Math 353 | Spring 2010 | HW 5 Solutions

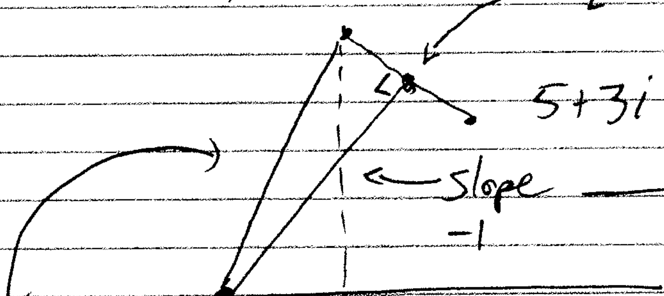
7.4, 7.5, 7.6, 7.11, 7.12, 7.32, 7.34, 7.36,

~~7.37~~, 7.38, 7.39, 7.52, 7.53, 7.54, 7.59

7.4 $P = 4+4i$ $Q = 5+3i$ Find the endpts of the Poincare line thru P & Q.

Sol'n:

$(4,4) = 4+4i$ $\quad \frac{9}{2} + \frac{7}{2}i$

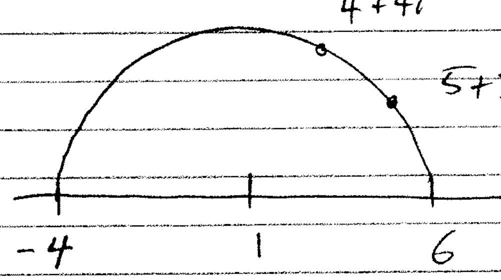


eqn: $y - \frac{7}{2} = -1\left(x - \frac{9}{2}\right)$

$y = 0$ so $x = 1$

so $r = \sqrt{4^2 + 3^2} = 5$

So

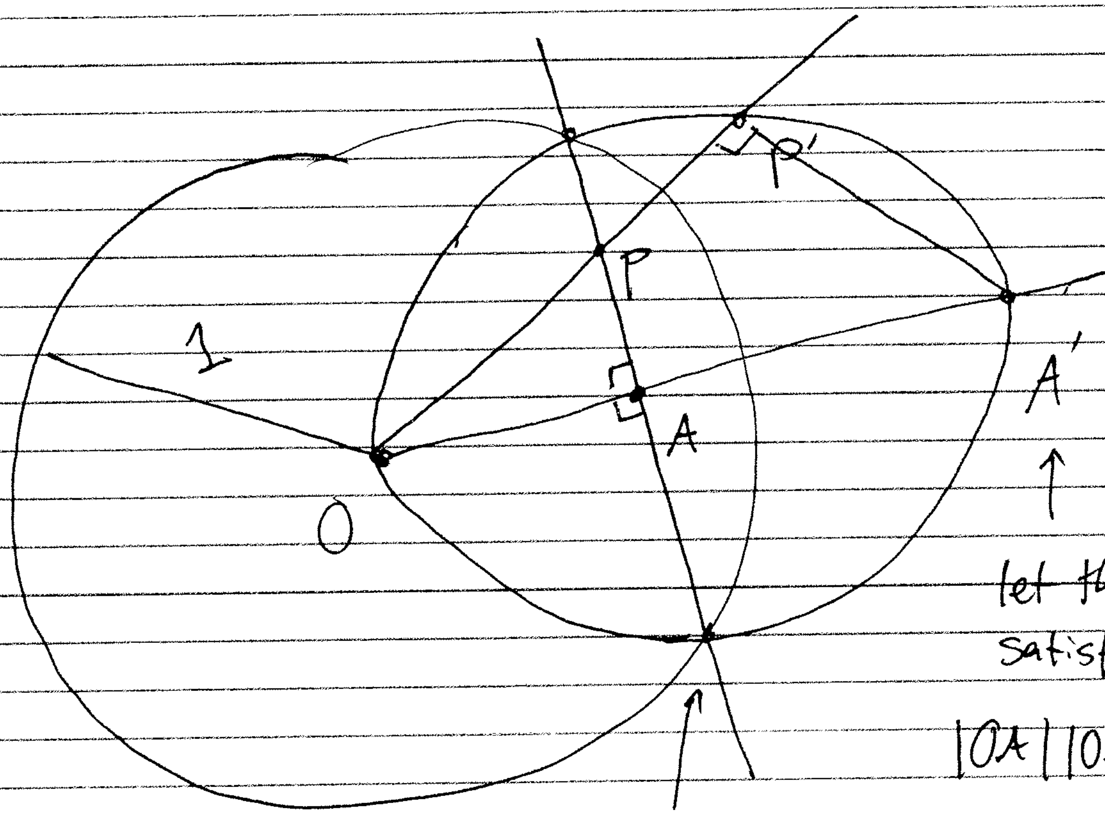


$x = -4$
$x = 6$

7.5 is similar.

Prove the other cases of Lemma 7.4.1

7.6 I'll do one case:



let this satisfy $|OA||OA'| = 1^2$

you don't actually know that this is true, but it turns out to be true

$\angle OP'A' = \pi/2$ by Star trek.

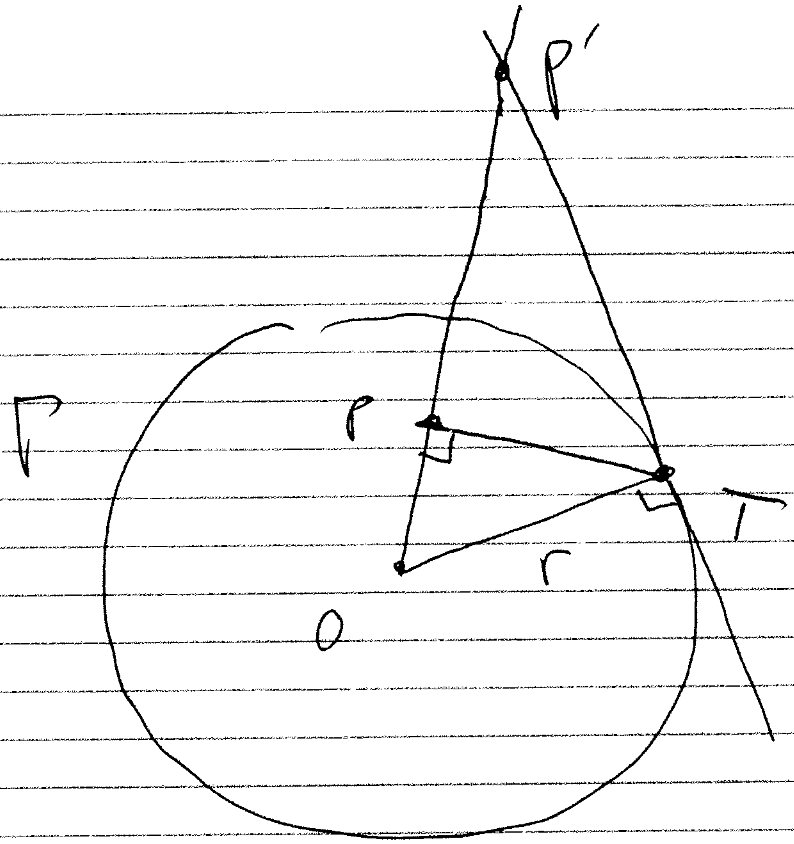
So $\triangle OAP \sim \triangle OP'A'$.

So $\frac{|OA|}{|OP|} = \frac{|OP'|}{|OA'|} \implies |OA||OA'| = |OP||OP'|$

$|OA||OA'| = 1$

$\therefore$ P & P' are the images of each other under inversion. $\square$

7.11

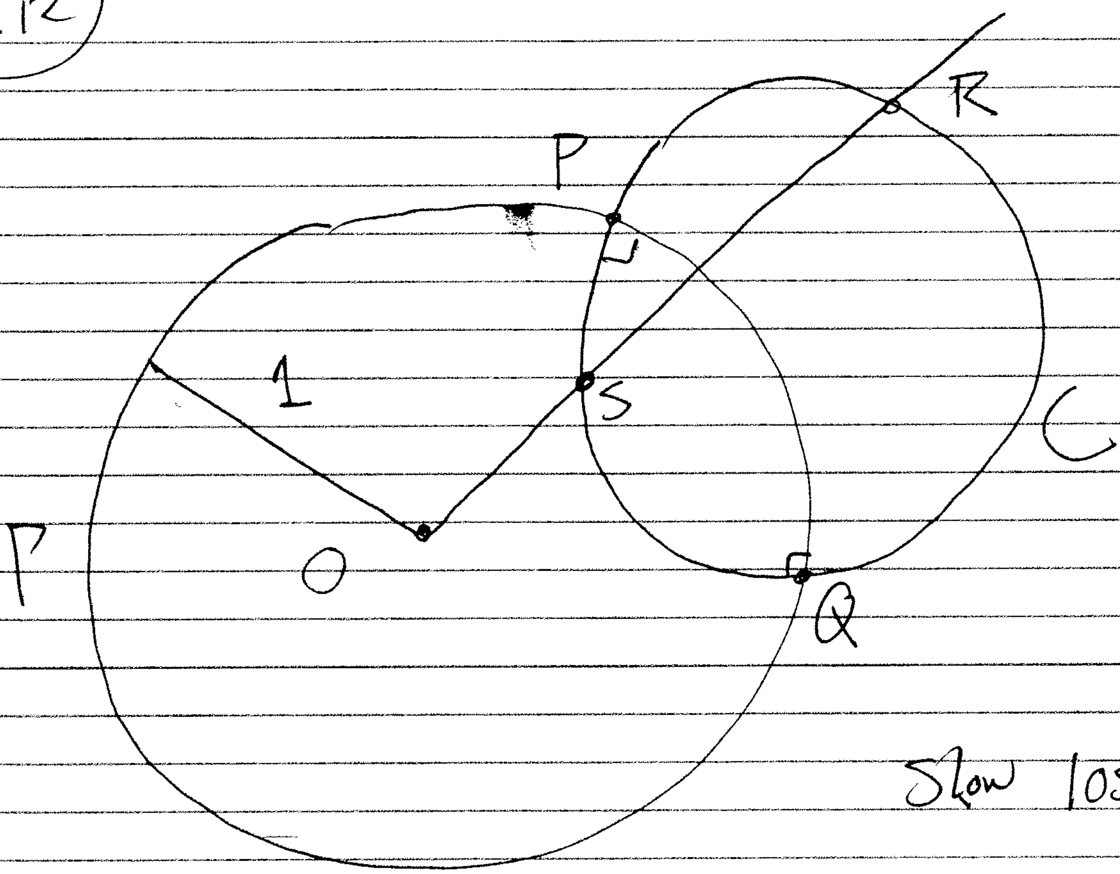


Show P & P' are images of each other under inversion.

Sol'n: $\Delta OPT \sim \Delta OTP'$. This implies

$$|OP||OP'| = |OT|^2 = r^2. \quad \blacksquare$$

7.12



Show $|OS||OR| = 1$.

Sol'n: Since i_Γ preserves Γ and preserves angles, we know $i_\Gamma(C) = C$. So R & S are the images of each other under i_Γ. so

$|OS||OR| = 1^2$. $\square$

7.32 This is an exercise in matrix multiplication

$$\gamma_1 = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad \gamma_2 = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

$$T_{\gamma_1} z = \frac{az+b}{cz+d} \qquad T_{\gamma_2} z = \frac{\alpha z+\beta}{\gamma z+\delta}$$

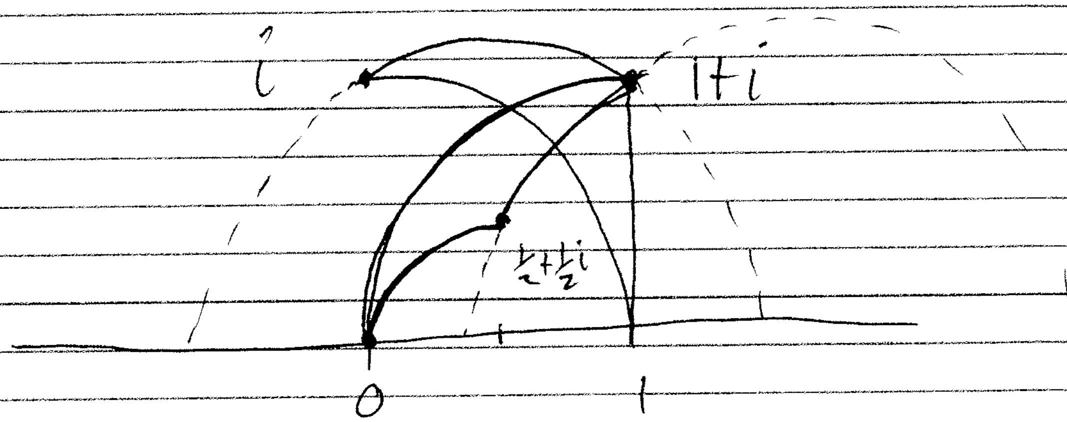
$$\gamma_1\gamma_2 = \begin{pmatrix} a\alpha + b\gamma & a\beta + b\delta \\ c\alpha + d\gamma & c\beta + d\delta \end{pmatrix}$$

$$T_{\gamma_1\gamma_2} z = \frac{(a\alpha + b\gamma)z + (a\beta + b\delta)}{(c\alpha + d\gamma)z + (c\beta + d\delta)} \qquad (*)$$

$$T_{\gamma_1}\left(T_{\gamma_2} z\right) = \frac{a\left(\dfrac{\alpha z + \beta}{\cancel{\alpha}\gamma z + \delta}\right) + b}{c\left(\dfrac{\alpha z + \beta}{\gamma z + \delta}\right) + d} \qquad (**)$$

Show $(*) = (**)$. ∎

7-34 Draw the triangle with vertices i, $1+i$, 1.



$$T$$

$$\begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix} \rightsquigarrow z \mapsto \frac{z-1}{z}$$

is an isometry b/c det = 1 and entries are real #'s.

$T(i) = 1+i$

$T(1) = 0$

$T(1+i) = \frac{1}{2} + \frac{1}{2} i$

7.36 $\quad P = \dfrac{8+i}{13} \quad Q = \dfrac{13+i}{20}$

$$\gamma = \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}.$$

$\det \gamma = 1$ so it is an isometry.

$$\gamma P = \frac{2\left(\frac{8+i}{13}\right) - 1}{-3\left(\frac{8+i}{13}\right) + 2} = \frac{\left(\frac{3}{13} + \frac{2i}{13}\right)}{\left(\frac{2}{13} - \frac{3}{13}i\right)} \cdot \frac{\left(\frac{2}{13} + \frac{3}{13}i\right)}{\left(\frac{2}{13} + \frac{3}{13}i\right)}$$

$$= \frac{13i}{4+9} = i$$

$$\gamma Q = \frac{2\left(\frac{13+i}{20}\right) - 1}{-3\left(\frac{13+i}{20}\right) + 2} = \frac{(6+2i)}{(1-3i)} \cdot \frac{(1+3i)}{(1+3i)}$$

$$= \frac{20i}{1+9} = 2i$$

so $\boxed{\begin{matrix} |\gamma P \, \gamma Q| = \ln(2/1) \\ \shortparallel \qquad\qquad \shortparallel \\ |PQ| \qquad\quad \ln 2 \end{matrix}}$

7.38 T a linear frac such that

$T(1) = 1$, $T(0) = 0$, $T(\infty) = \infty$.

Then T = identity.

Sol'n:

$$Tz = \frac{az+b}{cz+d}.$$

$$T(0) = b/d = 0 \quad \text{so} \quad b = 0$$

So $Tz = \dfrac{az}{cz+d}$. But $T(\infty) = \dfrac{a\infty}{c\infty+d} = \infty$

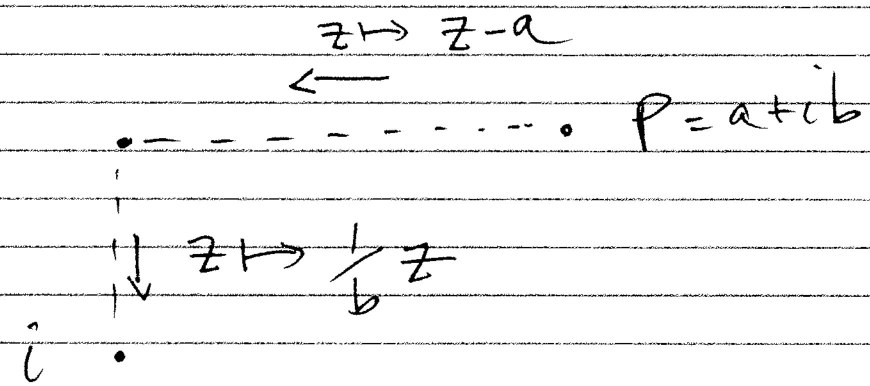
so $c = 0$. So $T(z) = \dfrac{az}{d}$. But

$$T(1) = a/d = 1 \quad \text{so} \quad a = d.$$

So $Tz = \frac{a}{a}z = z$. Identity. $\square$

7.39 $z \mapsto 2z$ is an isometry because

$$\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$$ has determinant $2 > 0$.



So $T(z) = \underbrace{\frac{1}{b}(z-a)}$ takes $a+ib$ to i. ∎

$$\begin{pmatrix} 1/b & -a/b \\ 0 & 1 \end{pmatrix}$$

7.52 $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2\mathbb{R}$ and $c=0$.

When is γ parabolic?

Sol'n: γ is parabolic if and only if it fixes exactly one point on $\mathbb{R} \cup \{\infty\}$.

If $c=0$ then $z \mapsto \dfrac{az+b}{d}$ fixes ∞.

So we have to find when this fixes no other real #:

$$z = \frac{az+b}{d} \longrightarrow z = \frac{b}{d-a}$$

$\dfrac{b}{d-a} \in \mathbb{R}$ unless $d=a$.

So $\begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \rightsquigarrow z \mapsto z + b/a$ fixes only ∞.

So $d=a$ for γ to be parabolic. $\boxtimes$

(7.53) $\gamma \in SL_2\mathbb{R}$ and $\gamma\infty = \infty$. Then γ is a translation. $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

pf: $\gamma\infty = \infty$ means $c = 0$.

So $\gamma = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \rightsquigarrow z \mapsto \dfrac{az+b}{d}$

Since $\gamma \in SL_2\mathbb{R}$, $ad = 1$, so $a = \frac{1}{d}$.

If $a = d$ then γ is a parabolic translatio (previous exercise).

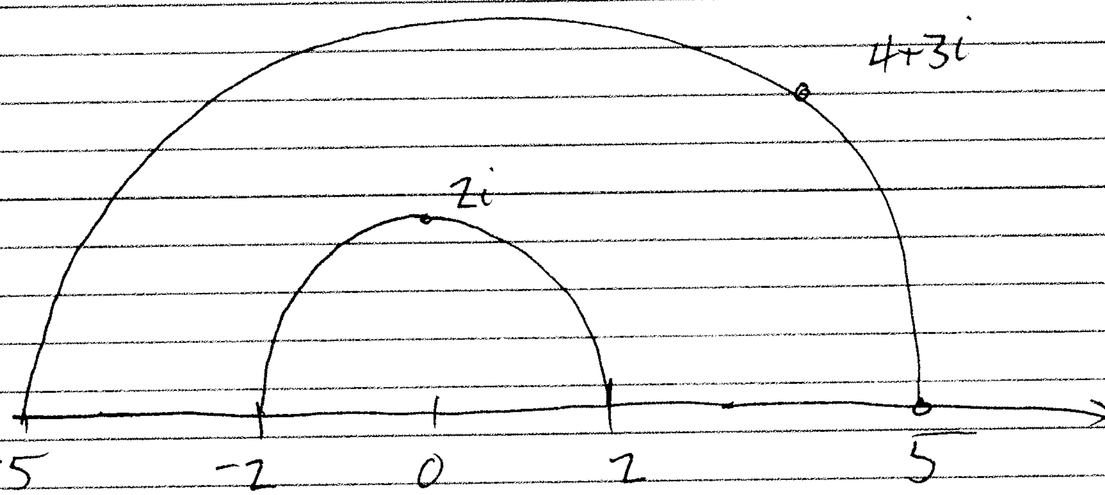
If $a \neq d$ then γ fixed $\dfrac{b}{d-a} \in \mathbb{R}$.

So γ fixes two points ∞ & $\dfrac{b}{d-a} \in \mathbb{R}$.

So γ is a hyperbolic translation. $\boxed{/\!/}$

7.54 τ_a is parabolic.

7.59 Find $\gamma \in GL_2\mathbb{R}$ (isometry) sending $2i \to 3i+4$ and 2 to -5. Does this have fixed points in $\mathcal{H}$?



take -2 to 5 also, b/c isometries must take lines to lines.

$2i \to 3i+4$

$2 \to -5$

$-2 \to 5$

so solve for w

$$(z, 2i; 2, -2) = (w, 3i+4, -5, 5)$$

$$\left.\frac{z-2}{z+2}\right| \frac{2i-2}{2i+2} \qquad = \qquad \left.\frac{w+5}{w-5}\right| \frac{3i+4+5}{3i+4-5}$$

$$\frac{z-2}{z+2} = \frac{w+5}{w-5} \cdot \frac{(9+3i)}{(-1+3i)} \cdot \frac{(2+2i)}{(-2+2i)}$$

$$\frac{z-2}{z+2} = \frac{w+5}{w-5} \cdot \left(\frac{12+24i}{-4-8i}\right) = -3\left(\frac{w+5}{w-5}\right)$$

$$(w-5)(z-2) = (-3w-15)(z+2)$$

$$w(z-2) - 5(z-2) = -3(z+2)w - 15(z+2)$$

$$w\left[z-2+3(z+2)\right] = -15(z+2) + 5(z-2)$$

$$w = \frac{-10z-40}{4z+4} = \boxed{\frac{-5z-20}{2z+2}}$$

fixed points:

$$z = \frac{-5z-20}{2z+2} \longrightarrow 2z^2+7z+20=0$$

$$\downarrow$$

$$z = \frac{-7 \pm \sqrt{49-160}}{4} = \boxed{\frac{-7 \pm \sqrt{111}\, i}{4}}$$