9/24

- **Outline:**
  - Continuity (2.5)

Announcements:
- Exam 1: Wed 9/30
  Here (our usual time)
- Review/Study Session
  Time/Place TBA

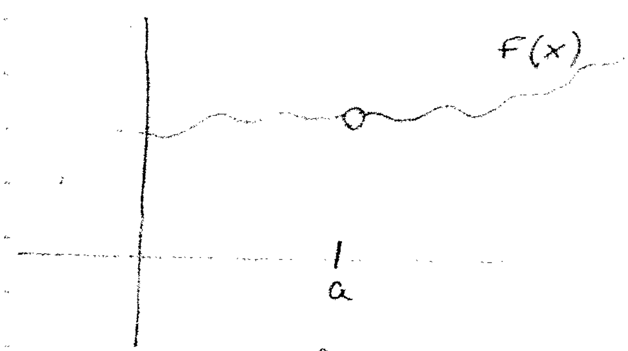
- **Continuity:**

Def: A function f is continuous @ the point $x = a$,
if
$$\lim_{x \to a} f(x) = f(a)$$

- What this says about f?
  1. $f(a)$ is defined
  2. $\lim_{x \to a} f$ exists
  3. $\lim_{x \to a} f = f(a)$

$\Rightarrow$ "A function is continuous at a if you can draw the graph of f through the point $x = a$ w/out lifting your pencil of the paper"

- Examples of Discontinuity:
  1. $f(a)$ is not defined



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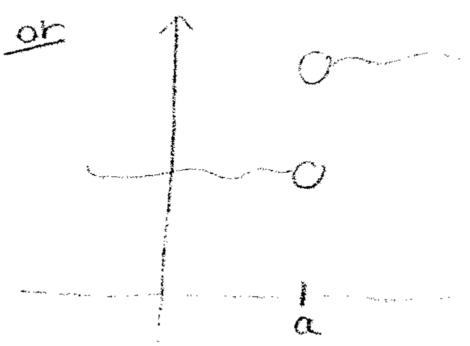

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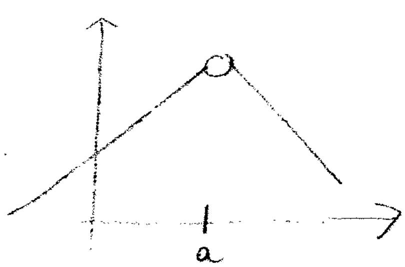

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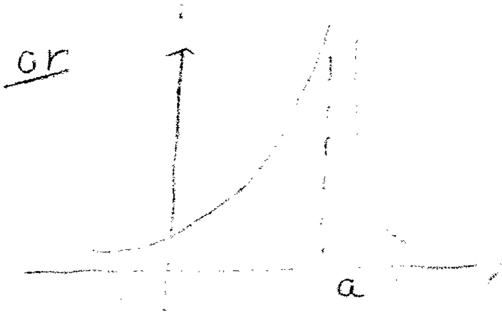
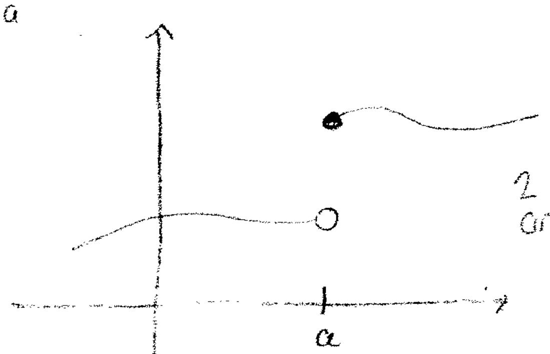
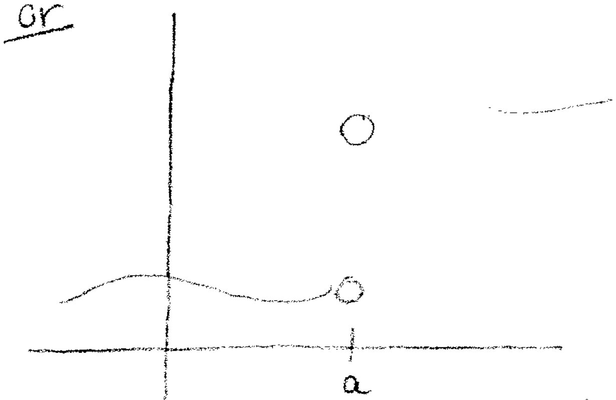


$\lim_{x \to a} F$ DNE



2 different one-sided limits

or



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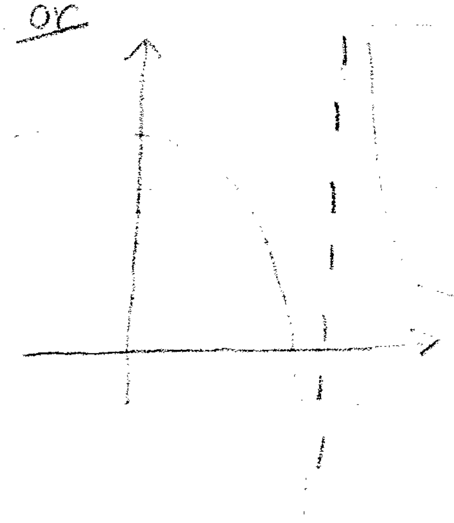
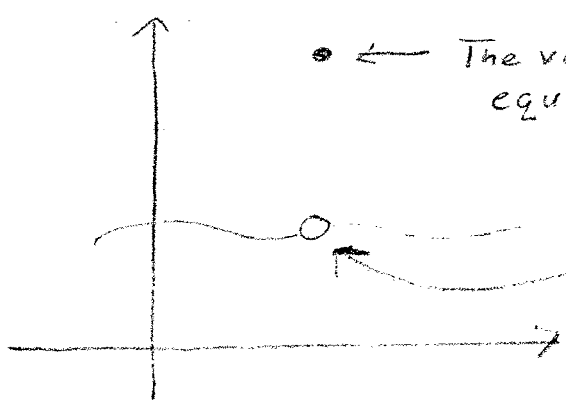
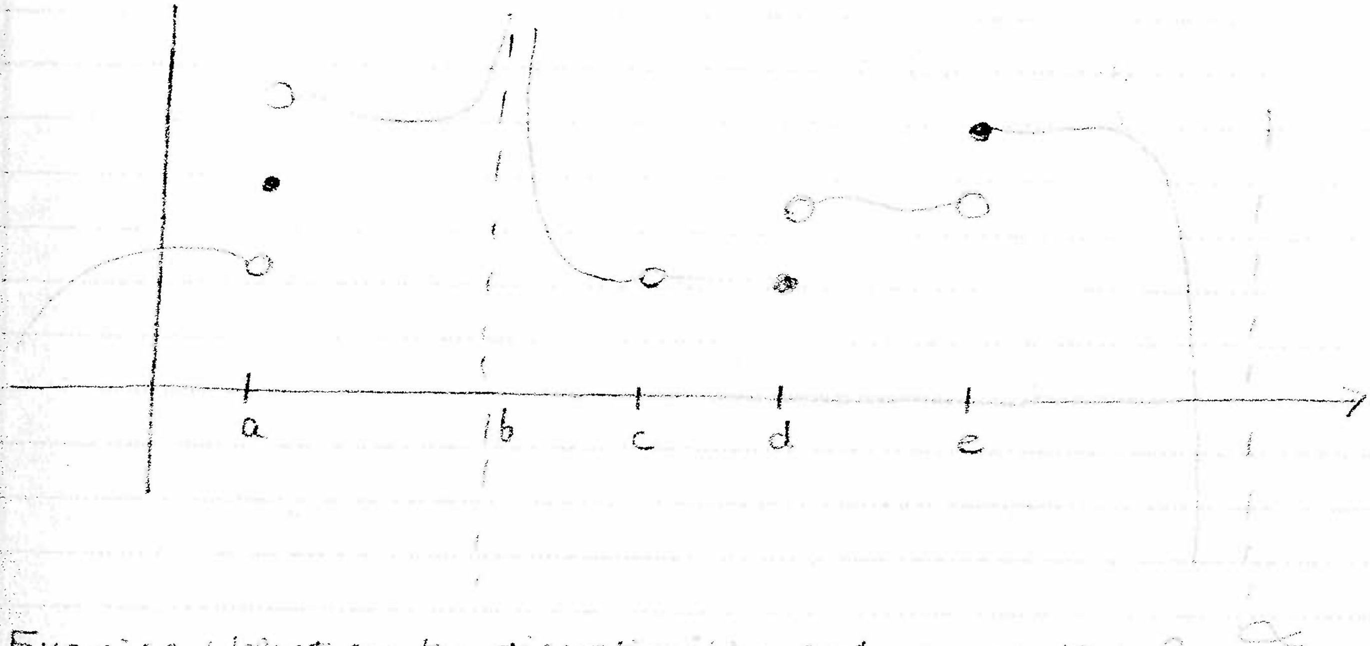


(3) $\lim_{x \to a} F(x) \neq F(a)$



← The value of F at x = a not equal to the limit



<u>Exercise</u>: What are the discontinuities and why are they?

Why is $b = x$ a "discty"?

b/c f is not defined at b.

Why is $e = x$ a discty?

b/c $\lim_{x \to e} f$ D.N.E.

A function f is continuous on the interval $[a, b]$ ($a \leq x \leq b$) if f is cts. on every pt. in (a, b) ← this means $a < x < b$

and if f is continuous from the right at $x = a$ ① and f is continuous from the left @ $x = b$ ②

① $\lim_{x \to a^+} f = f(a)$

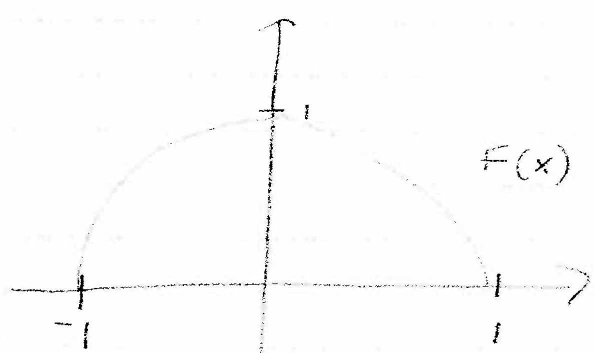
② $\lim_{x \to b^-}$

<u>Example</u>:

$f(x) = \sqrt{1 - x^2}$

Domain: $1 - x^2 \geq 0$

$1 \geq x^2$



~~[illegible]~~

$-1 \quad 0 \quad 1$

exist $+ =$ ... and b/c 2 one-sided

f(x) is continuous on $[-1, 1]$ b/c f is cts at

· Theorem : If $f(x)$ and $g(x)$ are cts. at $x=a$, then all of the following functions are continuous at $x=a$:

$$f+g,\ f-g,\ fg,\ \underset{\substack{\uparrow \\ \text{number}}}{c}f,\ \frac{f}{g} \leftarrow \text{if } g(a)\neq 0$$

· Theorem :

Polynomials are cts. on $(-\infty,\infty)$

Rational functions are cts. on their $\underbrace{\text{domains}}_{\text{everything that doesn't make denom} = 0.}$ $\quad \frac{x^3+2x+17}{x-3}$

Root functions cts. on their domains.

$$f(x) = \sqrt{\underbrace{h(x)}_{\text{Domain is wherever } h(x)\geq 0}}$$

Trig functions are cts. on their domains. $\left.\begin{matrix}\sin x \\ \cos x\end{matrix}\right\}$ cts. on $(-\infty,\infty)$

$\tan x$ (exercise: what is its domain?)

· Example :

a) $f(x) = x^{100} - 2x^{37} + 75$

b) $g(x) = \frac{x^2+2x-8}{x^2-1}$

c) $h(x) = \sqrt{x} + \frac{x+1}{x-1} + \frac{x+2}{x^2+1}$

Q: What are the intervals of continuity for these?

a) $(-\infty,\infty)$

b) All real #'s except for $x=\pm 1$

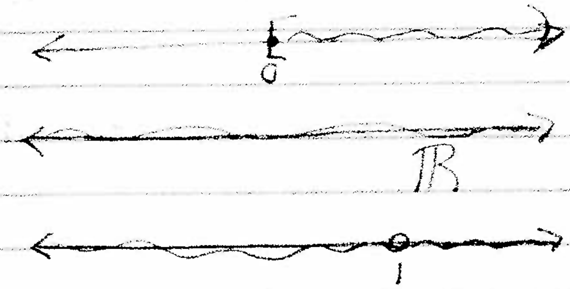
$(-\infty,-1) \cup (-1,1) \cup (1,\infty)$ intervals of cty.

c) $\sqrt{x}$ is cts. on $[0,\infty)$

$\frac{x+1}{x-1}$ cts. on $(-\infty,1)\cup(1,\infty)$

$\frac{x+2}{x^2+1}$ is cts. on $(-\infty, \infty)$

b/c denom $\neq 0$ ever



$[0,1) \cup (1, \infty)$ interval of continuity

Ex: $Q(x) = \frac{\sin x}{2 + \cos x}$

Where is Q continuous?

$\sin x$ is cts. on $(-\infty, \infty)$

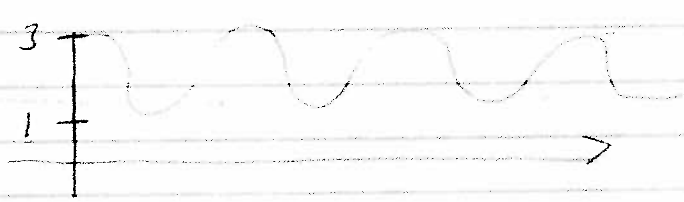
$2 + \cos x$ is cts. on $(-\infty, \infty)$

b/c $f(x) = 2$

$h(x) = \cos x$

are cts. everywhere

Just check when $2 + \cos x = 0$: $\boxed{\text{Never}}$



$2 + \cos x \neq 0$

So $Q(x)$ is cts. on $\mathbb{R}$

- Theorem: If f is continuous @ b and $\lim_{x \to a} g(x) = b$ then $f(g(x))$ is cts. at a

- Theorem*: If f is continuous at b and g is cts. at a and $g(a) = b$ then $f(g(x))$ is cts. at a.

- Ex: $Q(x) = \sin(x^2)$

  $Q(x) = f \circ g(x)$ $\quad f(x) = \sin x$, $g(x) = x^2$

  Recall: $f \circ g(x) = f(g(x))$

- Question: Where is Q cts.?

  Check where $\sin x$ cts? $\leftarrow \mathbb{R}$

  x^2 cts? $\leftarrow \mathbb{R}$

  Therefore the composition is cts. everywhere.

- Ex: $f(x) = \dfrac{1}{\sqrt{x^2+7} - 4}$ Where cts.?

  $f(x) = i \circ m \circ r \circ p(x)$

  $p(x) = x^2 + 7$

  $r(x) = \sqrt{x}$

  $m(x) = x - 4$

  $i(x) = \frac{1}{x}$

  $$x \xrightarrow{p} x^2+7 \xrightarrow{r} \sqrt{x^2+7} \xrightarrow{M} \sqrt{x^2+7} - 4 \xrightarrow{i} \frac{1}{\sqrt{x^2+7} - 4} = f(x)$$

- As an exercise:

$$p \circ m \circ i \circ r(x) = p(m(i(r(x))))$$
$$= p(m(i(\sqrt{x})))$$
$$= p\left(m\left(\frac{1}{\sqrt{x}}\right)\right)$$
$$= p\left(\frac{1}{\sqrt{x}} - 4\right)$$
$$= \left(\frac{1}{\sqrt{x}} - 4\right)^2 + 7$$

- A: $p(x)$ is cts. $\mathbb{R}$

$r(x) = \sqrt{x}$ cts. $[0, \infty)$

So $\therefore$ $r(p(x)) = \sqrt{x^2+7}$ is cts. $\mathbb{R}$ b/c x^2+7 is always positive.

$m(x)$ is cts. everywhere

So $\therefore$ $m(r(p(x))) = \sqrt{x^2+7} - 4$ is cts. on $\mathbb{R}$

$i(x) = \frac{1}{x}$ cts. on $x \neq 0$

So $i(m(r(p(x))))$ is cts. when $m \circ r \circ p(x)$ is $\neq 0$

$$\sqrt{x^2+7} - 4 = 0$$
$$\sqrt{x^2+7} = 4$$
$$x^2+7 = 16$$
$$x^2 = 9$$
$$x = \pm 3$$

$\therefore$ $f(x) = \frac{1}{\sqrt{x^2+7} - 4}$ is cts. on $(-\infty, -3) \cup (-3, 3) \cup (3, \infty)$

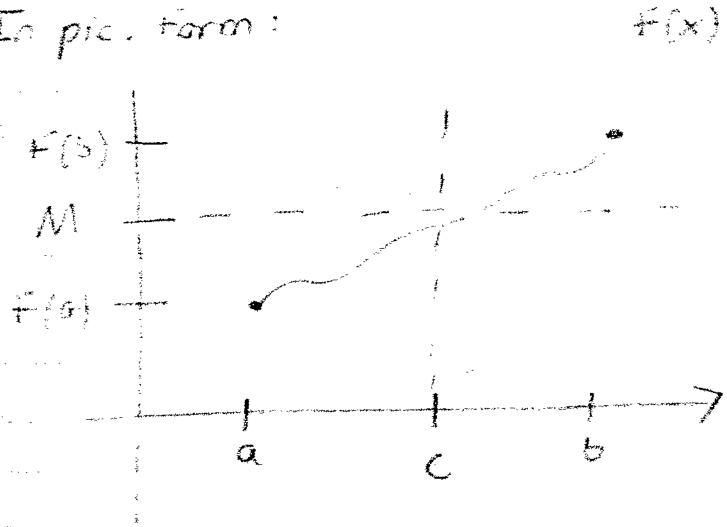
Intermediate Value Theorem:

If f is cts. $[a, b]$ and $f(a) < M < f(b)$

then there is another # c $(a < c < b)$

so that $f(c) = M$

In pic. form:



Ex. from Textbook: Section 2.5

15) $f(x) = \frac{1}{(x-1)^2}$, $a = 1$

Function is discontinuous b/c $f(1)$ is not defined.
Goes against rules of continuity.