Mariam I.

10/~~5~~19

Outline:

- vector derivatives & integrals
- arclength:

Annoncements:

- wednesday office hours switched to 5-6pm
- Quiz on thursday
- no Class on wednesday

Homework & Quiz:

- key idea is to always aim for data that you need (when finding lines and planes) and use geometry to guide you.
- normal vector is $\perp$ to plane

#10 - line through $(2,1,0)$ and $\perp$ to $\hat{i}+\hat{j}$ and $\hat{j}+\hat{k}$

$$(\hat{i}+\hat{j}) \times (\hat{j}+\hat{k})$$

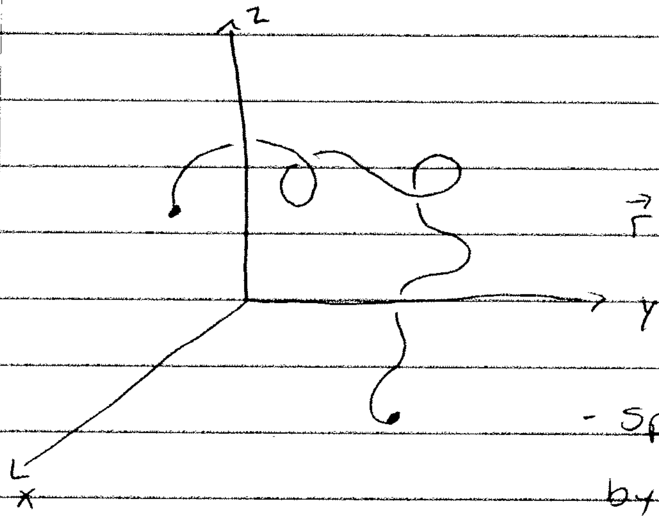
$$\langle 1,1,0\rangle \times \langle 0,1,1\rangle$$

$$[\hat{i}\times(\hat{j}+\hat{k}) + \hat{j}\times(\hat{j}+\hat{k})]$$

$$[(\hat{i}\times\hat{j}) + (\hat{i}\times\hat{k}) + (\hat{j}\times\hat{j}) + (\hat{j}\times\hat{k})]$$

$$\hat{k} + (-\hat{j}) + 0 + \hat{i}$$

$$\hat{i} - \hat{j} + \hat{k} = (1,-1,1)$$



$\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$

$a \le t \le b$

- Space curve determined by vector function $\vec{r}(t)$

* Last time: $\vec{v}(t) = \vec{r}\,'(t)$ is the tangent vector to the curve.

$\vec{r}\,'(t) = \langle f', g', h' \rangle$

rules:

① $\frac{d}{dt}(\vec{u}(t) \pm \vec{v}(t)) \Rightarrow \vec{u}\,'(t) \pm \vec{v}\,'(t)$

② $\frac{d}{dt}(c\,\vec{u}(t)) \Rightarrow c\,\vec{u}\,'(t)$

c constant

③ $\frac{d}{dt}(\vec{u}(t) \cdot \vec{v}(t)) \Rightarrow \vec{u}\,'(t) \cdot \vec{v}(t) + \vec{u}(t) \cdot \vec{v}\,'(t)$

④ same for cross product.

⑤ $\frac{d}{dt}(\underbrace{f(t)}_{\text{1 variable fct.}}\,\underbrace{\vec{u}(t)}_{\text{vector fct}}) = f'(t)\vec{u}(t) + f(t)\vec{u}\,'(t)$

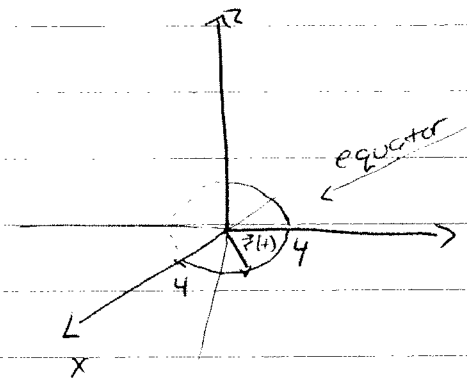
⑥ $\frac{d}{dt}\vec{u}(f(t)) = f'(t)\,\vec{u}\,'(f(t))$

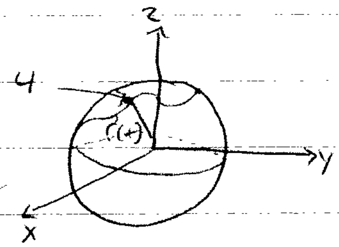
(cont'd)

- Suppose $|\vec{r}(t)| = c$ (for all t)

  Q: What does this same about the geometry of the space curve?

ex.





$$|\vec{r}(t)|^2 = c^2$$

$$\downarrow \frac{d}{dt}$$

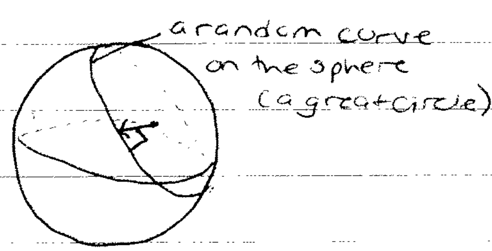
$$\frac{d}{dt}\left(\vec{r}(t)\cdot\vec{r}(t)\right) = 0 \qquad \boxed{\text{recall } \vec{r}\cdot\vec{r} = |\vec{r}|^2}$$

$$\downarrow$$

$$\vec{r}\,'(t)\cdot\vec{r}(t) + \vec{r}(t)\cdot\vec{r}\,'(t) = 0$$

$$2\vec{r}(t)\cdot\vec{r}\,'(t) = 0$$

$\therefore$ $\vec{r}(t) \perp \vec{r}\,'(t)$ - if $\vec{r}(t)$ describes a spherical curve, then the tangent vector is always $\perp$ to position.



Vector integration:

$$\vec{r}(t) = f(t)\,\vec{i} + g(t)\,\vec{j} + h(t)\,\vec{k} \qquad a \le t \le b$$

Definition:

$$\int_a^b \vec{r}(t)\,dt = \left(\int_a^b f(t)\,dt\right)\vec{i} + \left(\int_a^b g(t)\,dt\right)\vec{j} + \left(\int_a^b h(t)(dt)\right)\vec{k}$$

* Fundamental theorem of calculus

- if $\vec{R}'(t) = \vec{r}(t)$, then $\int_a^b \vec{r}(t)\,dt = \vec{R}(b) - \vec{R}(a)$

- you don't need $a+b$ in this def'n:

$$\int \vec{r}(t)\,dt = \left(\int f(t)\,dt\right)\vec{i} + \left(\int g(t)\,dt\right)\vec{j} + \left(\int h(t)\,dt\right)\vec{k}$$

ex. $\vec{r}(t) = \langle 2\cos t, \sin t, 2t \rangle$

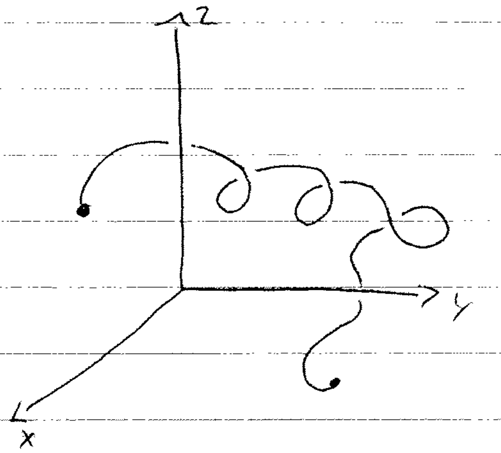
$$\int \vec{r}(t)\,dt = \left\langle \int 2\cos t\,dt, \int \sin t\,dt, \int 2t \right\rangle$$

$$= \langle 2\sin t + c_1, -\cos t + c_2, t^2 + c_3 \rangle$$

$$= \langle 2\sin t, -\cos t, t^2 \rangle + \underbrace{\langle c_1, c_2, c_3 \rangle}_{\vec{C}}$$

$$\boxed{\text{so } \int \vec{r}(t)\,dt = \langle 2\sin t, -\cos t, t^2 \rangle + \vec{C}}$$

Ch. 14.3: Arclength and curvature



$\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$

$a \le t \le b$

- if f', g', h' are all continuous functions and if the space curve C is traversed exactly once as t goes from a to b.

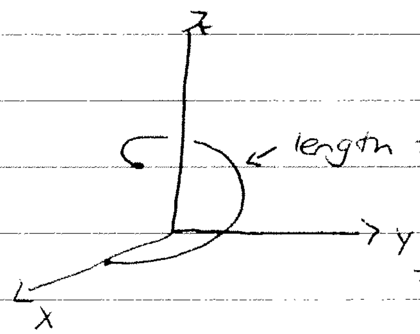
then:

$$\text{length}(C) = \int_a^b |\vec{r}\,'(t)|\,dt$$

$|\vec{r}\,'(t)| = \text{speed}$

$$= \int_a^b \sqrt{\left(\frac{df}{dt}\right)^2 + \left(\frac{dg}{dt}\right)^2 + \left(\frac{dh}{dt}\right)^2}\,dt$$

ex. $\vec{r}(t) = \langle \cos t, \sin t, t \rangle \quad 0 \le t \le 2\pi$



$$\text{length} = \int_0^{2\pi} \sqrt{(-\sin t)^2 + (\cos t)^2 + (1)^2}\,dt$$

$$= \int_0^{2\pi} \sqrt{2}\,dt = \boxed{\sqrt{2} \cdot 2\pi}$$

practice problem on the back →

#38. evaluate the integral.

$\vec{r}(t) = \int (\cos\pi t\, i + \sin\pi t\, j + t\, k)\, dt$

$\langle -\pi \sin\pi t + C_1, \pi \cos\pi t + C_2, \frac{t^2}{2} + C_3 \rangle$

$\langle -\pi \sin\pi t, \pi\cos\pi t, \frac{t^2}{2} \rangle + \underbrace{\langle C_1, C_2, C_3 \rangle}_{\vec{C}}$

so $\int \vec{r}(t)\, dt = \langle -\pi \sin\pi t, \pi \cos\pi t, \frac{t^2}{2} \rangle + \vec{C}$

#5. ~~Find the length of the curve~~

$\cancel{r(t) = i + t^2 j + t^3 k \quad 0 \le t \le 1}$

$\cancel{s(t) = \int_0^t \sqrt{(0^2) + (2t)^2 + (3t^2)^2}\, dt = \int_0^t \sqrt{4t^2 + 9t^4}\, dt}$

$\cancel{= \int_0^t 2t + 3t^2\, dt = t^2 + t^3 \Big|_0^t}$

$\cancel{s(t) = t^2 + t^3}$