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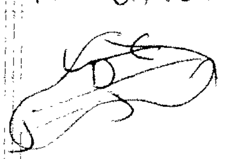
3/23

Greens Thm

Let C be a simple closed curve in $\mathbb{R}^2$, and D is the region bounded by C, assume C is positivly oriented. Then $(\vec{F}(x,y)) = P(x,y)\hat{i} + Q(x,y)\hat{j}$

$$\int_C \vec{F}\cdot d\vec{r} = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

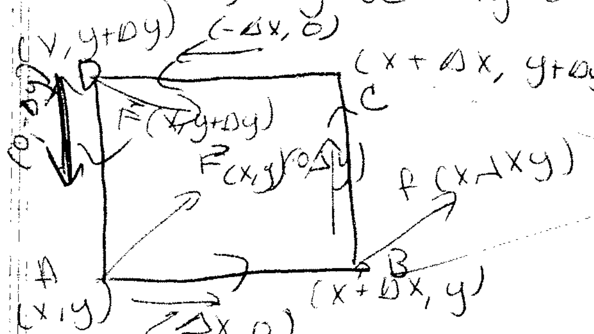
notation: $\oint_C$ = C is closed and positivly oriented

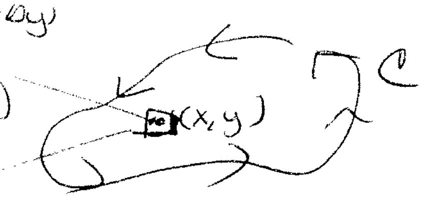


* Theorem is also true if C is a collection of curves

Idea of Why this is true

Let Δx, Δy be very small numbers





goal: to approximate $\int \vec{F}\cdot d\vec{r}$ around this tiny rectang

I) $\vec{F}(x,y)\cdot\langle\Delta x, 0\rangle = \langle P(x,y), Q(x,y)\rangle \cdot \langle\Delta x, 0\rangle = P(x,y)\Delta x \approx$ the work done by $\vec{F}$ moving a particle from A to B

II) $\vec{F}(x+\Delta x, y)\cdot\langle 0, y\rangle = \langle P(x+\Delta x, y), Q(x+\Delta x, y)\rangle \cdot \langle 0, \Delta y\rangle =$
$Q(x+\Delta x, y)\Delta y \approx$ the work done moving from B to C

III) $\vec{F}(x,y)\cdot\langle 0, -\Delta y\rangle = -Q(x,y)\Delta y \approx$ work D to A

IV) $\vec{F}(x, y+\Delta y)\cdot\langle -\Delta x, 0\rangle = -P(x, y+\Delta y)\cdot\Delta x \approx$
work C to D

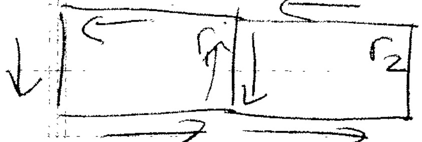
$\oint_{\text{tiny rectangle}} \vec{F}\cdot d\vec{r} \approx P(x,y)\Delta x + Q(x+\Delta x, y)\Delta y - P(x, y+\Delta y)\Delta x - Q(x,y)\Delta y =$

$$\frac{[Q(x+\Delta x, y) - Q(x,y)]\Delta y\Delta x}{\Delta x} - \frac{[P(x, y+\Delta y) - P(x,y)]\Delta x\Delta y}{\Delta y}$$

$$\left[\underbrace{\frac{Q(x+\Delta x,y)-Q(x,y)}{\Delta x}}_{\approx \frac{\partial Q}{\partial x}} - \underbrace{\frac{P(x,y+\Delta y)-P(x,y)}{\Delta y}}_{\approx \frac{\partial P}{\partial y}}\right] \underbrace{\Delta x \Delta y}_{=\text{area of rectangle}}$$

so $\oint \vec{F}\cdot dr \approx \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)$ area rectangle $= \iint_{rect} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$

So suppose you cut D into tiny rectangles, consider two of these side by side

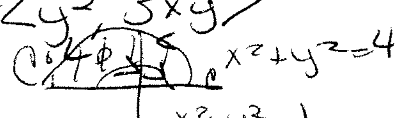


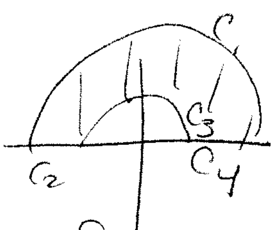
work cancles out

The quanity $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$ is called the <u>circulation density</u> because $\oint_C \vec{F}\cdot d\vec{r} =$ total circulation around C of F

ex) $\vec{F}(x,y) = \langle y^2, 3xy\rangle$

$\oint F\cdot dr$





as a line integral this is

$\int_{C_1} \vec{F}\cdot d\vec{r} + \int_{C_2} F\cdot dr + \int_{C_3} F\cdot dr + \int_{C_4} F\cdot dr$

$C_1 = \vec{r}(t) = \langle 2\cos(t), 2\sin(t)\rangle \quad 0 \le t \le \pi$

$C_2 : \vec{r}(t) = \langle t, 0\rangle \quad -2 \le t \le 1$

$\int_{C_1} \vec{F}\cdot d\vec{r} = \int_0^{\pi} \langle (2\sin(t))^2, 3\cos(t)\cdot 2\sin(t)\rangle \cdot \langle -2\sin(t), 2\cos(t)\rangle$

$= \int_0^{\pi} -8\sin^3(t) + 24\cos^2(t)\sin(t)\, dt \ldots$

AHHHH!!!!

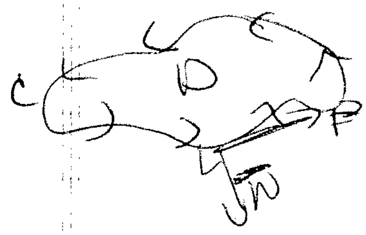
Greens: describe D: $0 \le \theta \le \pi \quad 1 \le r \le 2$

$\oint_C \vec{F}\cdot d\vec{r} = \iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = \iint_D 3y - 2y\, dA =$

$\iint_D y\, dA = \int_0^{\pi}\int_1^2 r\sin\theta \cdot r\,dr\,d\theta =$

$\int_0^{\pi}\int_1^2 r^2\sin\theta\, dr\, d\theta = \boxed{\frac{14}{3}}$

Another form of Greens Theorm



$\vec{F}(x,y) = P(x,y)\hat{i} + Q(x,y)\hat{j}$

$\vec{n}$ - is the unit <u>normal</u> vector at (x,y) on C, the component of $\vec{F}$ pushing on the direction of $\vec{n}$ is $\vec{F} \cdot \vec{n}$

$\oint_C \vec{F} \cdot \vec{n}\, ds$ = outward flux of $\vec{F}$ along C

It can be shown that $\oint_C \vec{F} \cdot \vec{n}\, ds =$

$\iint_D \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right) dA$ ← greens theorem of flux

↑

Divergence of $\vec{F}$

notation: $\operatorname{div}(\vec{F})$