Sanley

IMPLICIT DIFFERENTIATION — October 14

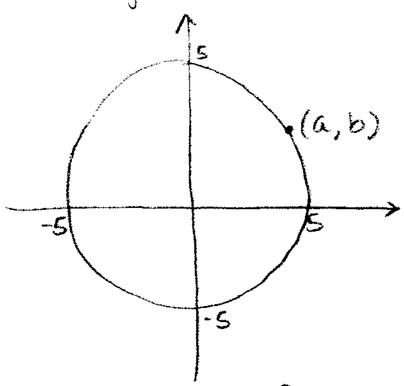
Motivation & Goal of implicit differentiation
be able to find $\frac{dy}{dx}$ even when it's not obvious how y is a function of x.

Here, it's obvious how y is a function of x -
$y = \sin x$

Here, it's less obvious -
$x^3 + y^3 = 6xy$
↳ this equation is said to define y as a function of x implicitly

Fact: any equation with y's & x's defines an implicit function

Ex. $x^2 + y^2 = 25$



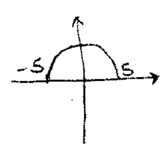
$a^2 + b^2 = 25$

• (c,d) $c^2 + d^2 \neq 25$

y is an implicit function of x in this case. We could solve for y and get 2 explicit functions:

$y = +\sqrt{25 - x^2}$

$y = -\sqrt{25 - x^2}$



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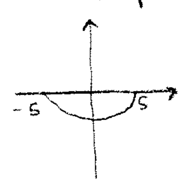
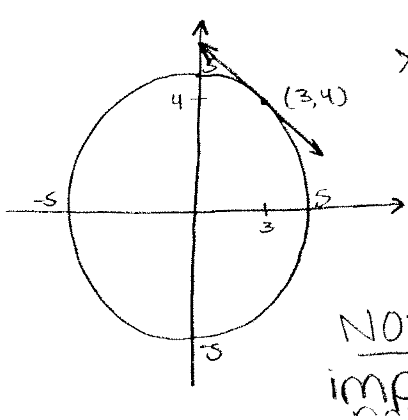




$x^2 + y^2 = 25$

Exercise: Find the slope of the tangent line to the circle at (3,4).

NOTE - just because y is defined implicitly, doesn't mean that y is

So, it makes sense to ask what $\frac{dy}{dx}$ is.

<u>Solution</u>: $x^2 + y^2 = 25$

$\downarrow \frac{d}{dx}(\cdot)$

$$\frac{d}{dx}(x^2 + y^2) = \frac{d}{dx}(25)$$

$$\frac{d}{dx}(x^2) + \frac{d}{dx}(y^2) = 0$$

$$2x + \frac{d}{dx}(y^2) = 0$$

$\frac{d}{dx}(y^2) = 2y\frac{dy}{dx}$ because of the chain rule & y is a function of x

(y^2)

$\downarrow$

$$\frac{d}{dx}(f(x))^2 = 2\,f(x)\cdot f'(x)$$

So,

$$2x + 2y\frac{dy}{dx} = 0$$

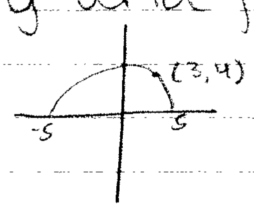
$$2y\frac{dy}{dx} = -2x$$

$$\boxed{\frac{dy}{dx} = \frac{-x}{y}}$$

$$\left.\frac{dy}{dx}\right|_{(3,4)} = \frac{-(3)}{4} = \boxed{\frac{-3}{4}}$$

Not that you want to do this, <u>but</u> you could have solved for y and just taken $\frac{d}{dx}$:

$$y = \sqrt{25 - x^2}$$



$$\frac{dy}{dx} = \frac{-2x}{2\sqrt{25-x^2}} = \frac{-x}{\sqrt{25-x^2}}$$

$$\left.\frac{dy}{dx}\right|_3 = \frac{-3}{\sqrt{25-9}} = \frac{-3}{4}$$

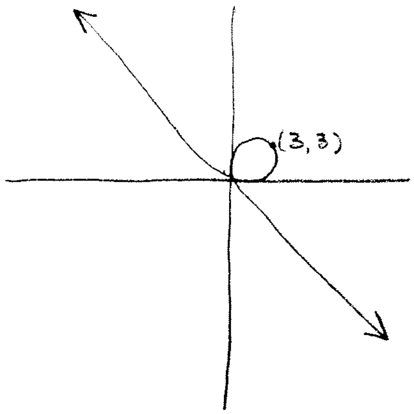
But! Try to solve for y:

$$x^3 + y^2 = 6xy$$

(don't try this.)

Example:

$x^3 + y^3 = 6xy$ Find the slope at $(3,3)$



$$x^3 + y^3 = 6xy$$

$$\downarrow \frac{d}{dx}$$

$$\frac{d}{dx}(x^3 + y^3) = \frac{d}{dx}(6xy)$$

product rule $(6x)\cdot(y)$

$$3x^2 + 3y^2 \frac{dy}{dx} = 6x \cdot \frac{dy}{dx} + 6y$$

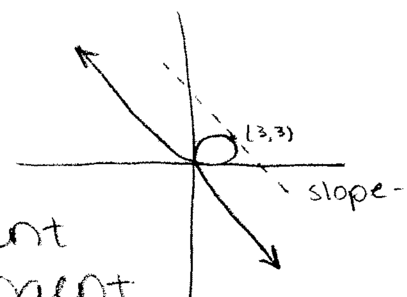
$$3y^2 \frac{dy}{dx} - 6x \frac{dy}{dx} = 6y - 3x^2$$

$$(3y^2 - 6x) \frac{dy}{dx} = 6y - 3x^2$$

$$\frac{dy}{dx} = \frac{6y - 3x^2}{3y^2 - 6x}$$

* you can write $\frac{dy}{dx}$ as y' or $\frac{dy}{dx}$

$$\left.\frac{dy}{dx}\right|_{(3,3)} = \frac{6(3) - 3(9)}{3 \cdot 9 - 6 \cdot 3} = -1$$



Q: where in the first quadrant ($x > 0$ & $y > 0$), is the tangent line horizontal?

A: set $\frac{dy}{dx} = 0$ and find (x, y) that satisfies that

$$0 = \frac{6y - 3x^2}{3y^2 - 6x}$$ This means $\boxed{6y - 3x^2 = 0}$

$$6y = 3x^2$$

$$y = \frac{x^2}{2}$$ ← x must satisfy this

&

$$x^3 + y^3 = 6xy$$

substitute $x^3 + \left(\frac{x^2}{2}\right)^3 = 6x\left(\frac{x^2}{2}\right)$

$$x^3 + \frac{x^6}{8} = 3x^3$$

$$\frac{x^6}{8} = 2x^3 \qquad x^6 = 16x^3$$

$$x^3 = 16$$

$$x = \sqrt[3]{16}$$

b/c $x = 0$ is not in the first quadrant, we can disregard that solution and

$y = \frac{x^2}{2} = \frac{(\sqrt[3]{16})^2}{2}$

so the answer is:
$\sqrt[3]{16}, \frac{(\sqrt[3]{16})^2}{2}$

$\sin(x+y) = y^2 \cos x$ find $\frac{dy}{dx}$

find d/dx ↳ $\cos(x+y) \cdot (1 + \frac{dy}{dx}) = -y^2 \sin x + 2y \frac{dy}{dx} \cos x$

↓ exercise (solve for $\frac{dy}{dx}$)

A: $\frac{\cos(x+y) + y^2 \sin x}{2y\cos x - \cos(x+y)}$

example problem -

find $\frac{dy}{dx}$ by implicit differentiation -

$2x^3 + x^2 y - xy^3 = 2$

$6x^2 + x^2 \frac{dy}{dx} + y - x\,3y^2 \frac{dy}{dx} + y^3 = 0$

$x^2 \frac{dy}{dx} - 3xy^2 \frac{dy}{dx} = -6x^2 - y^3$

$(x^2 - 3xy^2)\frac{dy}{dx} = -6x^2 - y^3$

$$\boxed{\frac{dy}{dx} = \frac{-6x^2 - y^3}{x^2 - 3xy^2}}$$