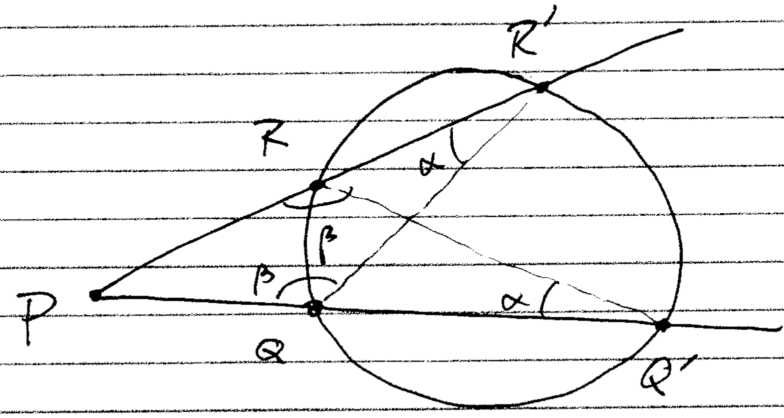
M 383 | Spring 2010 | HW 3 Solutions

1.51, 1.52, 1.55, 1.56, 1.67, 1.69, 1.72–1.77

1.81 (b) & (c), 1.92

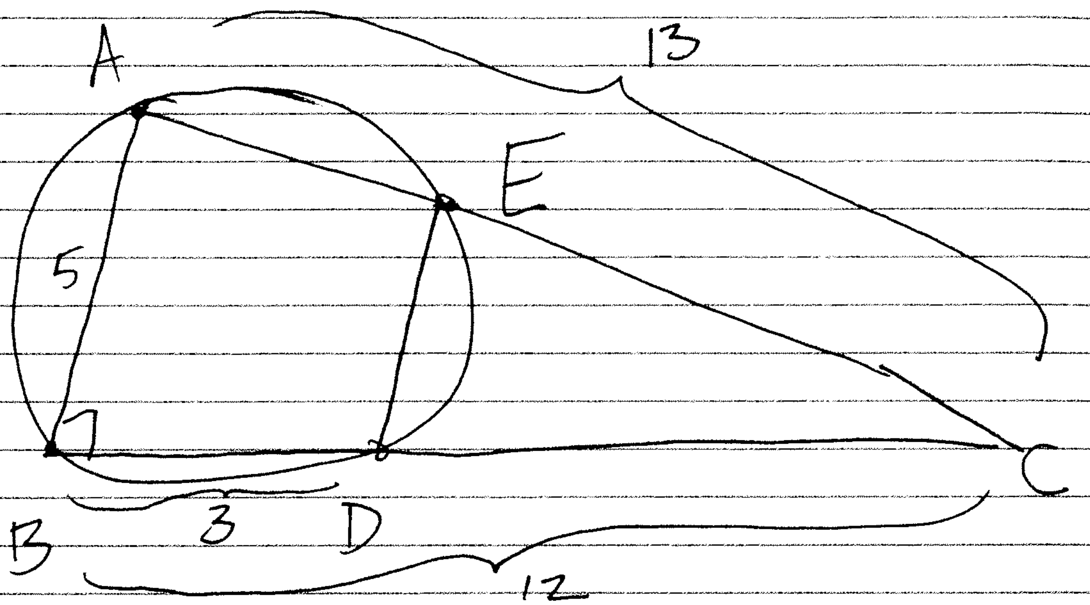
1.51 Prove the Power of a point Theorem when P lies outside the circle



pf: $\measuredangle PR'Q = \measuredangle PQ'R$ by star trek lemma. So $\measuredangle PQR' = \measuredangle PRQ'$ b/c $\Delta PQR'$ & $\Delta PRQ'$ share two equal angles. Therefore $\Delta PQR' \sim \Delta PRQ'$.

So $\dfrac{|PQ|}{|PR|} = \dfrac{|PR'|}{|PQ'|}$. ∎

1.52



If $|AB| = 5$, $|BC| = 12$, $|AC| = 13$, $|BD| = 3$

then what is $|DE|$?

<u>Solution:</u> $|BD| = 9$ so $\Pi(C) = 9 \cdot 12 = 108$. so

$|CE| \cdot 13 = 108$, and thus $|CE| = \frac{108}{13}$ and $|AE| = 13 - \frac{108}{13}$

Since $12^2 + 5^2 = 13^2$, we have $\measuredangle ABC = 90°$ by Pythagoras

By Startrek, AD is a diameter, because the arc AD is subtended by a $90°$ angle. So $\measuredangle AED$ is also $90°$.

$|AD| = \sqrt{5^2 + 3^2} = \sqrt{34}$. So

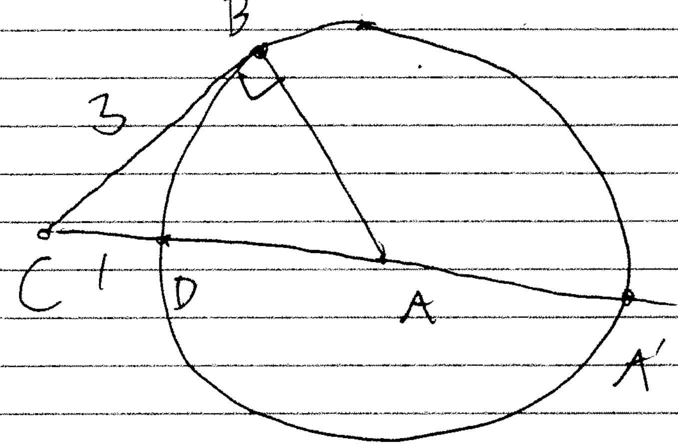
$$|ED|^2 = |AD|^2 - |AE|^2 = |AD|^2 - \left(13 - \frac{108}{13}\right)^2$$

$$= 34 - 169 + 216 - \frac{108^2}{13^2} = 81 - \frac{108^2}{13^2} = \frac{45^2}{13^2}$$

Answer

$\boxed{|DE| = \frac{45}{13}}$

1.55



What is radius of the circle centered at A?

Sol'n: By Power of Point, $|CD| \cdot |CA'| = 3^2 = 9$

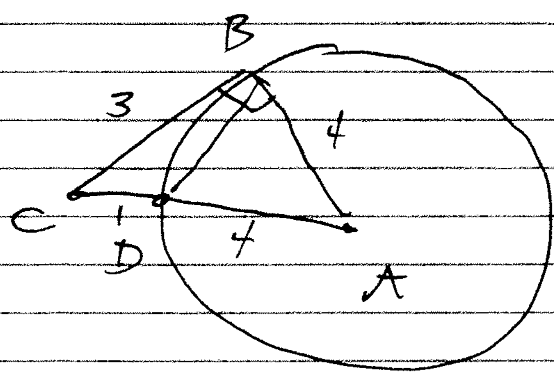
So $|CA| + |AA'| = 9$, but $|CA| = |CD| + |AA'|$ so

$\underline{1} + 2|AA'| = 9$, so $|AA'| = 4$. ← radius.

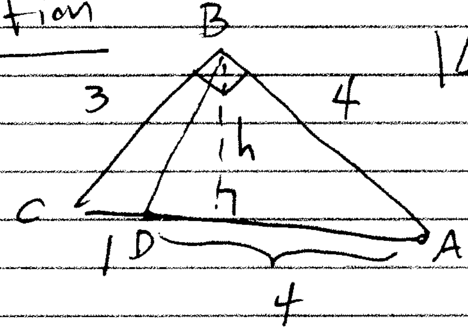
So $|AB| = 4$. By Pythagoras, $3^2 + 4^2 = |CA|^2$.

So $|CA| = 4$. ☒

1.56 Same as 1.55: What is Area(ΔABD)?



Solution



$|\Delta ABD| + |\Delta CDB| = 6$

But $|\Delta ABD| = \frac{1}{2} \cdot 4 \cdot h = 2h$

$|\Delta CDB| = \frac{1}{2} \cdot 1 \cdot h = h/2$

So $2h + \frac{h}{2} = 6 \longrightarrow 5h = 12$

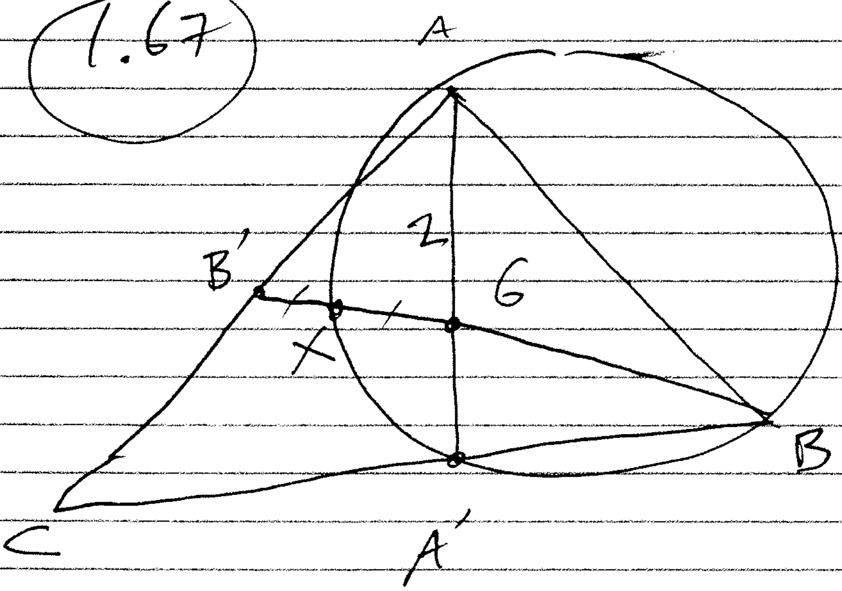
$\downarrow$

$h = 12/5$

$$|ABD| = \frac{1}{2} \cdot 4 \cdot \frac{12}{5} = \boxed{24/5}$$

∎

1.67



A', B' are midpts.
Suppose $|AG| = 2$ and that X is the midpt of BG.

What is $|BG|$?

Solution: G is the centroid of ΔABC. So

$$|A'G| = \frac{1}{2}|AG| = 1. \text{ So } \Pi(G) = 1 \cdot 2 = 2.$$

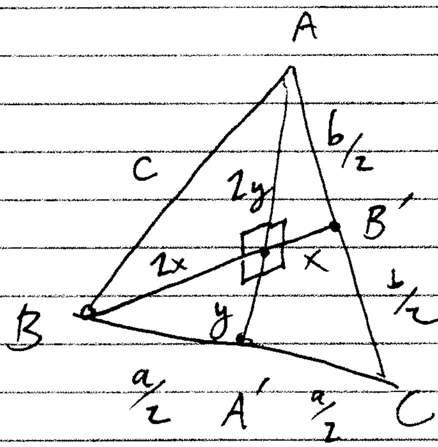
So $|BG| \cdot x = 2$. Also $2 \cdot x = |B'G| = \frac{1}{2}|BG|$.

So $|BG| = \frac{2}{x}$ and $|BG| = 4x$. So

$$\frac{2}{x} = 4x, \text{ or } x = \frac{1}{\sqrt{2}}.$$

So $|BG| = 2\sqrt{2}$. ■

1.69 Suppose median AA' & BB' intersect at right angles, in ΔABC. If $a=3$, $b=4$, then what is c?



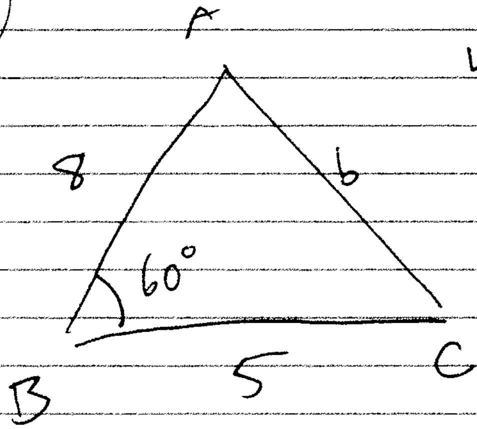
Sol'n: The medians are divided by the centroid as in the figure.

So
$$\left.\begin{aligned} c^2 &= 4x^2 + 4y^2 \\ \left(\frac{b}{2}\right)^2 &= x^2 + 4y^2 \\ \left(\frac{a}{2}\right)^2 &= 4x^2 + y^2 \end{aligned}\right\} \text{ by Pythagoras}$$

So $\left(\frac{b}{2}\right)^2 + \left(\frac{a}{2}\right)^2 = 5x^2 + 5y^2 = 5 \cdot \frac{1}{4} \cdot (4x^2 + 4y^2) = \frac{5}{4} c^2$

So $\frac{5}{4} c^2 = \left(\frac{4}{2}\right)^2 + \left(\frac{3}{2}\right)^2 = 4 + \frac{9}{4} = \frac{25}{4}$. So $\boxed{c = 5}$. ■

1.72



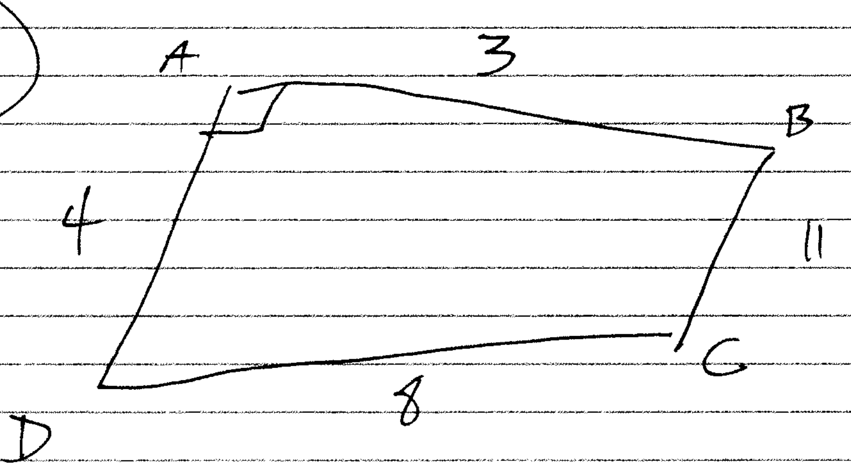
What is b?

Solution: Law of cosines.

$$b^2 = 5^2 + 8^2 - 2 \cdot 5 \cdot 8 \cos 60^\circ = 25 + 64 - 80 \cdot \tfrac{1}{2} = 49$$

so $\boxed{b = 7}$

1.73



What is the area of quadrilateral $ABCD$?

Solution: ΔABD has area $\frac{1}{2} \cdot 3 \cdot 4 = 6$ by Pythagoras.

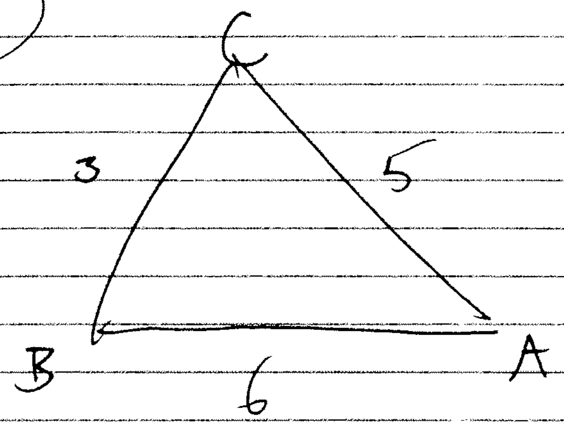
So the semiperimeter of ΔDBC is

$s = \frac{1}{2}(5 + 8 + 11) = 12$. So $|\Delta DBC|$ is by Heron

$$\sqrt{12(12-5)(12-8)(12-11)} = \sqrt{12 \cdot 28} = \boxed{4\sqrt{21}}$$

so $\boxed{6 + 4\sqrt{21}}$ Answer

1.74



$|\Delta ABC| = ?$

Solution: Heron's formula.

$s = \frac{1}{2} \cdot 14 = 7.$

So $|\Delta ABC| = \sqrt{7(7-3)(7-5)(7-6)} = \boxed{2\sqrt{14}}$ ☒

1.75 What is the area of the incircle of ΔABC if $a=5$, $b=6$, $c=7$?

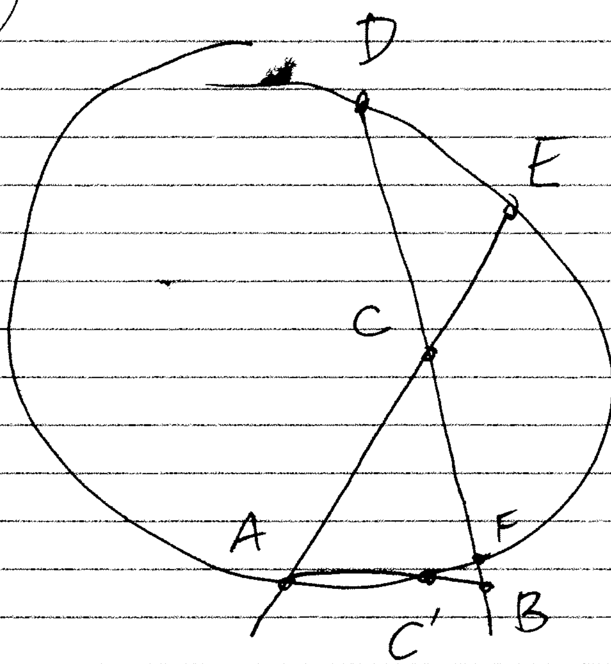
Solution: inradius $r = \frac{|\Delta ABC|}{s}$. $s = \frac{1}{2} \cdot 18 = 9$

$|\Delta ABC| = \sqrt{9 \cdot (9-5)(9-6)(9-7)} = 6\sqrt{6}.$

So $r = \frac{6\sqrt{6}}{9} = \frac{2\sqrt{6}}{3}$. So $\text{Area(Incircle)} = \pi\left(\frac{2\sqrt{6}}{3}\right)^2$

$$= \frac{4 \cdot 6}{9} \cdot \pi = \frac{8}{3}\pi$$

1.76



$|AC'| = |C'B| = |CE| = 2$,

$|CD| = 3$, $|BF| = 1$.

What is area(ΔABC)?

Sol'n: $\Pi(B)$

$$= |BC'| \cdot \left(|BC'| + |AC'| \right) = 2 \cdot (2 + 2) = 8$$

So $|BF| \cdot \left(|BF| + |FC| + |CD| \right) = 8$. (Power of Point)

So $1 \cdot (1 + |FC| + 3) = 8$. So $|FC| = 4$.

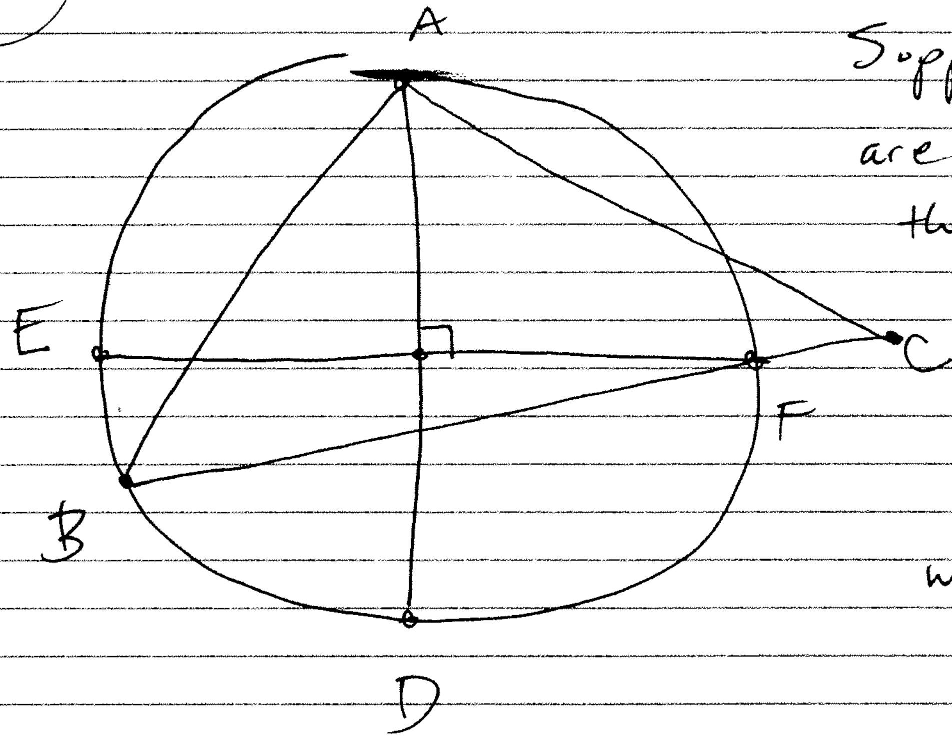
So $\Pi(C) = |CD| \cdot |CF| = 3 \cdot 4 = 12$.

Since $\Pi(C) = |CE| \cdot |CA| = 2 \cdot |CA|$, we have

$|CA| = 6$. So ΔABC has sides of length 6, 5, 4.

Heron's formula says Area$(\Delta ABC) = \boxed{\frac{15}{4}\sqrt{7}}$ ▲

1.77



Suppose AD & EF are $\perp$ and intersect the center of the circle.

If $|AB| = 2$ & $|BC| = 3$, then what is $|AC|$?

Sol'n : $\measuredangle ABC = 45°$ b/c it subtends a $90°$ arc.

So law of cosines:

$$|AC|^2 = 2^2 + 3^2 - 2 \cdot 2 \cdot 3 \cos 45°$$

$$= 13 - 12 \cdot \frac{\sqrt{2}}{2} = 13 - 6\sqrt{2}$$

So $|AC| = \sqrt{13 - 6\sqrt{2}}$ ∎

1.81

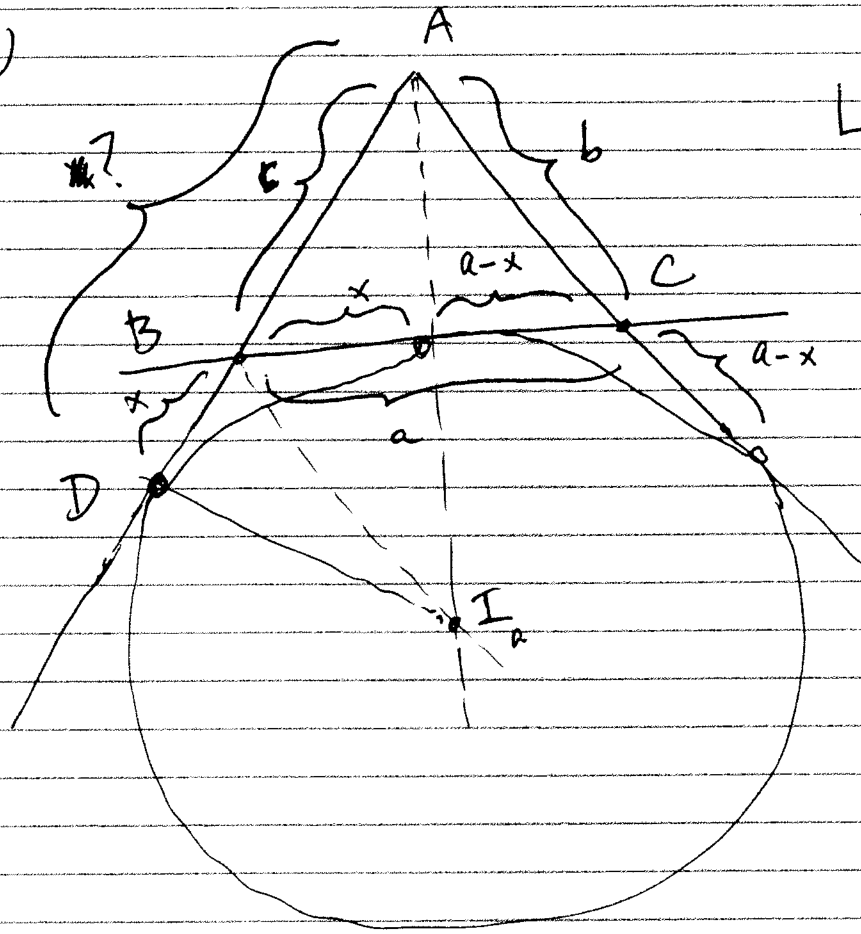
~~(a)~~ Show that the distance from A (in ΔABC) to the tangent on the side AB of

(b) the excircle centered at I_a is s.

(c) the excircle centered I_c is $s-b$.

Solution:

(b)



Looking at the figure, we have

$c+x = b+a-x$

So $x = \frac{1}{2}(a+b-c)$

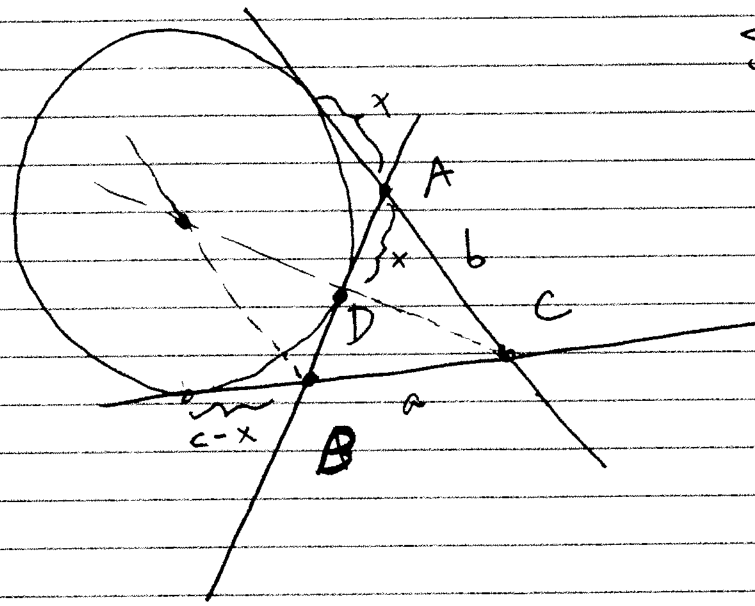
The whole distance from A to D is

$c+x$

$=$

$\frac{1}{2}(a+b+c) = s$ ∎

(c)



Similarly to (b), we have

$$b + x = a + c - x$$

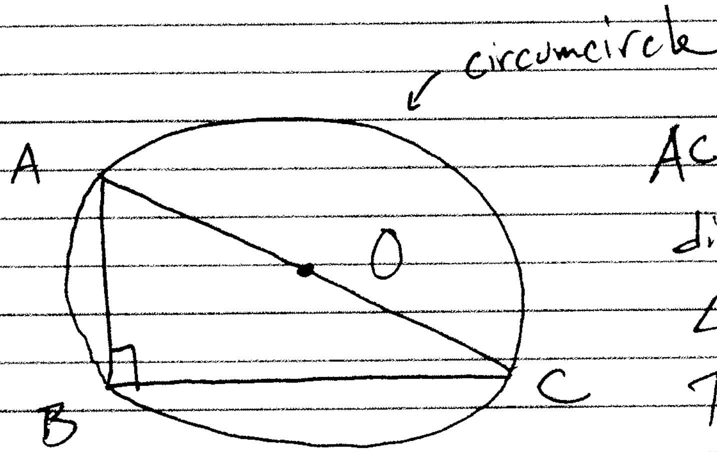
$$\Downarrow$$

$$x = \frac{1}{2}(a - b + c) = s - b.$$

$\square$

(1.92) Where is the circumcenter of a right triangle and why?

Solution:



AC must be a diameter b/c $\angle B = 90°$. Then the circumcenter has to lie on this side.

$\square$