St Patrick's Day! 3/17/2010

17.3 Fundamental Theorem of Line Integrals

Thm: $\int_C \nabla f \cdot d\vec{r} = f(B) - f(A)$

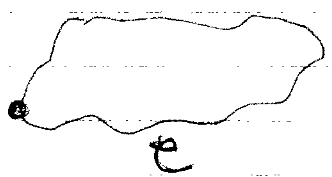
$\boxed{\text{Recall: } \vec{F} \text{ is conservative means } \vec{F} = \nabla f.}$

$\int_C \vec{F} \cdot d\vec{r}$ is called independent of path if

$\int_{C_1} \vec{F} \cdot d\vec{r} = \int_{C_2} \vec{F} \cdot d\vec{r}$ whenever C_1 and C_2 have the same endpoints

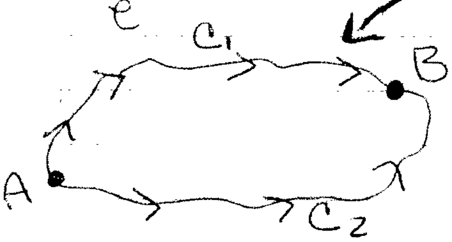
$\boxed{\text{Result: } \vec{F} \text{ conservative} \rightarrow \int_C \vec{F} \cdot d\vec{r} \text{ is independent of path}}$

Definition: A curve (in 2D or 3D) is called closed if it starts and ends at the same point.



suppose $\int_C \vec{F} \cdot d\vec{r}$ is independent of path.

$\int_C \vec{F} \cdot d\vec{r} = ?$



let B be any other point on C

because path independent

$\int_{C_1} \vec{F} \cdot d\vec{r} = \int_{C_2} \vec{F} \cdot d\vec{r}$

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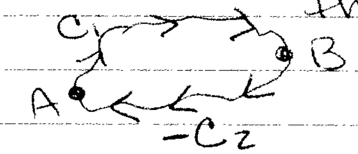
$$\int_{C_1} \vec{F}\cdot d\vec{r} = \int_{C_2} \vec{F}\cdot d\vec{r}$$

$$\Downarrow$$

$$\int_{C_1} \vec{F}\cdot d\vec{r} - \underbrace{\int_{C_2} \vec{F}\cdot d\vec{r}}_{\to \int_{-C_2} \vec{F}\cdot d\vec{r}} = 0$$

$\boxed{-C_2} \to C_2$ oriented the other wa[y]

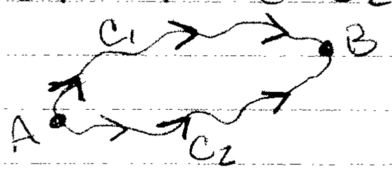


So (if $\int_C \vec{F}\cdot d\vec{r}$ is independent of path) then

$$\int_{C_1} \vec{F}\cdot d\vec{r} + \int_{-C_2} \vec{F}\cdot d\vec{r} = \int_{\mathcal{C}} \vec{F}\cdot d\vec{r} = 0$$

Now Suppose that $\int_C \vec{F}\cdot d\vec{r} = 0$ anytime C is a closed curve.

Let C_1 and C_2 be two paths from A to B:



Consider $C_1 \cup (-C_2) = \mathcal{C}$

$$\int_{\mathcal{C}} \vec{F}\cdot d\vec{r} = 0$$

$$0 = \int_{\mathcal{C}} \vec{F}\cdot d\vec{r} = \int_{C_1 \cup (-C_2)} \vec{F}\cdot d\vec{r} = \int_{C_1} \vec{F}\cdot d\vec{r} - \int_{C_2} \vec{F}\cdot d\vec{r}$$

$$\boxed{\int_{C_1} \vec{F}\cdot d\vec{r} = \int_{C_2} \vec{F}\cdot d\vec{r}}$$

$\int_C \vec{F}\cdot d\vec{r}$ ind. path

$\Updownarrow$

$\int_C \vec{F}\cdot d\vec{r} = 0$, $C \leftarrow$ closed

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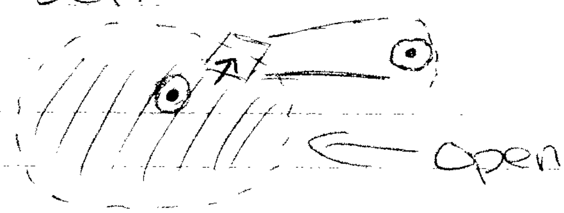
Definitions (2D)

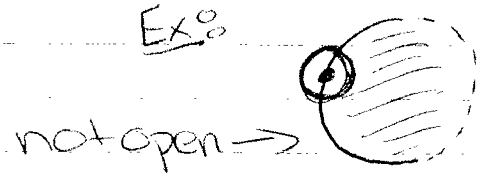
★ A set in the plane is called __open__ if every point in it has a small disk around it that is also contained in the set.

Ex:

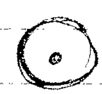


- Not open is anything that has any sort of boundary

Ex:



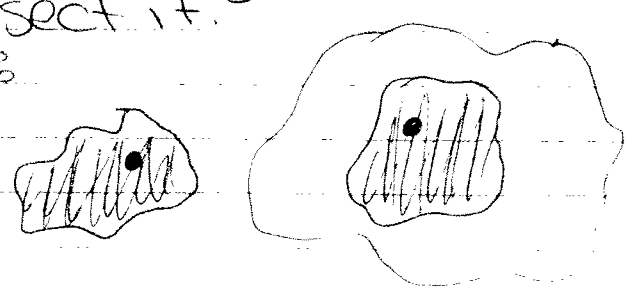
- __Open__ (the whole plane).



- __Open__ $\mathbb{R}^2 \to$ any finite collection of points

★ A set is called __connected__ if it can't be seperated by a curve that doesn't intersect it.

- __Idea:__



- Ok to think "__it__ __only__ __has__ __one__ __piece__."

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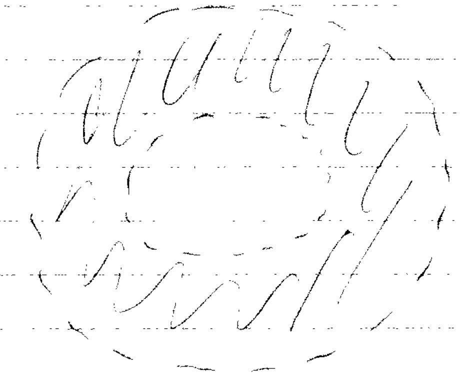
Ex:



OPEN and CONNECTED

Theorem: If $\vec{F}$ is a continuous vector field on an open and connected set, then $\int_C \vec{F} \cdot d\vec{r}$ (path independent) is $\vec{F} = \nabla f$ (conservative)

$\boxed{\text{2D ONLY}}$

Question: What can we say about $\vec{F}$ if we know it's conservative?

$$\vec{F} = \nabla f = f_x \vec{i} + f_y \vec{j}$$

So if $\vec{F} = \langle P(x,y), Q(x,y) \rangle$ then there is a function $f(x,y)$ satisfying $f_x = P(x,y)$, $f_y = Q(x,y)$.

We know: $\boxed{f_{xy} = f_{yx}} \longleftrightarrow \boxed{\dfrac{\partial P}{\partial y} = \dfrac{\partial Q}{\partial x}}$

This says that if $\vec{F} = \langle P, Q \rangle$ and $\dfrac{\partial P}{\partial y} \neq \dfrac{\partial Q}{\partial x}$, then $\vec{F}$ is not conservative

Warning: It's not necessarily true that $P_y = Q_x \rightarrow \vec{F}$ conservative.