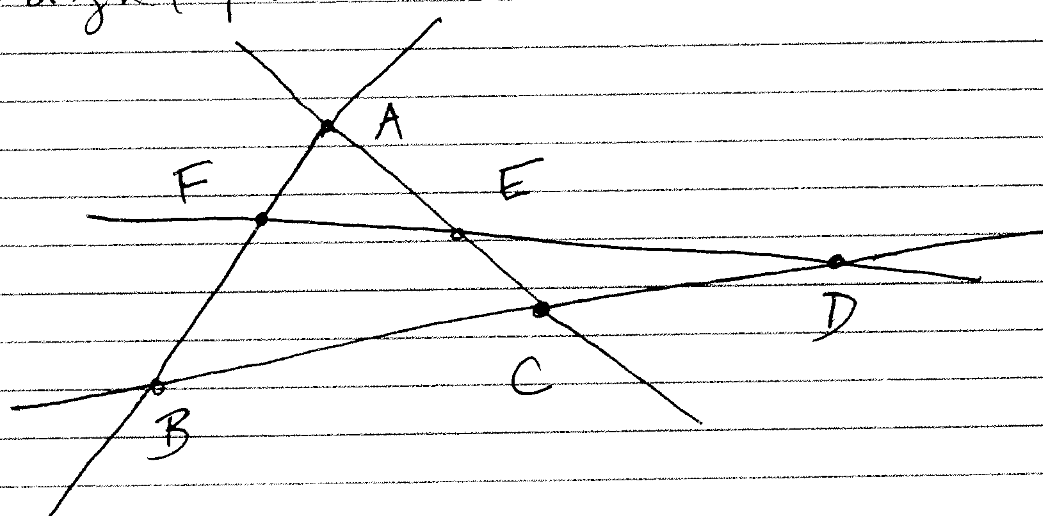
Math 383 | Spring 2010 | HW 4 Solutions

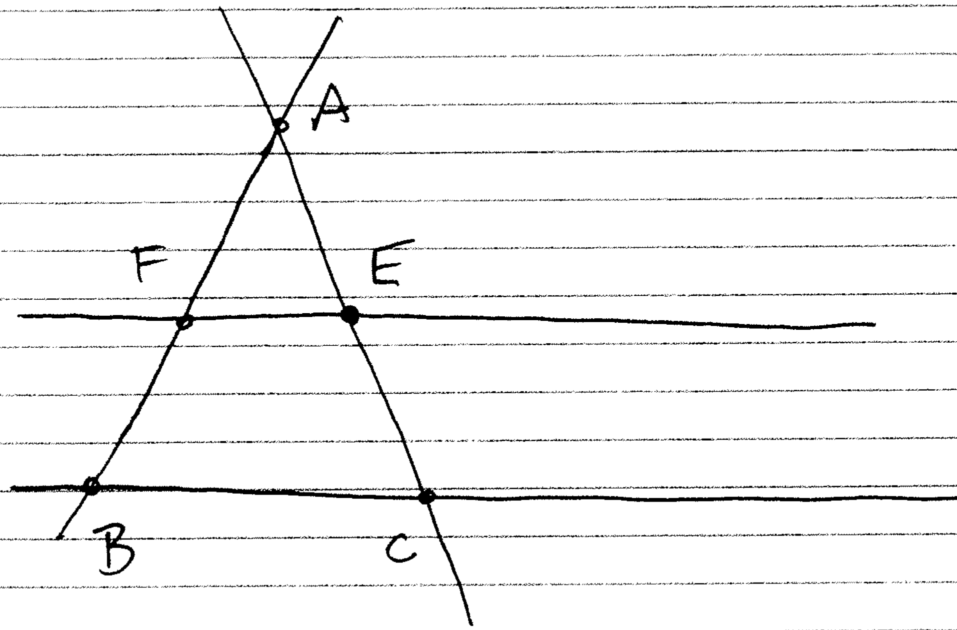
1.121 – 1.124, 3.2 – 3.4, 7.1 – 7.3

1.21 The analogue of Menelaus' theorem when D is at ∞.

This is the original pic:



As is moved further along the line BC, FE becomes closer to being parallel to BC:



The ratio product $\dfrac{|AF|}{|FB|} \cdot \dfrac{|BD|}{|CD|} \cdot \dfrac{|CE|}{|EA|} \longrightarrow -\dfrac{|AF|}{|FB|}\dfrac{|CE|}{|EA|}$

because $\dfrac{|BD|}{|CD|} \to -1$. So

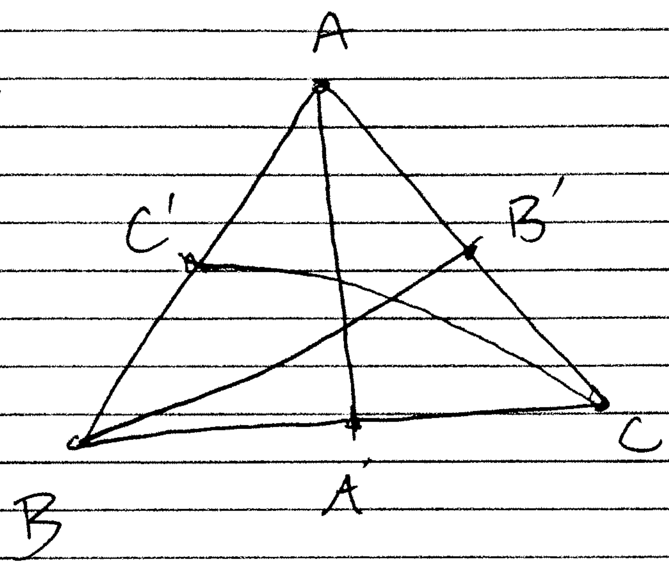
$$\boxed{\frac{|AF|}{|FB|}\frac{|CE|}{|EA|} = 1 \quad \text{iff} \quad FE \parallel BC}$$

should be the analogous statement.

This should remind you of similar triangles.

1.22 Use Ceva to show the medians of a triangle intersect at a common point.

pf:



Because A', B', C' are midpoints, we know

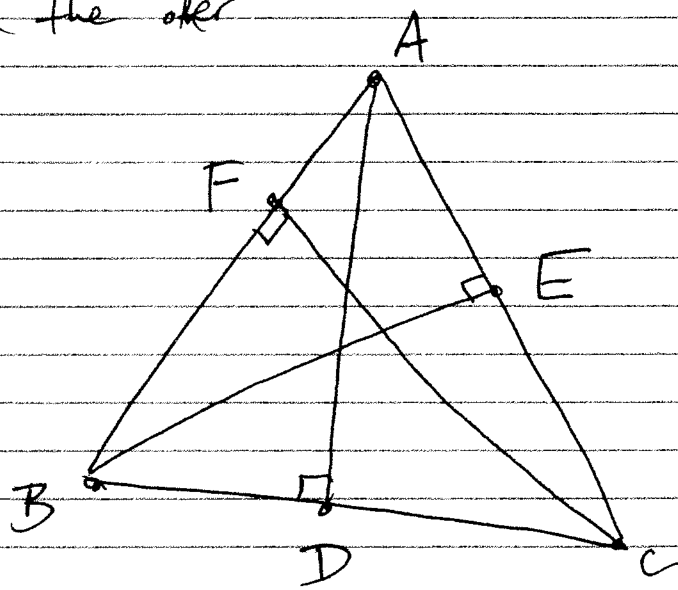
$$\frac{|AC'|}{|C'B|} = \frac{|BA'|}{|A'C|} = \frac{|CB'|}{|B'A|} = 1$$

So the product of these equals 1. So by Ceva's

1.23 Use Ceva to show a triangles altitudes are concurrent.

pf: (this pic assumes the triangle has no obtuse or right angles. The proof goes the same way in the other cases)

$\boxed{\text{We want to show that } AD, BE, CF \text{ are concurrent.}}$



Notice that $\triangle AFC \sim \triangle ACB$ and

$\triangle BDA \sim \triangle BFC$ and

$\triangle CDA \sim \triangle CEB$.

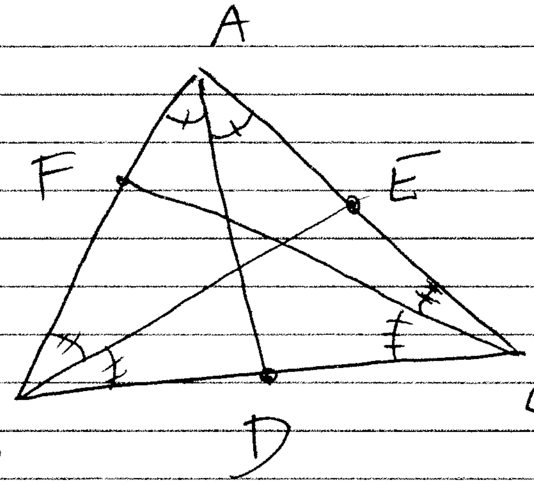
So $\dfrac{|AF|}{|FC|} = \dfrac{|AE|}{|BE|}$, $\dfrac{|FB|}{|CF|} = \dfrac{|BD|}{|AD|}$, $\dfrac{|DC|}{|AD|} = \dfrac{|CE|}{|BE|}$

So $\underbrace{\dfrac{|AF|}{\cancel{|FC|}} \cdot \dfrac{\cancel{|BE|}}{|AE|}}_{1} \cdot \underbrace{\dfrac{\cancel{|CF|}}{|FB|} \cdot \dfrac{|BD|}{\cancel{|AD|}}}_{1} \cdot \underbrace{\dfrac{|CE|}{\cancel{|BE|}} \cdot \dfrac{\cancel{|AD|}}{|DC|}}_{1} = 1$.

$\therefore$ by Ceva, we get the concurrency we want. ∎

1.24 Use Ceva to show the angle bisectors intersect at a common point.

pf:



We use the angle bisector theorem (Exercise 1.43).

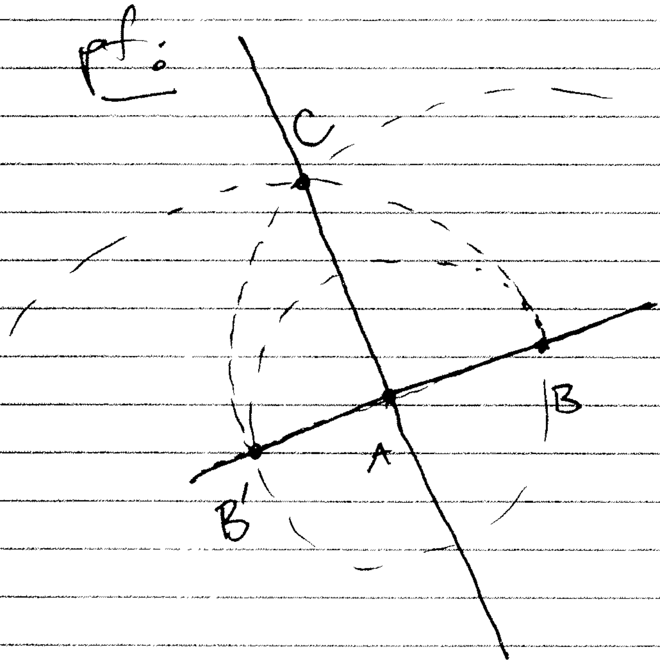
This theorem tells us the following:

$$\frac{|AF|}{|AC|} = \frac{|FB|}{|BC|}, \quad \frac{|BD|}{|AB|} = \frac{|DC|}{|AC|}, \quad \frac{|CE|}{|BC|} = \frac{|AE|}{|AB|}.$$

So
$$\underbrace{\underbrace{\frac{|AF|}{|FB|}\cdot\frac{|BC|}{|AC|}}_{1}\cdot\underbrace{\frac{|BD|}{|DC|}\cdot\frac{|AC|}{|AB|}}_{1}\cdot\underbrace{\frac{|CE|}{|EA|}\cdot\frac{|AB|}{|BC|}}_{1}}_{=1} = \frac{|AF|}{|FB|}\cdot\frac{|BD|}{|DC|}\cdot\frac{|CE|}{|EA|}$$

So the interior segments are concurrent by Ceva's Theorem. ∎

3.2 Given a constructed segment AB, construct the perpendicular at A.



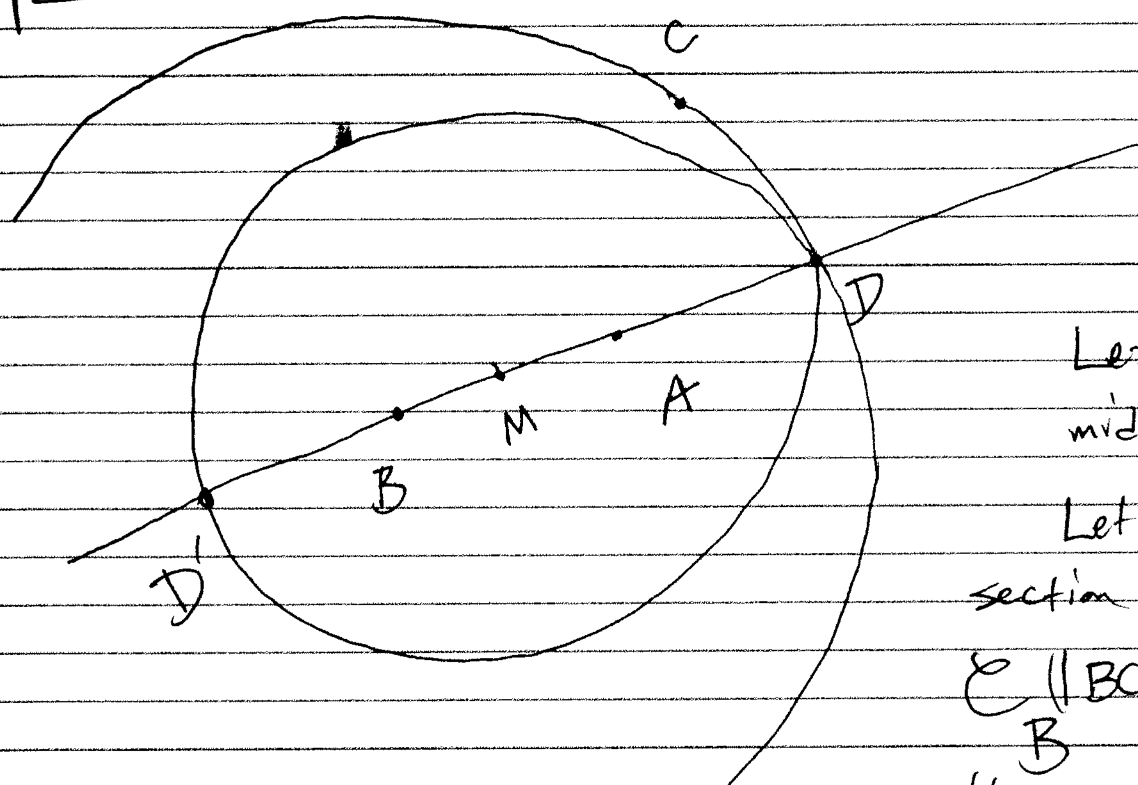
pf: Given AB, draw the circle $\mathcal{C}_A(|AB|)$ and let it intersect the line AB at B'.

Now let $\mathcal{C}_{B'}(|BB'|)$ and $\mathcal{C}_B(|BB'|)$ intersect at C.

Draw CA. This is a perpendic[ular] because $\triangle B'CA \cong \triangle BAC$. ∎

(3.3) We can draw $\mathcal{C}_A(|BC|)$ if A, B, C are constructed points with $|AB| \le 2|BC|$.

pf:



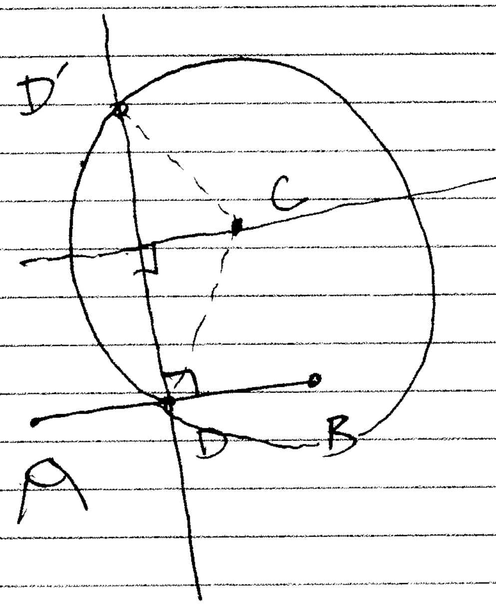
Let M be the midpoint of AB. Let D be the intersection point of $\mathcal{C}_B(|BC|)$ that lies on the same side of B as A does. Let D' be the intersection of the circle $\mathcal{C}_M(|MD|)$ that is not D.

Then $\mathcal{C}_A(|AD'|)$ is the circle we want. $\blacksquare$

3.4 Given AB and a point C, we can construct the line through C parallel to AB.

pf: Let D be the intersection of the line AB and the perpendicular bisector of the segment AB.

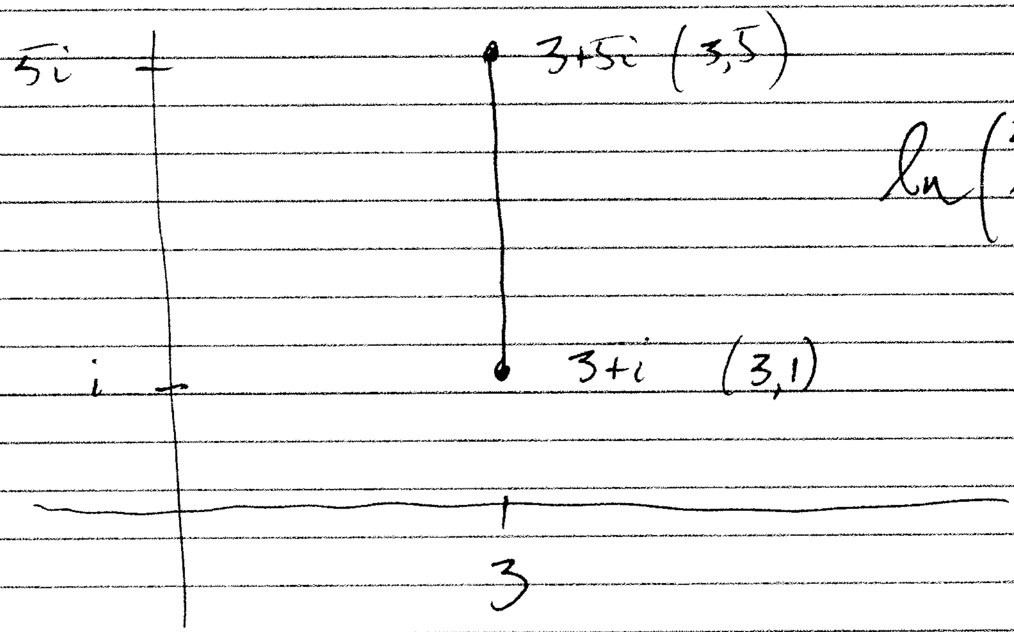
Draw $\mathcal{C}_C(|CD|)$, and let D' be the other point of intersection on the perp. bisector.



Now draw the perp. bisector of DD'.

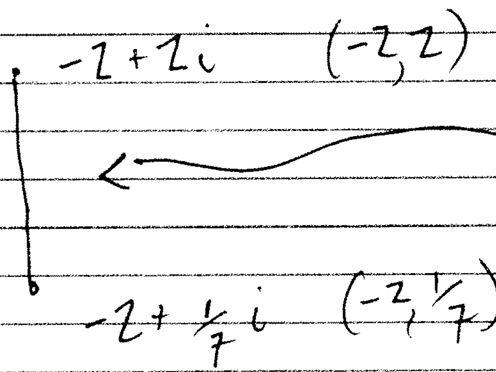
This passes through C because DD' is a chord of ther circle, and is parallel to AB because it is perpendicular to the perpendicular bisector of AB. ∎

7.1 What is the $\mathbb{H}$-distance between $3+i$ and $3+5i$?



$$\ln\left(\frac{5}{1}\right) = \boxed{\ln 5}$$

7.2 What about $-2+2i$ and $-2+\frac{i}{7}$?



$\boxed{\ln(14)}$.

7.3 Prove dilation $\delta_\lambda(x,y) = (\lambda x, \lambda y)$ is an isometry.

pf: $u = \lambda x$, $v = \lambda y$.

$du = \lambda\, dx \quad dv = \lambda\, dy$

$$\frac{du^2 + dv^2}{v^2} = \frac{\lambda^2 dx^2 + \lambda^2 dy^2}{(\lambda y)^2}$$

$$\shortparallel$$

$$\frac{dx^2 + dy^2}{y^2}$$

✓ Isometry of $\mathcal{H}$.