9/28/09

Lauren Balmert Ahhhhh!!

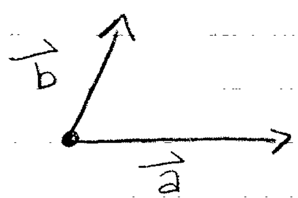
Outline:

- Cross Products

Announcements

- Exam Monday
- Study Session Sun 7-?? (TBA)
- Shape of Space
  T. Pizza @ 12 in BNW 319

Cross Products of 13.4

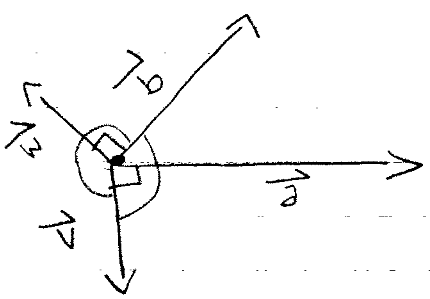


Definition (geometric):
cross product of $\vec{a}$ and $\vec{b}$
$\vec{a} \times \vec{b}$ is the vector in 3D space that satisfies the following properties:

① $\vec{a} \times \vec{b}$ is $\perp$ to both $\vec{a}$ and $\vec{b}$

② $|\vec{a} \times \vec{b}|$ = Area of the parallelogram formed by $\vec{a}, \vec{b}$

③ $\vec{a} \times \vec{b}$ points in the direction according to the "right hand rule"



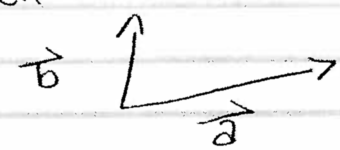
$\vec{v} \perp \vec{a}$ but not $\vec{b}$

$\vec{w} \perp \vec{b}$ but not $\vec{a}$

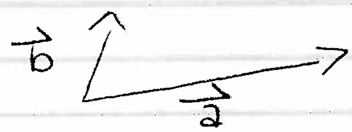
So... it has to leave the paper

①: $\vec{a} \times \vec{b}$ points in one of the two directions perpendicular to your paper

③: ex:

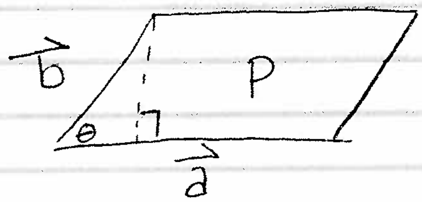


$\vec{a} \times \vec{b}$ points at you



$\vec{b} \times \vec{a}$ points into paper

②:



$|\vec{a} \times \vec{b}| = \text{Area}(P)$

$\text{Area}(P) = |\vec{a}||\vec{b}| \sin\theta$

These three properties determine the cross product.

Algebraic Definition:

$\vec{a} \langle a_1, a_2, a_3 \rangle \qquad \vec{b} \langle b_1, b_2, b_3 \rangle$

$\langle a_2 b_3 - a_3 b_2,\ a_3 b_1 - a_1 b_3,\ a_1 b_2 - a_2 b_1 \rangle$

$$\begin{array}{cccccc} \vec{i} & \vec{j} & \vec{k} & \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 & a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 & b_1 & b_2 & b_3 \end{array}$$

<u>Exercise</u>

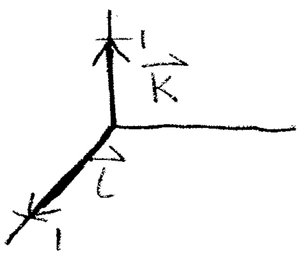
Take the dot product of $\vec{a} \times \vec{b}$ with $\vec{a} \langle a_1, a_2, a_3 \rangle$ and show it $= 0$.

Ex: $\vec{i} \times \vec{k} = ?$

$\vec{i} = \langle 1, 0, 0 \rangle$

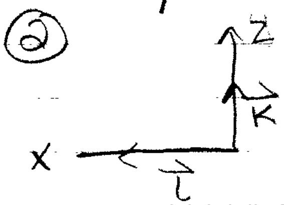
$\vec{k} = \langle 0, 0, 1 \rangle$

$$\vec{i} \times \vec{k} = \langle 0\cdot 1 - 0\cdot 0,\ 0\cdot 0 - 1\cdot 1,\ 0 - 0 \rangle$$
$$\langle 0, -1, 0 \rangle = -\vec{j}$$



① $\vec{i} \times \vec{k}$ points in direction of y axis

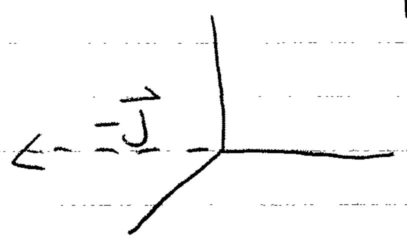
②



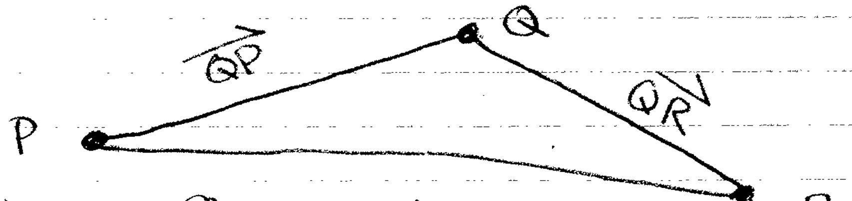
thumb points into paper

$\vec{i} \times \vec{k}$ points in the minus y direction

③ $\vec{i}, \vec{k}$ form a unit square so $|\vec{i} \times \vec{k}| = 1$



Ex: $P(1,4,6)$ $Q(-2,5,1)$ $R(1,-1,1)$



<u>Question</u>: ① compute the area of the ΔPQR

② Decide if the angle Q is 90°

Solution:

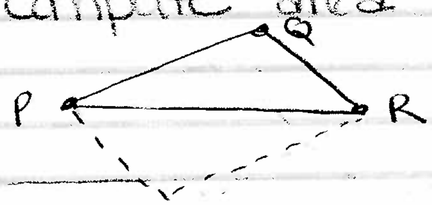
(2) $\overrightarrow{QP} = \langle 1-(-2),\ 4-5,\ 6-(-1) \rangle = \langle 3,-1,7 \rangle$

$-\overrightarrow{QR} = \langle -3, 6, -2 \rangle$, $\overrightarrow{QR} = \langle 3,-6,2 \rangle$

Compute dot product, if answer = 0, then angle is 90°

$\overrightarrow{QP} \cdot \overrightarrow{QR} = 9 + 6 + 14 \neq 0$ therefore, they are not perpendicular $\perp$

Exercise: compute area of ΔPQR



$|\overrightarrow{QP} \times \overrightarrow{QR}| = A(\text{Par})$

so $A(\Delta PQR) = \frac{1}{2}\text{Area}(\text{Par})$

$= \frac{1}{2}|\overrightarrow{QP} \times \overrightarrow{QR}|$

$\overrightarrow{QP} = \langle 3,-1,7 \rangle \quad \overrightarrow{QR} = \langle 3,-6,2 \rangle$

$\overrightarrow{QP} \times \overrightarrow{QR} = \langle a_2b_3 - a_3b_2,\ a_3b_1 - a_1b_3,\ a_1b_2 - a_2b_1 \rangle$

$\langle (-1)(2) - (7)(-6),\ (7)(3) - (3)(2),\ (3)(-6) - (-1)(3) \rangle$

$\langle -2+42,\ 21-6,\ -18+3 \rangle$

$\langle 40,\ 15,\ -15 \rangle$

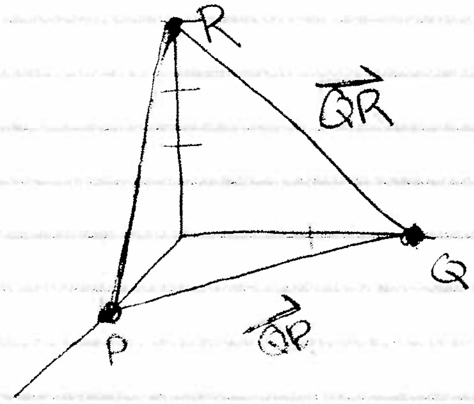
$|\overrightarrow{QP} \times \overrightarrow{QR}| = \sqrt{(40)^2 + (15)^2 + (-15)^2} =$

$\frac{1}{2}(45.276) \approx 22.64$ Area of ΔPQR

or $\frac{5}{2}\sqrt{82} =$

Example 13.4
Pg 829 #29
a) Find a nonzero vector orthogonal to the plane through the points P, Q, R
b) Find the area of triangle PQR

a) $P(1,0,0)$ $Q(0,2,0)$ $R(0,0,3)$



The vector determined by the cross product of two sides of the triangle, will be perpendicular to the two sides; and therefore, it will also be perpendicular or orthogonal to the plane through the points P, Q, R

The vector $\vec{QP} = \langle 1-0, 0-2, 0-0 \rangle$
$= \langle 1, -2, 0 \rangle$
The vector $\vec{QR} = \langle 0-0, 2-0, 0-3 \rangle$
$= \langle 0, 2, -3 \rangle$
The cross product formula is
$a \times b = \langle a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1 \rangle$

which can be obtained from the table:

$$\begin{array}{cccccc}
① & ② & ③ & -① & -② & -③ \\
\vec{i} & \vec{j} & \vec{k} & \vec{i} & \vec{j} & \vec{k} \\
a_1 & a_2 & a_3 & a_1 & a_2 & a_3 \\
b_1 & b_2 & b_3 & b_1 & b_2 & b_3
\end{array}$$

Using the cross product with the vectors, we get:

$$\vec{QP} \times \vec{QR} = \langle (-2)(-3)-(0)(0),\ (0)(0)-(1)(-3),\ (1)(2)-(-2)(0) \rangle$$
$$= \langle 6-0,\ 0-(-3),\ 2-0 \rangle$$
$$= \langle 6,\ 3,\ 2 \rangle$$

This vector is thus orthogonal to the plane

B) The area of the triangle PQR is half the area of the parallelogram. The area of the parallelogram is determined by finding the length of the cross product (or the length of the vector just calculated).

$$|\vec{QP} \times \vec{QR}| = \sqrt{(6)^2+(3)^2+(2)^2} = \sqrt{49} = 7$$

Area of the Δ is half this number

$$= 7/2$$