Stephen Flanagan

10/26

Outline: Finishing Ch. 14 (curvature)

Announcements: Tues, Oct. 27
Prof. R "slicing, dicing... scary stuff"
Library Mult. Media Room 5-6pm *Extra Credit

Exam II Thurs. 10/29
Review Session Wed. TBA

Curvature

$\vec{r}(t)$ is a vector function, determines a spacecurve.

$\vec{T}(t)$ is its unit tangent vector.

$s(t) = \int_a^t |\vec{r}\,'(u)|\, du$ arc length function

Notion: Parametrizing a curve by its length.
(time traveled = distance traveled).

Assume $\vec{r}(s)$ is param'd by arc length.
Def'n: The curvature of the curve determined by $\vec{r}(s)$ at the point s_0 is $K(s) = \left| \frac{d\vec{T}}{ds} \right| \Bigg|^{\vec{r}(s)}_{s=s_0}$

Note: $K(s_0)$ is a measure of how quickly $\vec{T}(s)$ is turning at point on curve $[\vec{r}(s_0)]$.

Ex
$\vec{r}(s) = \left\langle \cos\frac{s}{\sqrt{2}}, \sin\frac{s}{\sqrt{2}}, \frac{s}{\sqrt{2}} \right\rangle$ parametrized by length.

$$K(s) = \left| \frac{d}{ds}\Big(\underbrace{\left\langle \frac{-1}{\sqrt{2}}\sin\frac{s}{\sqrt{2}}, \frac{1}{\sqrt{2}}\cos\frac{s}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle}_{\vec{T}(s) = \vec{r}\,'(s)} \Big) \right|$$

b/c paramid by length means speed = 1.

$$= \left| \left\langle -\frac{1}{2}\cos\left(\frac{s}{\sqrt{2}}\right), \frac{-\sin}{2}(s/\sqrt{2}), 0 \right\rangle \right|$$

$$= \sqrt{\frac{1}{4}\cos^2(\ ) + \frac{1}{4}\sin^2(\ ) + 0} = \frac{1}{2}$$

$$\boxed{K(s) = 1/2}$$

curvature of the helix.

Note: Impossible by hand to parametrize some (most) curves by arc length. So we need a formula for K when $\vec{r}$ is <u>not</u> parametrized by length.

If $\vec{r}(t)$ is not length param'd, then let $s(t)$ be its arc length function. Then s is a function of t and we can substitute into $\vec{T}$:

$$\vec{T}(s(t)) \xrightarrow{d/dt} \underbrace{\frac{d\vec{T}}{ds}}_{\substack{\text{need}\\ \text{this}\\ \text{for } K}} \left(\frac{ds}{dt}\right) = \frac{d\vec{T}}{dt}$$

$$\boxed{s(t) = \int_a^t |\vec{r}\,'(u)|\,du \rightsquigarrow \frac{ds}{dt} = |\vec{r}\,'(t)| \quad \text{by FTC}}$$

So $$\frac{d\vec{T}}{dt} = \frac{d\vec{T}}{ds} \cdot |\vec{r}\,'(t)|$$

$$\left|\frac{d\vec{T}}{dt}\right| = K \cdot |\vec{r}\,'(t)|$$

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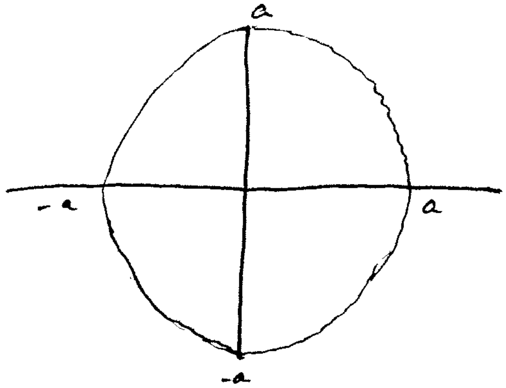
$$\boxed{K(t) = \frac{|\vec{T}\,'(t)|}{|\vec{r}\,'(t)|}}$$

Ex

$\langle a\cos t, a\sin t \rangle$ circle radius a in $\mathbb{R}^2$

(not param'd in arc length)



$$K(t) = \frac{|\vec{T}'(t)|}{|\vec{r}'(t)|}$$

$\vec{r}'(t) = \langle -a\sin t, a\cos t \rangle$

✓ $|\vec{r}'(t)| = a$

$\vec{T}(t) = \frac{1}{a}\vec{r}'(t) = \langle -\sin t, \cos t \rangle$

✓ $\vec{T}'(t) = \langle -\cos t, -\sin t \rangle$

$$K(t) = \frac{|\langle -\cos t, -\sin t \rangle|}{a} = \frac{1}{a}$$

Curvature of a circle is $\boxed{\frac{1}{\text{radius}}}$

Useful: $K(t) = \dfrac{|\vec{r}'(t) \times \vec{r}''(t)|}{|\vec{r}'(t)|^3}$

Ex

$\vec{r}(t) = \langle t, t^2, t^3 \rangle$

$K(0) = ?$

$\vec{r}\,'(t) = \langle 1, 2t, 3t^2 \rangle$

$\vec{r}\,''(t) = \langle 0, 2, 6t \rangle$

Curvature at point $K(0)$

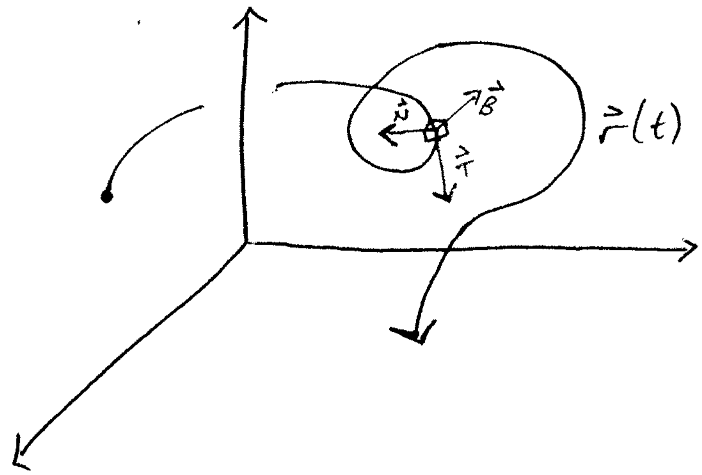
$$K(0) = \frac{|\langle 1,0,0\rangle \times \langle 0,2,0\rangle|}{|\langle 1,0,0\rangle|^3} = \frac{|2\vec{k}|}{1} = 2$$

$\vec{r}(t)$ is a vector function determining curve C.

$\vec{T}(t)$ is its unit tangent vector

$$\vec{T} \cdot \vec{T}\,' = 0 \quad \left(\text{recall: by } \frac{d}{dt}|\vec{T}|^2 = 0\right)$$

$$\vec{N}(t) = \frac{\vec{T}\,'(t)}{|T'(t)|}$$ This is called the unit normal vector to C at time t.



FACT: $\vec{N}(t)$ points $\perp$ to $\vec{T}$ in the direction of turning of $\vec{T}$.

$$\vec{B}(t) = \vec{T} \times \vec{N}$$

(binormal vector)

<u>Def'ns:</u>

- The plane that contains $\vec{r}(t)$ and vectors $\vec{T}$ and $\vec{N}$ is called the <u>osculating plane</u>.
- The plane determined by $\vec{N}$ & $\vec{B}$ is called the normal plane.

FACT: The osculating plane is the unique plane that comes closest to containing the curve at that point.

Def'n: The osculating circle (circle of curvature) is the circle contained in the osc. plane whose radius is $\frac{1}{K}$ and whose center is $\frac{1}{K}$ units away from the point $\vec{r}(t)$ on C along $\vec{N}(t)$

The osculating plane comes very close to containg the curve at this point.

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Find an equation of the osculating plane of the curve: $x = \sin 2t$, $y = t$, $z = \cos 2t$ at point $(0, \pi, 1)$.
$\therefore t = \pi$

$\vec{r}\,'(\pi) = \langle 2\cos 2t, \ 1, -2\sin 2t \rangle$
$= \langle -2, 1, 0 \rangle$

$-2(x-0) + 1(y-\pi) + 0(z-1) = 0$

or $-2x + y - \pi = 0$
$y = 2x + \pi$

$|r'(t)| = \sqrt{(2\cos 2t)^2 + 1^2 + (-2\sin 2t)^2}$
$= \sqrt{4\cos^2 2t + 1 + 4\sin^2 2t}$
$= \sqrt{4(1) + 1} = \sqrt{5}$

$T(t) = \frac{1}{\sqrt{5}} \langle 2\cos 2t, 1, -2\sin 2t \rangle$

$T'(t) = \frac{1}{\sqrt{5}} \langle -4\sin 2t, 0, -4\cos 2t \rangle$

$|T'(t)| = \sqrt{(-4\sin 2t)^2 + (-4\cos 2t)^2}$
$= \sqrt{16\sin^2 2t + 16\cos^2 2t} = \sqrt{16} = 4\sqrt{5}$

$$N(t) = \frac{T'(t)}{|T'(t)|} = \frac{1}{4}\langle -4\sin 2t, 0, -4\cos 2t \rangle$$

$$B = T \times N = \frac{1}{4\sqrt{5}} \begin{pmatrix} i & j & k & i & j & k \\ 2\cos 2t & 1 & -2\sin 2t & 2\cos 2t & 1 & -2\sin 2t \\ -4\sin 2t & 0 & -4\cos 2t & -4\sin 2t & 0 & -4\cos 2t \end{pmatrix}$$

$$= \frac{1}{4\sqrt{5}} \langle -4\cos 2t - 0,\ 8\sin^2 2t + 8\cos^2 2t,\ 0 - 4\sin 2t \rangle$$

$$= \langle -4\cos 2t,\ 8,\ -4\sin 2t \rangle$$

$$B(\pi) = \langle -4, 8, 0 \rangle$$

$$-4(x-0) + 8(y-\pi) + 0(z-1) = 0$$

$$-4x + 8y - 8\pi = 0$$

$$-x + 2y - 2\pi = 0 \quad \text{or} \quad x - 2y + 2\pi = 0$$