9/9/09

Jennifer I. is thinking about Watagatapitusberry <3

Outline:

Calculus on Parametric Curves

Announcements:

HW due Tomorrow at the start of class.

Example

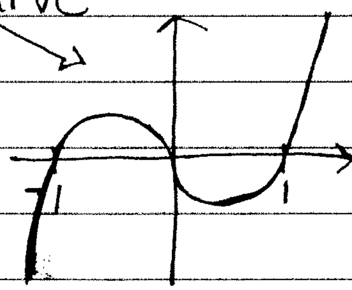
$$\begin{cases} x = t-1 \\ y = t^3 - 3t^2 + 2t \end{cases}$$

$\hookrightarrow t = x+1$

so,

$$y = (x+1)^3 - 3(x+1)^2 + 2(x+1)$$
$$= x^3 - x = x(x-1)(x+1)$$

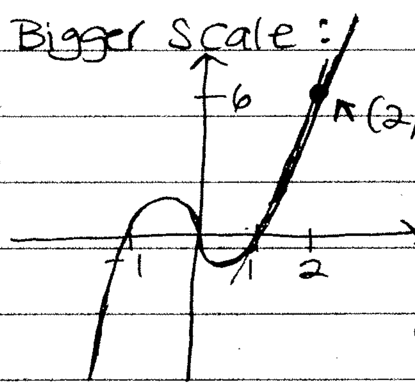
Curve



Exercise

Calculate the equation of the tangent line at $x=2, y=6$

Bigger Scale:



(2,6) Slope at (2,6) of $y = x^3 - x$

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$\frac{dy}{dx} = 3x^2 - 1$

at $x=2$

$\frac{dy}{dx} = 11$

So the tangent line passes through (2,6) and has slope 11.

$$\frac{y-6}{x-2} = 11$$

$y = 11(x-2)+6$

Goal: Let's do this by eliminating the parameter.

Q: Why would we want to do this?

A: Consider the example:

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$x = t^4 + 1 \qquad y = t^3 + t$

Start: $t = \sqrt[4]{x-1} \qquad y = (\sqrt[4]{x-1})^3 + \sqrt[4]{x-1}$

AHHHHA!!

Fact: Very often you are able to write

$y = F(x)$ (In our case $F(x) = x^3 - x$)

remember x, y are functions of t

so, $y(t) = F(x(t))$ $\frac{d}{dt}$

$$\frac{dy}{dt} = F'(x(t)) \cdot \frac{dx}{dt} \quad \text{(by the chainrule)}$$

Manipulate

$$\frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} = F'(x(t))$$ slope of the graph of $y = F(x)$ at the point $(x(t), y(t))$

$$\frac{dx}{dt} = 1 \qquad \frac{dy}{dt} = 3t^2 - 6t + 2$$

$$\frac{\frac{dy}{dt}}{\frac{dx}{dt}} = 3t^2 - 6t + 2$$
$$\parallel$$
$$F'(x(t))$$

$$\boxed{\begin{array}{l} \text{At } (2,6) \\ x = 2 \\ y = 6 \\ t = 3 \end{array}} \ *$$

so, $F'(\underset{x(3)}{2}) = \left.\frac{\left(\frac{dy}{dt}\right)}{\frac{dx}{dt}}\right|_{t=3} = 3(3)^2 - 6(3) + 2 = 11$

Recall: AHHHHH! $x = t^4 + 1 \quad y = t^3 + t$

Find the slope of the tangent line at $t = -1$

$$\frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{3t^2+1}{4t^3} \xrightarrow{t=-1} \frac{4}{-4} = -1$$ ← slope of the tangent at $x(-1) = 2$, $y(-1) = -2$

Rule:

a] If we can eliminate the parameter and write our parametric curve as $y = F(x)$ then

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} \quad \left(\text{as long as } \frac{dx}{dt} \neq 0\right)$$

b] If $x = G(y)$ then

$$\frac{dx}{dy} = \frac{dx/dt}{dy/dt} \quad \left(\text{as long as } \frac{dy}{dt} \neq 0\right)$$

c] At points where $\frac{dx}{dt} = 0$ there's a verticle tangent line.

d] At points where $\frac{dy}{dt} = 0$ it's horizontal

e] $$\frac{dy^2}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}}$$ $\boxed{\text{can use this to Check for concavity}}$

second derivative

Example

C : $x = t^2,\ y = t^3 - 3t$

a] Find the tangent slopes/lines at $(3,0)$

b] Find the horizontal/vertical tangent lines

c] Determine where concave up/down

d] Sketch the curve

Solutions

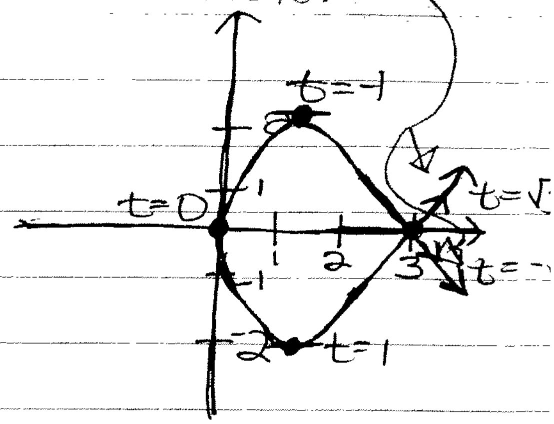
a] $t = \pm\sqrt{3}$ Plug in $t = \pm\sqrt{3}$

$$\frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{3t^2 - 3}{2t}$$

$$= \frac{3(\sqrt{3})^2 - 3}{2(\sqrt{3})} = \frac{6}{2\sqrt{3}} = \frac{3}{\sqrt{3}} = \sqrt{3}$$

$$\frac{3(-\sqrt{3})^2-3}{2(-\sqrt{3})} = -\sqrt{3}$$

Two lines & directions of the curve passing thru (3,0)



$\boxed{b}$ $\frac{dx}{dt} = 0$ $\quad$ $\frac{dx}{dt} = 2t$ equals zero at $t=0$

Check

or

$\frac{dy}{dt} = 0$ $\quad$ $(x,y) = (0^2, 0^3 - 3(0))$

$$\frac{dy}{dt} = 3t^2 - 3$$

$3t^2 - 3 = 0$ $\quad$ $(x,y) = (1,-2)$ $\quad$ $t = 1$

$t = \pm 1$ $\quad$ or $\quad$ or

$(1,2)$ $\quad$ $t = -1$

Homework

Exercise → Finish $\boxed{c}$

11.2 #3–5, 7, 9, 11–14