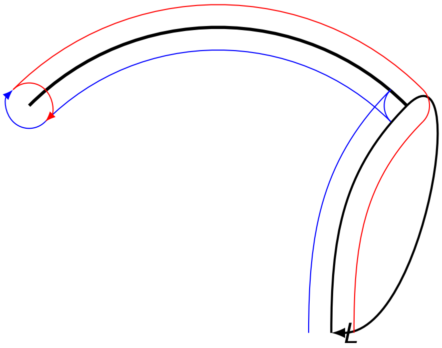

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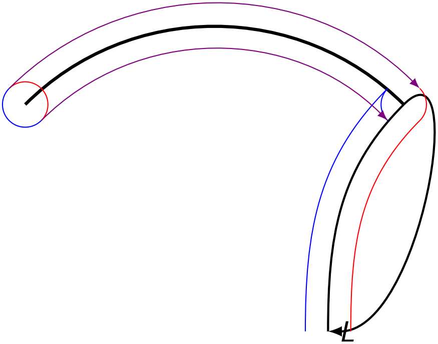

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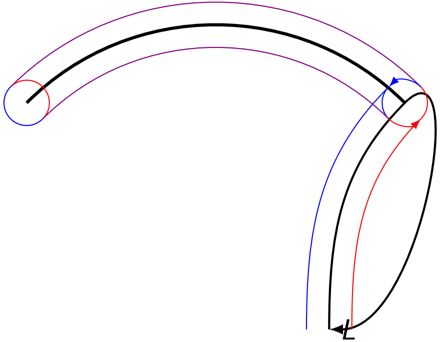

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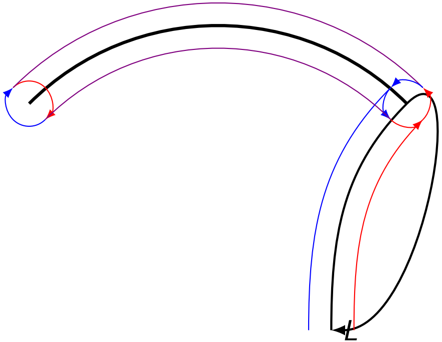
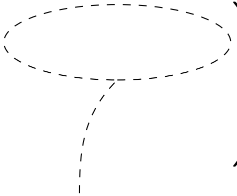


left wheel and right wheel travel the same distance during detour.

Holonomy - Another Version

$H\Bigg($  $\Bigg) = H\Bigg($  $\Bigg)$

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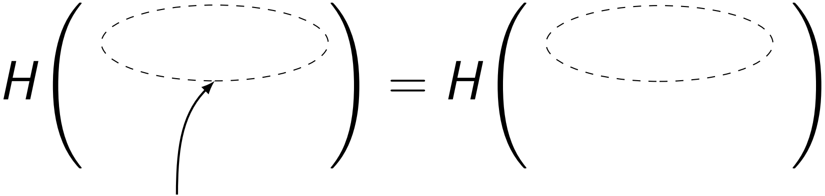

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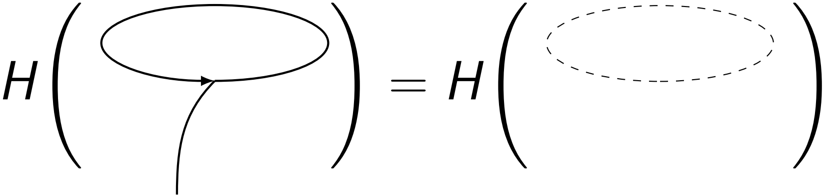

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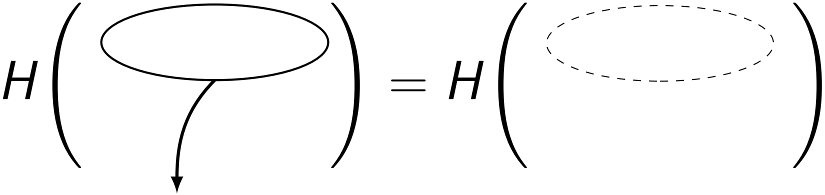

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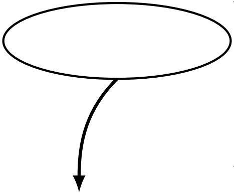
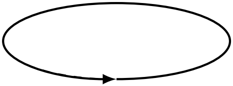
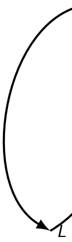
$H\Bigg($  $\Bigg) = H\Bigg($  $\Bigg)$

Holonomy - Cutting Loops Into Pieces



Consider a loop L surrounding a certain region of the surface. Add a Starbucks move back to the start to get L'. Which we can write as L_1 concatenated with L_2. Thus holonomy is sum of pieces just like area.

$$H(L) = H(L') = H(L_1) + H(L_2)$$

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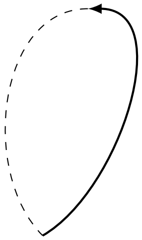

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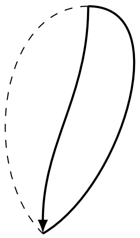

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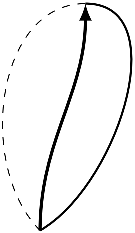

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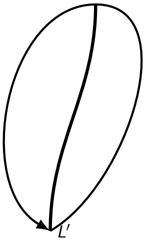

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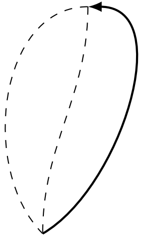

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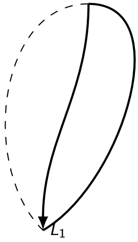

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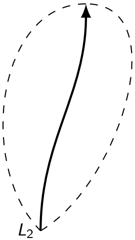

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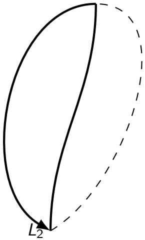

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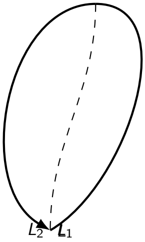

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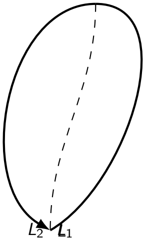

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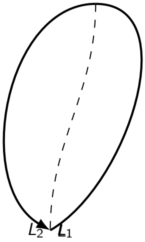

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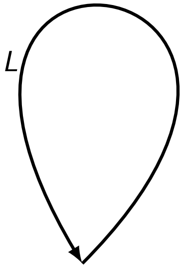
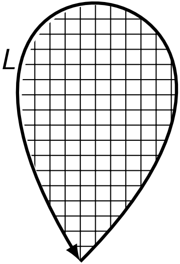
Holonomy - More Chopping into Pieces



So we can chop up a loop around a region as much as we want. Let's say we want a lot.

$$H(L) = \sum_{i,j} H\left[R_{\Delta x,\Delta y}(x_i, y_i)\right].$$

Holonomy - More Chopping into Pieces



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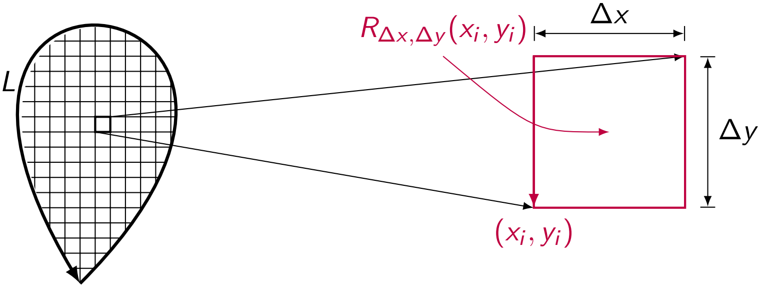
$$H(L) = \sum_{i,j} H\left[R_{\Delta x, \Delta y}(x_i, y_i)\right].$$

Holonomy - More Chopping into Pieces



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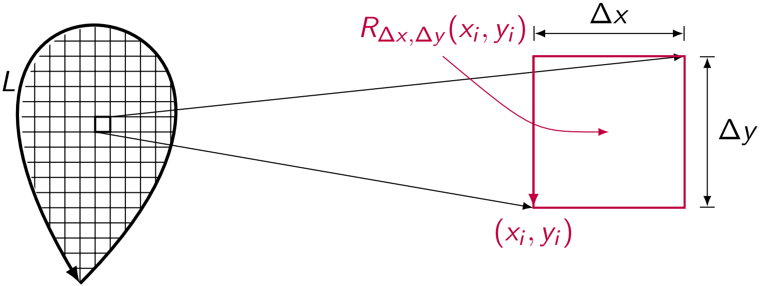

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Holonomy - Limits of Chopping in Pieces

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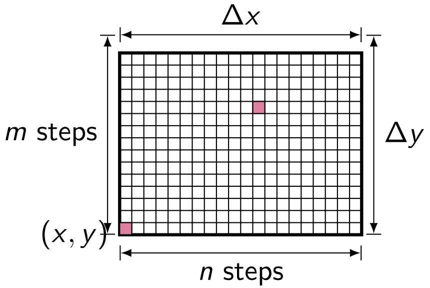

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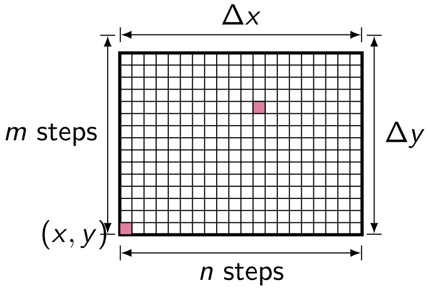

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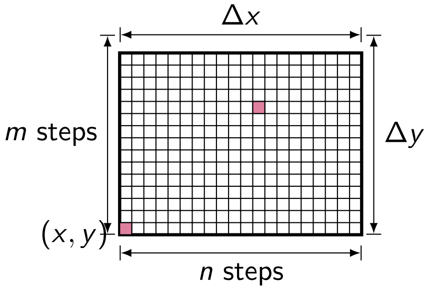

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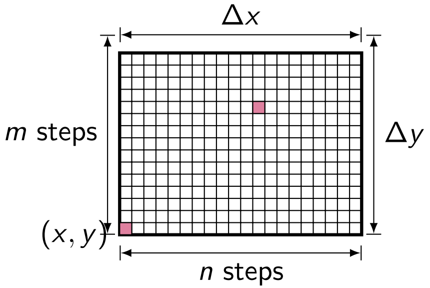

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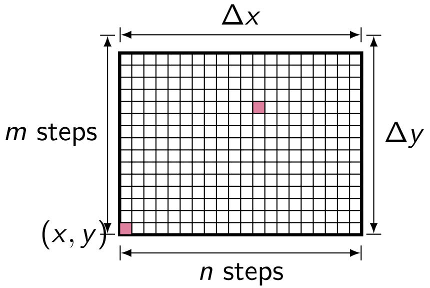

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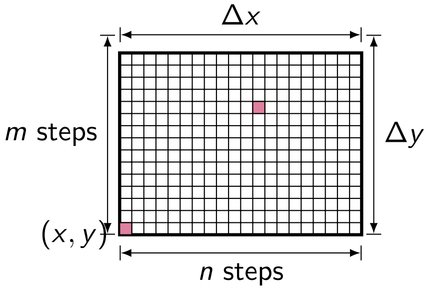

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We just argued that for small Δx and Δy the quantity $H\left[R_{\Delta,\Delta y}(x,y)\right]/\Delta x\Delta y$ doesn't depend on Δx and Δ_y. That is the limit of this quantity as Δx and Δy go to zero exists. Define

$$k(x,y)=\lim_{\Delta x,\Delta y\to 0}\frac{H\left[R_{\Delta,\Delta y}(x,y)\right]}{\Delta x\Delta y}.$$

k assigns a number to each point on the surface, which we call the *curvature at that point.* Notice

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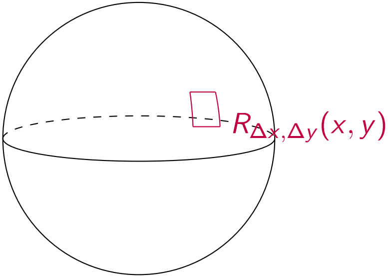

Any loop of the same size and shape on a sphere has the same holonomy.
So the limit $k(x, y)$ at any point on the sphere is the same: The sphere has constant curvature k. We know this loop has holonomy 90° or $\frac{\pi}{2}$.

$$\frac{\pi}{2} = H(L) = \iint k dx\, dy = k\text{Area} = \frac{4\pi r^2 k}{8}$$

since the loop takes up 1/8 the surface area of sphere. So

$$k = \frac{1}{r^2}$$

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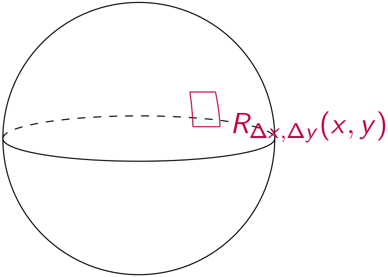

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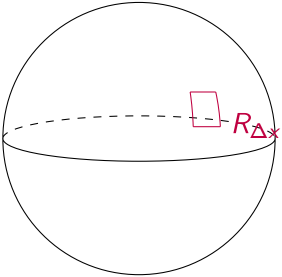

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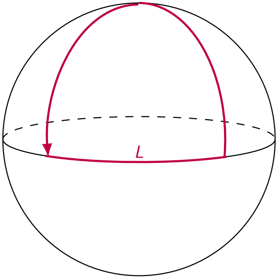

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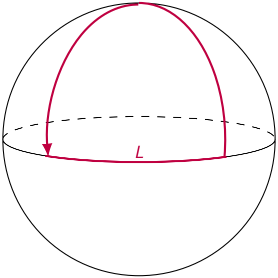

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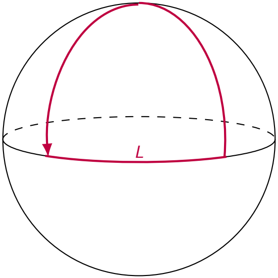

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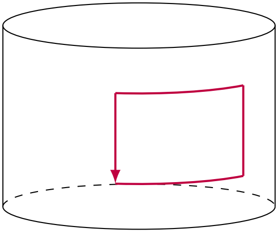
Curvature on a Cylinder



Consider a rectangular path on a cylinder. Clearly vertical lines and horizontal circles are geodesics. So the holonomy around such a rectangle is 0, which means the curvature k at each point is 0. So a cylinder is not curved?!??

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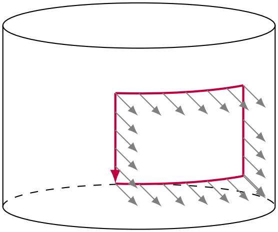

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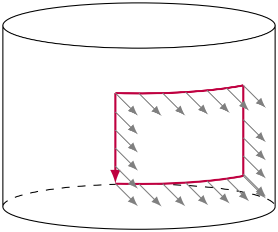

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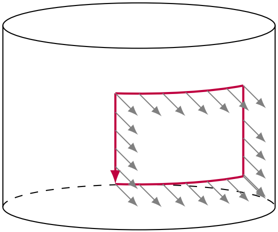

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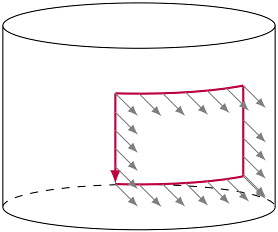

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Cloth, Rubber, and Curvature

Draw a line on your shirt. Put it on a hanger, or throw it on a chair, the line is the same length. Changes you can do to cloth do not change distances. SPC only measures distances, holonomy unchanged by distance preserving transformations.

A cylinder of cloth, if cut, can be flattened. So a "cloth invariant" notion of curvature would say a cylinder has no curvature. Could you flatten a (cut) sphere made of cloth?

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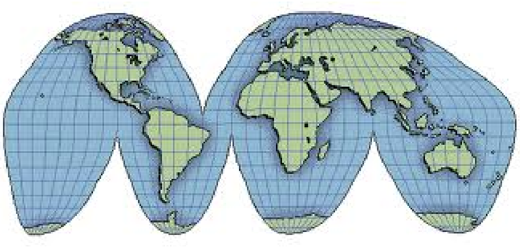

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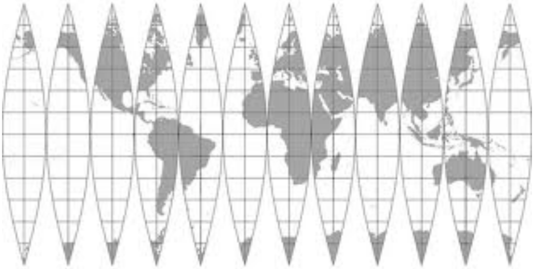

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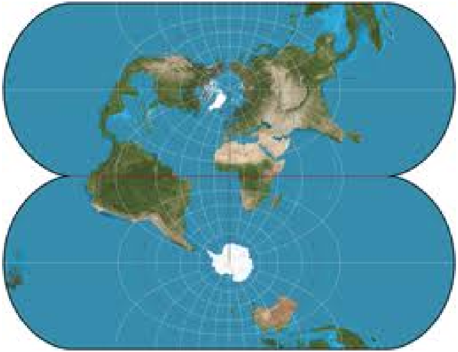

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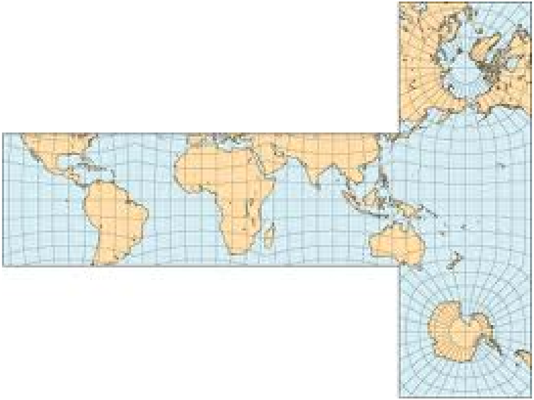

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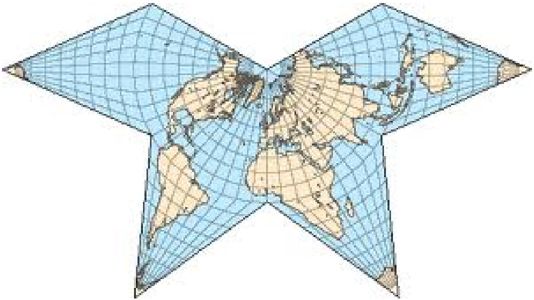

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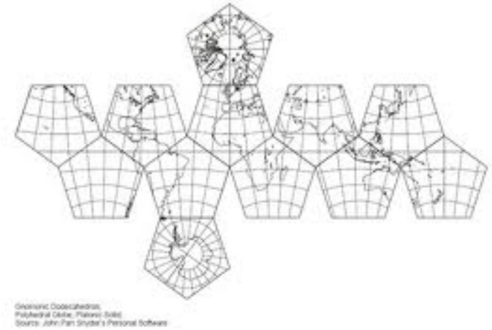

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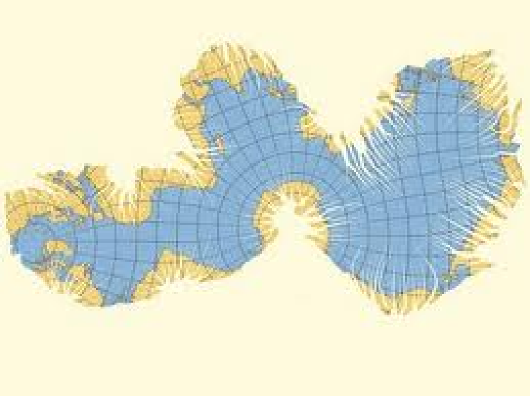

No!

As long as you can draw a loop on the projection, it will have holonomy on the sphere proportional to area, but holonomy 0 on the map. So the distances cannot agree, there is distortion. We have proven that it is not possible to make a map of the world without distortion.

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How To Do This For Real

Gauss defined curvature as the product of the maximum and minimum curvature of the intersection of the surface with all possible normal planes to surface at that point. This appears to depend on the embedding of the surface, i.e. is not a cloth embedding. Gauss proved in his Theorema Egregium that it was intrinsic, that is that it depended only on distances. His theorem effectively proved that his definition was equivalent to ours in terms of SPC. From this he could easily prove that you can't map the earth. Geodesics, holonomy and curvature can all be extended to higher dimensions and form the basis of modern differential geometry.

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From Cloth To Rubber

Now suppose you have a sphere made of rubber. Draw a very small clockwise loop near the north pole. The integral of curvature *outside* the loop is just about 4π, so the loop has holonomy $4\pi = 0$. Now hold a neighborhood of the loop fixed by pinching and stretch the rest of the sphere. The curvature at every point may change. What about the integral? Since the holonomy of the loop does not change, the integral remains 4π.

On any surface which can be continuously deformed into a sphere, the integral of the curvature over the whole surface is 4π. The integral of curvature is not just a cloth invariant, it is a rubber invariant! In general the integral of curvature of a surface is 2π times the Euler number of the surface. This is called the Gauss-Bonnet Theorem. The study of cloth invariant properties of an object is roughly speaking geometry. The study of rubber-invariant properties is called topology.

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