A TRICHOTOMY FOR HITTING TIMES AND ESCAPE RATES FOR A CLASS OF UNIMODAL MAPS

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ABSTRACT. We consider local escape rates and hitting time statistics for unimodal interval maps of Misiurewicz-Thurston type. We prove that for any point z in the interval there is a local escape rate and hitting time statistics which is one of three types. While it is key that we cover all points z, the particular interest here is when z is periodic and in the postcritical orbit which yields the third part of the trichotomy. We also prove generalised asymptotic escape rates of the form first shown by Bruin, Demers and Todd.

1. INTRODUCTION AND MAIN RESULT

Much attention has been paid recently to the connection between hitting times on the one hand [FFT2, GKV, AHV] and escape rates on the other [KL2, PU, DT2]. Given a map f of the interval preserving a probability measure μ, the principal connection is that in many situations the extremal index, defined by

$$-\lim_{\varepsilon \to 0} \log \mu(\{x : f^j(x) \notin B_\varepsilon(z), j = 0, 1, \ldots, \lfloor \mu(B_\varepsilon(z))^{-1}\rfloor\}),$$

where $B_\varepsilon(z)$ denotes the ball of radius ε centred at z, yields the same limit [BDT] as the local (or asymptotic) escape rate given by,

$$\lim_{\varepsilon \to \infty} \lim_{n \to \infty} \frac{-1}{\mu(B_\varepsilon(z))} \frac{1}{n} \log \mu(\{x : f^j(x) \notin B_\varepsilon(z), j = 0, 1, \ldots n\}).$$

The limit typically obeys a dichotomy: if z is not periodic, the limit is 1; if z is periodic with prime period p, then the limit is $1 - \frac{1}{|Df^p(z)|}$ if μ is an acip. More generally, when μ is an equilibrium state for a Hölder continuous potential ϕ, the limit in the periodic case is $1 - e^{S_p \phi(z)}$, where $S_p \phi(z) = \sum_{i=0}^{p-1} \phi(f^i(x))$.

Such dichotomies for asymptotic escape rates have been proved for all z in some uniformly hyperbolic contexts [BY, KL2, FP, HY]. In the non-uniformly hyperbolic setting, the asymptotic escape rate has been achieved for large sets of z in e.g. [DT1, PU, BDT, DT2].

In this paper we expand this dichotomy to a trichotomy by establishing conditions in which a third limit is possible. We do this principally by considering unimodal maps whose post-critical orbit is eventually periodic and centring holes at such periodic points z. We show that in this case, the asymptotic escape rate is a function of both $|Df^p(z)|$ as well as the order of the spike in the invariant density at z.

Before stating our main result, we precisely define the class of maps to which our results will apply.

We consider S-unimodal maps $f : [0, 1] \to [0, 1]$ in the sense of [NS]. That is

- f is C^2 with one critical point c;
- in a neighbourhood of c, $f(x) = f(c) - A(x - c)^\ell$ for $\ell > 1$ and $A > 0$;
- $|Df|^{-\frac{1}{2}}$ is convex on each of $[0, c]$ and $[c, 1]$.

We assume that the map is Misiurewicz-Thurston: there is a minimal $k_0 \geqslant 2$ such that $f^{k_0}(c)$ is periodic, i.e., there is a minimal $p \geqslant 1$ such that $f^p(f^{k_0}(c)) = f^{k_0}(c)$. Moreover this is a repelling periodic point: $|Df^p(f^{k_0}(c))| > 1$. This implies that the postcritical orbit of c, $\mathrm{orb}(f(c)) = \{f(c), f^2(c), \ldots, f^{k_0+p-1}(c)\}$, is finite and that there is an absolutely continuous f-invariant probability measure (acip) μ. Then the density of μ has a spike at each point of $\mathrm{orb}(f(c))$ of type $x^{\frac{1}{\ell}-1}$. We let $I := [f^2(c), f(c)]$ denote the *dynamical core*: all points except 0 and 1 eventually map into I and remain there, so μ is supported in I.

Define a hole centred at z by $H_\varepsilon(z) = (z - \varepsilon, z + \varepsilon)$. The *escape rate* of the open system with hole $H_\varepsilon(z)$ with respect to the measure μ is defined by

$$\mathfrak{e}(H_\varepsilon(z)) = -\lim_{n\to\infty} \frac{1}{n} \log \mu\left(\{x \in I : f^j(x) \notin H_\varepsilon(z), \ j = 0, \ldots, n-1\}\right), \tag{1.1}$$

when the limit exists. Define the *local (asymptotic) escape rate at z* by

$$\mathrm{esc}(z) := \lim_{\varepsilon \to 0} \frac{\mathfrak{e}(H_\varepsilon(z)))}{\mu(H_\varepsilon(z))}.$$

If z is a periodic point with prime period p, we write $\lambda_z := |Df^p(z)|$.

Our main result is the following trichotomy regarding possible values of the local escape rate. The only model where such a result has appeared previously is in [DT2, Remark 3.11], which is a very special case (the full quadratic map).

Theorem 1.1. *Let f be as defined above. For any $z \in I$,*

$$\mathrm{esc}(z) = \begin{cases} 1 & \text{if } z \text{ is not periodic,} \\ 1 - \frac{1}{\lambda_z} & \text{if } z \text{ is periodic and not in } \mathrm{orb}(f(c)), \\ 1 - \frac{1}{\lambda_z^{1/\ell}} & \text{if } z \text{ is periodic and in } \mathrm{orb}(f(c)). \end{cases}$$

The cases when z is outside the postcritical orbit $\mathrm{orb}(f(c))$ follow from [DT2, Theorem 3.7], so our focus here is on the case when $z \in \mathrm{orb}(f(c))$: when $z \in \mathrm{orb}(f(c))$ is preperiodic (which falls into the first case in the above theorem), and when $z \in \mathrm{orb}(f(c))$ is periodic. The techniques to show the former are essentially a subset of the latter. Our proofs exploit the fact that the finite post-critical orbit allows us to define a finite Markov partition for the map (albeit with unbounded distortion) which we then use to define a first return map to a conveniently chosen set.

1.1. **Hitting time statistics.** Setting $r_A(x) := \inf\{k \geqslant 1 : f^k(x) \in A\}$, the *first hitting time to A*, the *Hitting Time Statistics (HTS)* at z is given by

$$T_z(t) := \lim_{\varepsilon \to 0} \mu\left(r_{H_\varepsilon(z)} \geqslant \frac{t}{\mu(H_\varepsilon(z))}\right)$$

for $t > 0$, provided this limit exists. A generalisation of this limit is formulated in [BDT], where the link between HTS and asymptotic escape rates was explored by scaling the hitting time asymptotic as $t\mu(H_\varepsilon(z))^{-\alpha}$ for some $\alpha > 0$. Accordingly, define for $t, \alpha > 0$,

$$L_{\alpha,t}(z) := \lim_{\varepsilon \to 0} \frac{-1}{t\mu(H_\varepsilon(z))^{1-\alpha}} \log \mu\left(r_{H_\varepsilon(z)} \geqslant \frac{t}{\mu(H_\varepsilon(z))^\alpha}\right).$$

Note that $T_z(t)$ corresponds to $\alpha = 1$. Using Theorem 1.1 we prove the following in Section 6.

Theorem 1.2. *For any $z \in I$ and all $\alpha, t > 0$, $L_{\alpha,t}(z)$ exists and equals*

$$L_{\alpha,t}(z) = \begin{cases} 1 & \text{if } z \text{ is not periodic,} \\ 1 - \frac{1}{\lambda_z} & \text{if } z \text{ is periodic and not in } \mathrm{orb}(f(c)), \\ 1 - \frac{1}{\lambda_z^{\frac{1}{\ell}}} & \text{if } z \text{ is periodic and in } \mathrm{orb}(f(c)). \end{cases}$$

As with Theorem 1.1, the main focus here is on the case when $z \in \mathrm{orb}(f(c))$; the other cases follow from adaptations of [BDT], as we describe in Section 6.4.

We note that a full description of $T_z(t)$ in uniformly hyperbolic cases can be seen in [FFT1, AFV]. In the non-uniformly hyperbolic case, the only full dichotomy (i.e. the result applies to all z with no excluded values), and only for $T_z(t)$ rather than $L_{\alpha,t}(z)$ as above, has been demonstrated for Manneville-Pomeau maps in [FFTV]; see also the recent preprint [BF].

Remark 1.3. *Our techniques actually deal with holes of the form $(z - \varepsilon_L, z + \varepsilon_R)$ where $\frac{\varepsilon_L}{\varepsilon_R}$ is uniformly bounded away from 0 and ∞, so we are able to handle non-symmetric holes in both Theorems 1.1 and 1.2.*

1.2. **Structure of the paper.** In Section 2 we assume that we are in the $z \in \mathrm{orb}(f(c))$ periodic case and define a domain Y and a first return map F to Y so that the hole at z is outside Y: (Y, F) has exponential return times and a Markov structure. We outline how domains of F can map into the hole and then modify F to F_{ε} by introducing extra cuts at the boundary of the hole. In Section 3 again we initially assume $z \in \mathrm{orb}(f(c))$ is periodic and show how the density spike at z affects the scaling of the hole and its preimages. These scalings lead to the three quantities seen in Theorems 1.1 and 1.2. The section ends in Section 3.4 where the case $z \in \mathrm{orb}(f(c))$ is preperiodic is dealt with.

Section 4 sets up the functional framework for F_{ε} and its punctured counterpart $\mathring{F}_{\varepsilon}$. In particular a spectral gap is proved along with the relevant perturbation theory for the transfer operator corresponding to $\mathring{F}_{\varepsilon}$ for ε small and for certain nice 'β-allowable' holes. The convergence of the relevant spectral properties is proved, which prove a version of Theorem 1.1, see (4.16) and (4.17) for these specific holes. Section 5 then proves Theorem 1.1 for general holes and in all cases of $z \in \mathrm{orb}(f(c))$. In Section 6 we prove Theorem 1.2. The strategy is to use the induced map to construct a non-Markovian Young Tower and prove a spectral gap for the associated transfer operator in a space of weighted bounded variation.

Notation: we use $g_1(\varepsilon) \gtrsim g_2(\varepsilon)$ to mean $\frac{g_1(\varepsilon)}{g_2(\varepsilon)} \geqslant h(\varepsilon)$ where $h(\varepsilon) \to 1$ as $\varepsilon \to 0$. Similarly $g_1(\varepsilon) \lesssim g_2(\varepsilon)$ means $\frac{g_1(\varepsilon)}{g_2(\varepsilon)} \leqslant h(\varepsilon)$ where $h(\varepsilon) \to 1$ as $\varepsilon \to 0$ and $g_1(\varepsilon) \sim g_2(\varepsilon)$ if $\frac{g_1(\varepsilon)}{g_2(\varepsilon)} \to 1$ as $\varepsilon \to 0$.

2. First return map structure

Recall that $k_0 \geqslant 2$ denotes the minimal integer such that $f^{k_0}(c)$ is periodic with prime period p. From here until the end of Section 3.3 we will focus on the case that $z \in orb(f(c))$ is periodic; for the preperiodic case we make minor adjustments in Section 3.4. Suppose that for $k_1 \geqslant k_0$, $f^{k_1}(c) = z$. Then necessarily, $f^p(z) = z$ with prime period p.

Proposition 2.1. (1) *There exists $\lambda_{per} > 1$ such that if x is periodic of period n then $|Df^n(x)| \geqslant \lambda_{per}^n$.*

(2) *For each neighbourhood U of c there are $K_U > 0$ and $\lambda_U > 1$ such that if J is an interval with $U \cap \left(\cup_{i=0}^{n-1} f^i(J) \right) = \emptyset$ then $\inf_{x \in J} |Df^n(x)| \geqslant K_U \lambda_U^n$.*
(3) *There is a unique acip μ; this is supported on a finite union of intervals, which contain c in the interior.*

The first item here is in, for example [NS, Theorem A], while the second is more generally known as Mañé's Lemma (eg [MS, Theorem III.5.1]). The third item can be found for example as part of [MS, Theorem V.1.3].

Recall that we call a hyperbolic periodic point x of (prime) period n, *orientation preserving/reversing* if f^n preserves/reverses orientation in small neighbourhoods of x (i.e. $Df^n(x) > 0/Df^n(x) < 0$).

Denote the orbit of the critical point by $c_i = f^i(c)$. Then by the definition of k_0 and p above, $\text{orb}(c) = \{c_0, \ldots, c_{k_0+p-1}\}$. Let $\mathcal{A} := \{A_1, \ldots, A_M\}$ for $M = k_0 + p - 1$ be the set of ordered open intervals in $[f^2(c), f(c)]$ with boundary points from $\text{orb}(c)$. This forms a Markov partition for f on $[f^2(c), f(c)]$.[1] The partition induces a dynamical coding on $I' := [f^2(c), f(c)] \setminus \bigcup_{n \geqslant 0} f^{-n}(\text{orb}(c))$: to each $x \in I'$ there is a sequence $(x_0, x_1, \ldots) \in \{1, \ldots, M\}^{\mathbb{N}_0}$ with $f^i(x) \in A_{x_i}$. For each $n \in \mathbb{N}$, we refer to $[x_0, \ldots, x_{n-1}] := \{(y_0, y_1, \ldots) \in \Sigma : y_0 = x_0, \ldots, y_{n-1} = x_{n-1}\}$ as an *n-cylinder*, or *cylinder of depth n*. There will be a unique topologically transitive component: we then remove any component which does not intersect this. Let $\Sigma \subset \{1, \ldots, M\}^{\mathbb{N}_0}$ be the corresponding subshift of finite type, which is moreover locally eventually onto and thus topologically mixing.

We will abuse notation and refer to cylinders in Σ and the intervals in I they represent by $w = [x_0, \ldots, x_n]$. We will write the word $w \in \cup_{n \geqslant 1} \cup_{w' \in \{1, \ldots, M\}^n} w'$ and the corresponding cylinder $[w]$ as just w. Given cylinders $w_1 = [x_0, \ldots, x_n]$ and $w_2 = [y_0, \ldots, y_m]$, the concatenation $[w_1 w_2]$ is the cylinder $[x_0, \ldots, x_n, y_0, \ldots, y_m]$. When needed, we denote by π the projection from Σ to I.

2.1. **The inducing scheme.** For $N \in \mathbb{N}$ to be chosen below, we will take a collection of cylinders $w_1, \ldots, w_K \in \{1, \ldots, M\}^N$ containing z and consider the first return map $F = \sigma^\tau$ to $\Sigma \setminus \{w_1, \ldots, w_K\}$ where τ is the first return time. We consider the domains of this map to be cylinders $w \in \cup_{n \geqslant 1} \cup_{w' \in \{1, \ldots, M\}^n} w'$, rather than unions of these. Let $\Sigma_{N,K}$ denote the set of one-cylinders for F.

Lemma 2.2. *Given $K \in \mathbb{N}$, for any $\eta > 0$ there exists $N = N(\eta, K) \in \mathbb{N}$ such that for all $n \in \mathbb{N}$, $\#\{w \in \Sigma_{M,K} : \tau(w) = n\} = O(e^{\eta n})$.*

Proof. If $w \in \Sigma_{N,K}$ has $\tau(w) = n$, then w is an n-cylinder for σ and so must have the form $[x_0 w_{i_1} w_{i_2} \cdots w_{i_j} x_{n-1}]$ where $j \leqslant n/N$. So we can estimate the number of such cylinders by $K^{\frac{n}{N}}$ (note that this estimate comes from considering (Σ, σ) to be the full shift, which is a substantial over-estimate here). So the lemma is complete if we choose N large enough that $\frac{1}{N} \log K \leqslant \eta$. $\square$

We will use the above partly to ensure our inducing scheme has exponential tails (see Proposition 2.5). We will also choose the scheme to be compatible with the periodic structure at z, see Remark 2.8 below.

Lemma 2.3. *For any $\varepsilon_0, \eta > 0$, there exist $K, N \in \mathbb{N}$ such that $Y = \pi(\Sigma_{N,K})$ has the following properties:*

(a) *$\{f^i(c)\}_{i=1}^{k_0+p-1} \cap Y = \emptyset$; in particular, $z \notin Y$;*
(b) *if $x \in I \setminus Y$ and $i \geqslant 1$ is minimal such that $f^i(x) \in Y$, then $f^i(x) \in (z - \varepsilon_0, z + \varepsilon_0)$;*

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[1]Distortion is unbounded.

(c) $f(Y) \supset Y$;

(d) *let $F = \sigma^\tau$ denote the first return map to $\Sigma_{N,K}$; then $\#\{w \in \Sigma_{N,K} : \tau(w) = n\} = \mathcal{O}(e^{\eta n})$;*

(e) *$F : \Sigma_{N,K} \circlearrowleft$ is topologically mixing.*

Proof. Fix $\varepsilon_0, \eta > 0$. For $\tilde{N} \in \mathbb{N}$, we will choose a pair of $\tilde{N}$-cylinders w_L, w_R which are adjacent to z (to the left and right of this point respectively) and, observe that for $\tilde{N}$ large, $f^i(w_L), f^i(w_R)$ are $(\tilde{N} - i)$-cylinders for $i = 1, \ldots, p - 1$. If $i \in \{1, \ldots, p\}$ is minimal with $z = f^{i+k_0}(c)$ then define $w'_L = f^{p-i}(w_L), w'_R = f^{p-i}(w_R)$ and set $w_{j,L}, w_{j,R}$ to be the $(\tilde{N} - p + i + j)$-cylinders along the orbit segment $\{f(c), \ldots, f^{k_0-1}(c)\}$ mapping to w'_L, w'_R by f^j for $j = 0, \ldots, k_0 - 1$. Then define $Z_{\tilde{N}} := \{w_{j,L}, w_{j,R} : j = 0, \ldots, k_0 - 1\} \cup \{f^i(w_L), f^i(w_R) : i = 0, \ldots, p - 1\}$ be the sets we remove from Σ. The deepest cylinders here are either (i) $w_{k_0-1,L}, w_{k_0-1,R}$; or (ii) w_L, w_R. In case (i), these are N-cylinders, where $N = \tilde{N} - p + i + k_0 - 1$. Since, for example $w_{k_0-1-i,L}$ consists of at most M^i N-cylinders (a significant over-estimate), we have removed at most $K = 2\frac{M^{k_0+p-1}-1}{M-1}$ N-cylinders. So for $\eta > 0$, let $N_0 = N(\eta, K) \in \mathbb{N}$ be as in Lemma 2.2. Set[2] $\Sigma' = \Sigma'(N_0) := \Sigma \setminus Z_{\tilde{N}_0}$ (here $\tilde{N}_0 = N_0 + p + i - k_0 + 1$) and let $F = \sigma^\tau$ be the first return map to Σ'. Lemma 2.2 implies that the number of n-cylinders with first return time n is $O(e^{\eta n})$. Case (ii) follows similarly.

Define $Y = \pi(\Sigma')$. Property (a) of the lemma is obvious. The choice of $\tilde{N}_0$ above guarantees property (d). The definition of $Z_{\tilde{N}_0}$ guarantees that returns to Y must occur in a neighbourhood of z and that $f(Y) \supset Y$, and choosing $\tilde{N}$ sufficiently large forces this neighbourhood to intersect $(z - \varepsilon_0, z + \varepsilon_0)$, so (b) and (c) hold. Finally, (e) follows from the mixing of σ. $\qquad \square$

We will make our final choice of Y once we choose ε_0 before Remark 2.8. We will abuse notation and let $F = f^\tau$ denote the first return map to Y as well.

Let $\{I_i\}_i$ be the domains (intervals) of monotonicity of the first return map and for brevity, write $\tau_i = \tau|_{I_i}$. Note that each I_i is associated to a cylinder as in the symbolic model.

Lemma 2.4. *There exists $P \in \mathbb{N}_0$ such that I_i contains a periodic point of prime period $\tau_i + k$ with $k \leqslant P$.*

Proof. Since I_i corresponds to some n-cylinder $[x_0 w_{i_1} w_{i_2} \cdots w_{i_j} x_{n-1}]$, the lemma is saying that there is a chain of allowable transitions $x_{n-1} \mapsto y_1 \mapsto y_{k-1} \mapsto x_0$, for $k \leqslant P$, which follows since the shift is an SFT. $\qquad \square$

Proposition 2.5.   (a) *F has finitely many images, i.e., $\{F(I_i)\}_i$ is a finite set;*

(b) *(bounded distortion) there exists $C_d > 0$ such that if x, y belong to the same n-cylinder for F, then*

$$\left| \frac{DF^n(x)}{DF^n(y)} - 1 \right| \leqslant C_d |F^n(x) - F^n(y)|;$$

(c) *$|\{\tau = n\}| = O\left(e^{-n(\log \lambda_{per} - \eta)}\right)$, where $|\cdot|$ denotes the Lebesgue measure of the set.*

Proof. Property (a) follows from the SFT structure and Property (b) follows from the Koebe Lemma added to the fact that we removed a neighbourhood of the orbit of the critical value to create Y.

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[2] Observe that we are removing a neighbourhood of the orbit of the critical *value*: the critical point is not removed.

To prove (c), by Lemma 2.4, given I_i, there is a periodic point $x \in I_i$ of period $\tau_i + k$ for $0 \leqslant k \leqslant P$. Thus, by Proposition 2.1(1),

$$|Df^{\tau_i}(x)| \geqslant \lambda_{per}^{\tau_i+k}|Df|_\infty^{-k} \geqslant \left(\frac{\lambda_{per}}{|Df|_\infty}\right)^P \lambda_{per}^{\tau_i} = \tilde{K}\lambda_{per}^{\tau_i}. \tag{2.1}$$

Since we also have bounded distortion, we see that $|I_i| = O(\lambda_{per}^{-\tau_i})$ and hence applying Lemma 2.2, $|\{\tau = n\}| = O\left(e^{-n(\log\lambda_{per}-\eta)}\right)$, as required. □

In the light of this result, we will assume $\eta \in \left(0, \frac{1}{2}\log\lambda_{per}\right)$ from here on (we make a final choice for η before Remark 2.8.

Remark 2.6. *The cylinder structure here means that* $\left(\cup_{k\geqslant 1}f^k(\partial Y)\right) \cap (Int(Y^c) \setminus orb(f(c))) = \emptyset$. *In particular supposing that I_i has $f^k(I_i) = (x,y)$ in a neighbourhood of z and $y < z$, then there will be a set of domains adjacent to I_i such that the closure of the union of their f^k iterates covers (x,z). (Similarly if $f^k(I_i)$ lies to the right of z.)*

Recall that μ is the unique acip for f according to Proposition 2.1. Define $\mu_Y := \frac{\mu|_Y}{\mu(Y)}$, which by Kac's Lemma is an F-invariant probability measure. We can recover μ from μ_Y by the well-known formula,

$$\mu(A) = \mu(Y)\sum_i \sum_{j=0}^{\tau_i-1} \mu_Y(I_i \cap f^{-j}(A)), \quad \text{for } A \subset I. \tag{2.2}$$

Note that f^{k_1} in a neighbourhood of c is a 2-to-1 map composed with a diffeomorphism. The Hartman-Grobman Theorem implies that f^p restricted to any small enough neighbourhood of z, the linearisation domain, is conjugated to the transformation $x \mapsto \lambda_z x$ and is indeed asymptotic to this for x close to z. Let $\varepsilon_0 > 0$ be such that this theorem holds in $(z - \varepsilon_0, z + \varepsilon_0)$.

For $\varepsilon \leqslant \varepsilon_0$, let $\delta = \delta(\varepsilon)$ be such that $f^{k_1}|_{(c-\delta,c+\delta)}$ is 2-to-1 onto either $(z - \varepsilon, z]$ or $[z, z + \varepsilon)$. Set $\lambda_\delta := \lambda_{(c-\delta(\varepsilon_0),c+\delta(\varepsilon_0))}$ and $K_\delta := K_{(c-\delta(\varepsilon_0),c+\delta(\varepsilon_0))}$ from Proposition 2.1. Then choose $\eta > 0$ such that $\eta < \min\{\frac{1}{2}\log\lambda_{per}, \log\lambda_\delta\}$. Now with ε_0 and η fixed, we choose $N_1 \geqslant N(\eta, K)$ so that the cylinders making up $Z_{\tilde{N}_1}$ from the proof of Lemma 2.3 are contained in $\cup_{i=1}^p f^i(z - \varepsilon_0, z + \varepsilon_0)$ and adjust our inducing domain to $\Sigma' = \Sigma'(N_1)$, thus also adjusting Y.

Definition 2.7. *With ε_0, η and N_1 fixed as above, we make our final choice of Y, which will remain fixed from here on.*

With Y fixed, we choose $\varepsilon_1 \in (0, \varepsilon_0]$ so that $Y \cap \left(\cup_{i=1}^p f^i(z - \varepsilon_1, z + \varepsilon_1)\right) = \emptyset$.

Remark 2.8.   (1) *Our choice of cylinders around $\mathrm{orb}(z)$ means that any domains of the first return map to Y can enter $(z - \varepsilon_1, z + \varepsilon_1)$ at most once, since the linear structure of the dynamics in this region means that any domain that does so must be 'spat out' into Y, and thus have made a return, before it can escape the linearisation domain.*
    (2) *Moreover, due to the periodic way our cylinders $Z_{\tilde{N}}$ were chosen, if $f^k(I_i)$ for $k < \tau_i$ is in the neighbourhood of z, then it cannot return to Y in less than p steps. Indeed, $\tau_i - k$ is a multiple of p.*

It will be convenient to treat the left and right neighbourhoods of z separately. For this purpose, we introduce the following notation for asymmetric holes. Given $\varepsilon_L, \varepsilon_R > 0$, we write $\boldsymbol{\varepsilon} = \{\varepsilon_L, \varepsilon_R\}$ and define the corresponding hole at z by $H_{\boldsymbol{\varepsilon}}(z) = (z - \varepsilon_L, z + \varepsilon_R)$. We will abuse notation in the following ways: we write $\boldsymbol{\varepsilon} = 0$ to mean $\varepsilon_L = \varepsilon_R = 0$, and we write $\boldsymbol{\varepsilon} < \varepsilon$ (also $\boldsymbol{\varepsilon} \in (0, \varepsilon)$) to mean

$\varepsilon_L, \varepsilon_R < \varepsilon$ (and $\varepsilon_L, \varepsilon_R \in (0, \varepsilon)$). We extend the definition of $\delta(\varepsilon)$ to $\delta(\boldsymbol{\varepsilon})$ in the natural way, noting that for example in the orientation preserving case $\delta(\boldsymbol{\varepsilon})$ will only depend on one of ε_L and ε_R.

For notation, we will denote by $(z - \varepsilon_L)_i$, $(z + \varepsilon_R)_i$ the local inverse of f^{ip} applied to $z - \varepsilon_L$, $z + \varepsilon_R$.

For a hole $H_{\boldsymbol{\varepsilon}}(z)$, we denote by $H'_{\boldsymbol{\varepsilon}}$ the set of intervals J in Y such that $f^s(J) \subset H_{\boldsymbol{\varepsilon}}$ for $s < \tau|_J$. Thus $H'_{\boldsymbol{\varepsilon}}$ represents the 'induced hole' for the first return map F. In fact, due to the construction here, each such J will be one of the domains I_i. The main expression we must estimate is $\frac{\mu(H'_{\boldsymbol{\varepsilon}})}{\mu(H_{\boldsymbol{\varepsilon}})}$ for appropriately chosen $\varepsilon_L, \varepsilon_R$: we will show how this allows us to estimate $\frac{\mu(H'_{\boldsymbol{\varepsilon}})}{\mu(H_{\varepsilon})}$ for all sufficiently small $\varepsilon > 0$ in Section 5.

2.2. Chain structure.

Let the interval of $I \setminus Y$ containing z be denoted by $[a, b]$. Now suppose I_i is a domain of Y with $f^s(I_i) \subset [a, b]$ with $1 \leqslant s \leqslant \tau_i$. Since there is an interval U closer to z than $f^s(I_i)$ with $f^p(U) = f^s(I_i)$, then by the first return structure, there must be some $I_{i'}$ with $f^s(I_{i'}) = U$. Iterating this, we call the resulting domains a *chain*. We consider I_i here to have some *depth* k, and the $I_{i'}$ above to have depth $k + 1$ and write these as I_i^k and I_i^{k+1} respectively. For each chain there is an 'outermost' interval which we consider to have depth 1: this is the interval $I_{i''} = I_i^1$, a domain in Y where $f^s(I_{i''}) \subset [a, b]$, but $f^{s+p}(I_{i''})$ not contained in $[a, b]$. Thus chains are denoted $\{I_i^k\}_{k=1}^\infty$ and $f^{s+kp}(I_i^k) = f^s(I_i^1)$.

Note that the cylinder structure implies $f^p(a), f^p(b) \notin (a, b)$.

Lemma 2.9. (1) If f^p is orientation preserving around z then two elements I_i^k and $I_i^{k'}$ of a chain have depths differing by 1 if and only if they are adjacent to each other. Here $f^{s+p}(I_i^{k+1}) = f^s(I_i^k)$ for s as above.

(2) If f^p is orientation reversing around z then a chain $\{I_i^k\}_k$ has a neighbouring chain $\{I_j^k\}_k$ so that two elements I_i^k and $I_j^{k'}$ have depths differing by 1 if and only if they are adjacent to each other. Here the $I_i^k, I_j^{k+1}, I_i^{k+2}$ are adjacent intervals and moreover $f^{2p}(I_i^{k+2}) = I_i^k$.
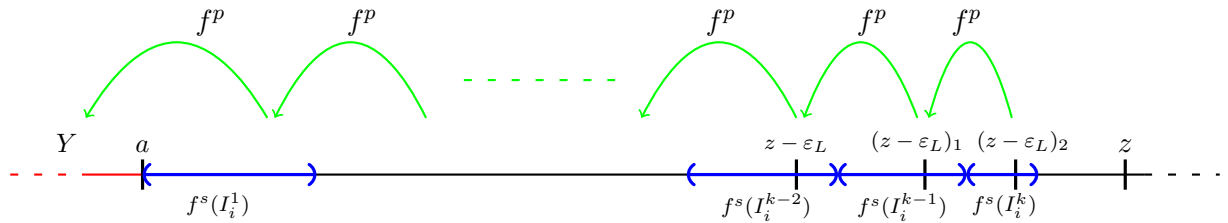
See Figure 1 for a sketch of case (1) of the lemma.

FIGURE 1. The chain structure mapped near z. We are assuming that f^p is locally orientation preserving and focussing attention on the left-hand side of z. We sketch some intervals of the images (by f^s) of the chain $\{I_i^k\}_{k \geqslant 1}$. Note that here $\tau|_{I_i^k} = s + k$.

Proof. We first assume that $\{I_i^k\}_k$ is a chain and that f^p is locally orientation preserving around z. Suppose that I_j^k is adjacent to I_i^1. Suppose for some $t \geqslant 1$ we have $f^{t-1}(I_i^1) \cap Y = \emptyset$, but $f^t(I_i^1) \subset Y$. As in Remark 2.8, according to our choice of $Z_{\tilde{N}_0}$ in the proof of Lemma 2.3, $f^t(I_i^1)$ can only reenter Y in an ε_0-neighbourhood of z. Note that $f^t(I_i^1)$ must contain either a or b in its boundary: we assume this is a. We observe that a is also a boundary point of $f^t(I_j^k)$ and $f^t(I_j^k) \cap Y = \emptyset$. But then we must have $f^{t+1}(I_j^k) \subset Y$, and moreover this domain must contain a

in its boundary. Indeed, by the first return structure it must coincide with $f^t(I_i^1)$. But this implies that actually $I_j^k = I_i^2$. This means that the left boundary point of $f^{t-1}(I_i^1)$ must map by f^p to the other boundary point a. In the same way we see that all the domains $\{I_i^k\}_k$ are adjacent to each other and have f^p mapping adjacent f^s-images of the chain to each other.

In the orientation reversing case, suppose that I_i^1 is adjacent to I_j^k. Then $f^{s+p}(I_i^1) \subset Y$. The Markov structure implies that $f^{s+p}(I_j^k) \subset [a,b]$ and that these intervals both contain either a or b in their boundary, which means that $f^{s+2p}(I_j^k) \subset Y$ and so $k = 2$. By the linear structure of f^{2p} in this region, if I_ℓ^k is adjacent to I_j^2, then $f^{s+2p}(I_\ell^k) = f^s(I_i^1)$, so $I_\ell^k = I_i^3$. To prove this starting with I_i^k with $k \geqslant 1$, we iterate the argument by $f^{(k-1)p}$. The mapping of the f^s-images to each other by f^{2p} follows as in the orientation preserving case. $\square$

In case (2) above we call $\{I_i^k\}_k$ and $\{I_j^k\}_k$ *alternating chains*. We also call any $\{I_i^k\}_{k \geqslant j}$, for some $j \geqslant 1$, a *subchain*.

Adding the result of this lemma to the first return structure we see that the images in $[a,b]$ of any chain must be of the same fixed form, i.e., if $\{I_i^k\}_k$, $\{I_j^k\}_k$ are two chains which map by f^s and f^t respectively into $[a,b]$, then assuming $f^s(I_i^k)$ and $f^t(I_j^k)$, for some depth $k \geqslant 1$, are on the same side of z then these images coincide. Moreover, $\overline{\cup_k f^s(I_i^k)}$ will be either $[a,z]$ or $[z,b]$ if f^p is orientation preserving around z, and $\overline{\cup_k f^s(I_i^k \cup I_j^k)} = [a,b]$ if f^p is orientation reversing around z, and the alternating chains are as in the statement of the lemma.

2.3. Extra cuts for a 'Markovian' property.

Suppose that f^p is orientation preserving around z. Let $a_0 = a$ and then inductively let $f^p(a_{i+1}) = a_i$, so that $a_i \to z$ as $i \to \infty$. We similarly define $(b_i)_i$ to the right of z, accumulating on z. Observe that domains I_i^k of F which map into $[a,b]$ by some f^s with $1 \leqslant s < \tau_i$ must map onto some (a_{k-1}, a_k) or (b_k, b_{k-1}).

In the orientation reversing case we define $a_0 = a$, and a_1 such that $f^p(b) = a_1$ and inductively define a_i by $f^{2p}(a_{i+2}) = a_i$. Similarly we define $(b_i)_i$. Here if $f^s(I_i^k) \subset [a,b]$ for $1 \leqslant s < \tau_i$ then $f^s(I_i^k) \in \{(a_{k-1}, a_k), (b_k, b_{k-1})\}$.

Note that by bounded distortion, the ratios $\frac{z-a_i}{b_i-z}$ will be comparable to $\frac{z-a}{b-z}$.

The above implies that if we were to take our holes around z as, say (a_i, b_j), these would be Markov holes in the sense that either an interval of the inducing scheme enters the hole before returning to Y, or it never intersects it. Note that by topological transitivity the only way z cannot be accumulated by domains on both the left and the right as above is if z is a boundary point of the dynamical core $[f^2(c), f(c)]$, a simpler case which needs only obvious modifications.

For generic $\varepsilon_L, \varepsilon_R \in (0, \varepsilon_1)$ the corresponding hole will not have this Markov property, so we make an extra cut here, namely if $z - \varepsilon_L \in f^s(I_i)$ (in the sketch in Figure 1, $I_i = I_i^{k-2}$) then we split I_i into two so that one of the resulting intervals maps by f^s into the hole and one maps outside, and similarly for $z + \varepsilon_R$, thus defining F_ε from $F = F_0$. For this map, in the generic case, the chains no longer have common boundary points, but now, in the orientation preserving case, they come in pairs $\{I_i^k\}_k, \{\hat{I}_i^k\}$, where $I_i^{k+1} \overline{\cup} \hat{I}_i^k$ is a domain of F, and $\overline{\cup}$ denotes the union of the open intervals along with their common boundary point.

In the case f^p is orientation reversing around z, we may need to cut a domain of F twice to produce F_ε. For simplicity, we will not address this case here, but all the ideas below go through similarly. From now on, unless indicated otherwise, when we discuss our induced map we mean F_ε.

Remark that due to the extra cuts, although the domains of F_{ε} have the property that each one either maps into $H_{\varepsilon}(z)$ or never intersects it, for generic holes this does not define a countable Markov partition for F_{ε}. This is because $F_{\varepsilon}(I_i)$ may not be a union of 1-cylinders if a boundary point of I_i is created by one of the extra cuts.

We next observe that in the orientation preserving case there is a unique pair of subchains we denote $\{I_L^k\}_{k \geqslant k_{L,\varepsilon}}, \{\hat{I}_L^k\}_{k \geqslant k_{L,\varepsilon}} \subset (c - \delta(\varepsilon), c)$ and a unique pair of subchains $\{I_R^k\}_{k \geqslant k_{R,\varepsilon}}, \{\hat{I}_R^k\}_{k \geqslant k_{R,\varepsilon}} \subset (c, c + \delta(\varepsilon))$. We call these the *principal subchains*. Note $\overline{\cup_{k \geqslant k_{L,\varepsilon}} \left(I_L^k \cup \hat{I}_{I_L}^k \right)} \cup \left(\cup_{k \geqslant k_{R,\varepsilon}} I_R^k \cup \hat{I}_{I_R}^k \right) = [c - \delta(\varepsilon), c + \delta(\varepsilon)]$. We can naturally extend this idea to the orientation reversing case. In either case, $\delta(\varepsilon)$ is determined by only one of ε_L or ε_R, depending on whether $f^{k_1}(c - \delta(\varepsilon), c + \delta(\varepsilon)) = (z - \varepsilon_L, z]$ or $f^{k_1}(c - \delta(\varepsilon), c + \delta(\varepsilon)) = [z, z + \varepsilon_R)$. Let us call ε_* this value in $\{\varepsilon_L, \varepsilon_R\}$.

3. Measuring the holes

The aim of this section is to estimate $\mu(H_{\varepsilon})$, $\mu(H'_{\varepsilon})$ and thus $\frac{\mu(H'_{\varepsilon})}{\mu(H_{\varepsilon})}$.

3.1. The contribution from principal subchains.

In order to estimate $\mu(H'_{\varepsilon})$ and $\mu(H_{\varepsilon})$, we first estimate the contribution from the principal subchains.

Lemma 3.1. *For x close to c,*

$$|x - c| \sim \left(\frac{|f^{k_1}(x) - z|}{A|Df^{k_1 - 1}(f(c))|} \right)^{\frac{1}{\ell}}.$$

Proof. By the Mean Value Theorem there exists $\theta \in (f(x), f(c))$ such that $|f^{k_1}(x) - f^{k_1}(c)| = Df^{k_1 - 1}(\theta)|f(x) - f(c)|$. Then,

$$\frac{|f^{k_1}(x) - f^{k_1}(c)|}{|x - c|} = \frac{|f(x) - f(c)|}{|x - c|} \frac{|f^{k_1}(x) - f^{k_1}(c)|}{|f(x) - f(c)|} = \frac{A|x - c|^{\ell}}{|x - c|}|Df^{k_1 - 1}(\theta)|,$$

where A is from Section 1. The result follows using $|Df^{k_1 - 1}(\theta)| \sim |Df^{k_1 - 1}(f(c))|$ by bounded distortion. $\square$

We can see from Lemma 3.1 that $\left| \overline{\cup_{k \geqslant k_{L,\varepsilon}} \left(I_L^k \cup \hat{I}_L^k \right)} \right| \sim C_p(\varepsilon_*)^{\frac{1}{\ell}}$ where $C_p := \frac{1}{(A|Df^{k_1 - 1}(f(c))|)^{\frac{1}{\ell}}}$, i.e. $\delta(\varepsilon) \sim C_p(\varepsilon_*)^{1/\ell}$. Since c is bounded away from the critical orbit, as in [M, Theorem 6.1] (see also [N, Theorem A]) the invariant density is continuous at c and defining $\rho := \frac{d\mu}{dm}(c)$, the μ_Y-measure of the set can be estimated by,

$$\sum_{k \geqslant k_{L,\varepsilon}} \mu_Y(I_L^k \cup \hat{I}_L^k) \sim \frac{C_p \rho}{\mu(Y)}(\varepsilon_*)^{\frac{1}{\ell}}. \tag{3.1}$$

Similarly for $\sum_{k \geqslant k_{L,\varepsilon}} \mu_Y(I_R^k \cup \hat{I}_R^k)$. As in (2.2), the sum of these two quantities scaled by $\mu(Y)$ is the relevant contribution to $\mu(H'_{\varepsilon})$.

Now for the contribution to $\mu(H_{\varepsilon})$, let us suppose that f^p is locally orientation preserving at z and $f^{k_1 - 1}$ is also orientation preserving at $f(c)$ (which implies that $\varepsilon_* = \varepsilon_L$), the other cases follow

similarly. Denote $J^k = \cup_{k'\geqslant k} I_L^{k'} \bar{\cup} \hat{I}_L^{k'}$. Then $f^{k_1}(J^{k_{L,\varepsilon}+k}) = ((z - \varepsilon_L)_{k-1}, z)$ so using Lemma 3.1 again, we have

$$\mu_Y(J^{k_{L,\varepsilon}+k}) \sim \frac{\rho C_p}{\mu(Y)}|(z - \varepsilon_L)_k - z|^{\frac{1}{\ell}} \sim \frac{\rho C_p}{\mu(Y)}\lambda_z^{-\frac{k}{\ell}}\varepsilon_L^{\frac{1}{\ell}} = \frac{\rho C_p}{\mu(Y)}\lambda_z^{-\frac{k}{\ell}}\varepsilon_*^{\frac{1}{\ell}}.$$

Then using (2.2), we estimate the (scaled) contribution to $\mu(H_\varepsilon)$ from the left of c by,

$$\sum_{k\geqslant k_{L,\varepsilon}} k\mu_Y(I_L^k \cup \hat{I}_L^k) = \sum_{k\geqslant k_{L,\varepsilon}} \sum_{k'\geqslant k} \mu_Y(I_L^{k'} \cup \hat{I}_L^{k'}) = \sum_{k\geqslant k_{L,\varepsilon}} \mu_Y(J^k)$$

$$\sim \frac{\rho C_p}{\mu(Y)}\varepsilon_*^{\frac{1}{\ell}} \sum_{k\geqslant 0} \lambda_z^{-\frac{k}{\ell}} = \frac{\rho C_\rho \varepsilon_*^{\frac{1}{\ell}}}{\mu(Y)(1 - \lambda_z^{-\frac{1}{\ell}})}. \tag{3.2}$$

An identical estimate follows for the domains $\{I_R^{k'}, \hat{I}_R^{k'}\}_{k'\geqslant k}$ from the right of c.

3.2. The contribution from non-principal subchains.

Now for the subchains that are not in $(c - \delta(\varepsilon), c + \delta(\varepsilon))$, but which map into H_ε before returning to Y, we will use a different, rougher, type of estimate. First notice that such subchains cannot be contained in $(c - \delta(\varepsilon_0), c + \delta(\varepsilon_0))$ since if $I_i \subset (c - \delta(\varepsilon_0), c + \delta(\varepsilon_0)) \setminus (c - \delta(\varepsilon), c + \delta(\varepsilon))$, then the repelling structure of our map around z means that I_i will return to Y before mapping into H_ε. With this in mind, recall $\lambda_\delta := \lambda_{(c-\delta(\varepsilon_0), c+\delta(\varepsilon_0))}$ and $K_\delta := K_{(c-\delta(\varepsilon_0), c+\delta(\varepsilon_0))}$ defined after Remark 2.6 when we fixed the definition of Y.

We will deal with the orientation preserving case here, the orientation reversing case is similar. Suppose that $\{I_i^k\}_{k\geqslant k_i}$, $\{\hat{I}_i^k\}_{k\geqslant k_i}$ is a pair of non-principal subchains such that $f^s(I_i^k \cup \hat{I}_i^k) \subset H_\varepsilon$ for any $k \geqslant k_i$ (so this is the first time that any element of the subchain enters H_ε). For $x \in I_i^k$, by the Mean Value Theorem, there is $x \in I_i^k$ such that

$$|I_i| = |Df^s(x)|^{-1}|f^s(I_i^k)| \leqslant K_\delta^{-1}\lambda_\delta^{-s}|f^s(I_i^k)|$$

by Proposition 2.1. So since $f^s\left(\cup_{k\geqslant k_i}(I_i^k \cup \hat{I}_i^k)\right)$ covers either $(z - \varepsilon_L, z)$ or $(z, z + \varepsilon_R)$, we thus obtain an analogue of (3.1): for $\varepsilon' = \max\{\varepsilon_L, \varepsilon_R\}$,

$$\mu(Y) \sum_{k\geqslant k_i} \mu_Y(I_i^k \cup \hat{I}_i^k) \leqslant K_\delta^{-1}\lambda_\delta^{-s}\varepsilon'.$$

We know that there are $O(e^{s\eta})$ domains I_j which have $f^s(I_j) = f^j(I_i)$, so, also accounting for the intervals which map to the other side of z, the total measure of non-principal chains contributing to $\mu(H'_\varepsilon)$ can be estimated by

$$2\sum_{s\geqslant 1} K_\delta^{-1} e^{s(\eta - \log\lambda_\delta)}\varepsilon' = O(1)\varepsilon', \tag{3.3}$$

and recall we have chosen $\eta \in (0, \log\lambda_\delta)$ in our definition of Y. Similarly to (3.2) we can obtain an analogous estimate, also $O(1)\varepsilon'$, for the contribution to $\mu(H_\varepsilon)$.

3.3. The limiting ratio in the periodic postcritical case.

We will assume that $\frac{\varepsilon_L}{\varepsilon_R}$ is uniformly bounded away from 0 and infinity (so in particular $\varepsilon_L, \varepsilon_R = O(\varepsilon_*)$), so that the density spike dominates the measure of our holes. Now recalling (2.2), (3.1) together with (3.3) imply

$$\mu(H'_\varepsilon) \sim 2C_p\rho(\varepsilon_*)^{\frac{1}{\ell}} + O(1)\varepsilon_*$$

while (3.2) and the analogue of (3.3) imply

$$\mu(H_\varepsilon) \sim 2C_p\frac{\rho(\varepsilon_*)^{\frac{1}{\ell}}}{1 - \lambda_z^{-\frac{1}{\ell}}} + O(1)\varepsilon_*,$$

so the two principal chains dominate this estimate and the limiting ratio is

$$\lim_{\varepsilon \to 0} \frac{\mu(H'_\varepsilon)}{\mu(H_\varepsilon)} = 1 - \lambda_z^{-\frac{1}{\ell}} . \tag{3.4}$$

3.4. **The preperiodic postcritical case.** For the case that $z = f^{k_1}(c)$ for $1 \leqslant k_1 < k_0$, we choose the inducing scheme Y via a minor adaptation of the method described in the proof of Lemma 2.3: properties (a) and (d) of that lemma will follow here along with the condition (b'): If $x \in I \setminus Y$ and i is minimal such that $f^i(x) \in Y$ then $f^i(x) \in (f^{k_0+p-1}(c) - \varepsilon_0, f^{k_0+p-1}(c) + \varepsilon_0)$ for some small $\varepsilon_0 > 0$. In the notation of the proof of that lemma we choose w_L and w_R adjacent to z (or just one of these if $z \in \{f(c), f^2(c)\}$ is one of the endpoints of I) and then choose the rest of the cylinders around orb$(f(c), Z_{\tilde{N}} := \{w_{j,L}, w_{j,R} : j = 0, \ldots, k_1-1\} \cup \{f^i(w_L), f^i(w_R) : i = 0, \ldots, k_0+p-k_1-1\}$ be the sets we remove from Σ, where $f^{k_1-j}(w_{j,L}) = w_L$ and $f^{k_1-j}(w_{j,R}) = w_R$. If w_L and w_R are chosen small enough then properties (a), (c) and (d) of Lemma 2.3 hold, as does Proposition 2.5.

This construction also guarantees for small enough $\varepsilon > 0$ each I_i passes at most once through $(z - \varepsilon_L, z + \varepsilon_R)$ before returning to Y, ensuring that $\mu(H'_\varepsilon) = \mu(H_\varepsilon)$. Moreover, note that an I_i which does pass through $(z - \varepsilon_L, z + \varepsilon_R)$, must also pass through a neighbourhood of $\{f^{k_1+1}(c), \ldots, f^{k_0+p-1}(c)\}$ before returning to Y.

4. FUNCTIONAL FRAMEWORK FOR β-ALLOWABLE HOLES

Our goal in this section is to formalise a functional framework for the transfer operator corresponding to the induced map F_ε and its punctured counterpart $\mathring{F}_\varepsilon$. In order to do this, we will work with a fixed higher iterate n_0 of the induced map and formulate a classification of holes depending on the minimal length of images of n_0-cylinders under $\mathring{F}_\varepsilon^{n_0}$ (see the definition of β-allowable in Definition 4.2). Using this control, we prove the punctured transfer operator enjoys a spectral in a space of functions of bounded variation (Theorem 4.7) and use it to prove a local escape rate for $\mathring{F}_\varepsilon$ (Lemma 4.9). We then use this prove the local escape rate for f needed for Theorem 1.1 via (4.16) by computing the limits in (4.17).

We begin by formally defining the induced open system.

Recall that given a hole $H_\varepsilon(z) = (z - \varepsilon_L, z + \varepsilon_R)$, the induced hole H'_ε for $\mathring{F}_\varepsilon$ is the collection of intervals that enter $H_\varepsilon(z)$ before returning to Y. Define the open system for $n \geqslant 1$ by

$$\mathring{F}_\varepsilon^n = F_\varepsilon^n|_{\mathring{Y}_\varepsilon^n}, \quad \text{where} \quad \mathring{Y}_\varepsilon^n := \cap_{i=0}^{n-1} F_\varepsilon^{-i}(Y \setminus H'_\varepsilon) , \tag{4.1}$$

i.e. the open system at time n is the induced map restricted to the set of points that have not entered H'_ε before time n. Note that $\mathring{Y}_\varepsilon^0 = Y$ and $\mathring{Y}_\varepsilon^1 = Y \setminus H'_\varepsilon$.

We first prove the following fact about the map F_ε.

Lemma 4.1. *For all $n \geqslant 0$, $|DF_\varepsilon^n(x)| \geqslant \tilde{K}\lambda_{per}^{\tau^n(x)} \geqslant \tilde{K}\lambda_{per}^n$, where $\tilde{K} > 0$ is from (2.1).*

Proof. This follows as in the proof of Proposition 2.5, using Lemma 2.4 to find a point of the right period. □

Using Lemma 4.1, we choose $n_0 \geqslant 1$ so that

$$\inf_x |DF_\varepsilon^{n_0}(x)| > 3. \tag{4.2}$$

Next we define a parameter to keep track of the minimum size of images of n_0-cylinders under the punctured map $\mathring{F}^{n_0}_{\boldsymbol{\varepsilon}}$.

Definition 4.2. *Let $\beta \in (0, 1/2)$ and let $\{J_i\}_i$ denote the set of one cylinders for $F^{n_0}_{\boldsymbol{\varepsilon}}$ (n_0-cylinders for $F_{\boldsymbol{\varepsilon}}$).*

- *For an interval $J = (x, y)$, we say that $t \in J$ is β-deep in J if $t \in [x + \beta|J|, y - \beta|J|]$.*
- *For our holes, we say that $\boldsymbol{\varepsilon}$ is β-left-allowable if there is a domain J_i of $F^{n_0}_{\boldsymbol{\varepsilon}}$ with $f^s(J_i) \subset (z - \varepsilon_1, z)$ with $1 \leqslant s < \tau_i$ and $z - \varepsilon_L$ β-deep in $f^s(J_i)$. We similarly define β-right-allowable with respect to $(z, z + \varepsilon_1)$. In case $\boldsymbol{\varepsilon}$ satisfies both of these conditions we call it β-allowable.*

The property of β-allowable is important for the following reason.

Lemma 4.3. *(Large images depending on β). Let n_0 be chosen as in (4.2). For each $\beta \in (0, 1/2)$, there exists $C_\beta > 0$ such that if $H_{\boldsymbol{\varepsilon}}$ is β-allowable, then $|F^{n_0}_{\boldsymbol{\varepsilon}}(J_i)|, |\mathring{F}^{n_0}_{\boldsymbol{\varepsilon}}(J_i)| \geqslant C_\beta$ for all n_0-cylinders J_i.*

Proof. The property follows immediately from the Markov structure of $F^{n_0}_{\boldsymbol{\varepsilon}}$ together with the assumption of β-admissible. Note that without extra cuts due to the boundary of the hole, the minimum length of the image of any one-cylinder for $F^{n_0}_0$ is bounded below away from 0 by a number depending only on $\Sigma'(N_0)$. Next, by definition of $F^{n_0}_{\boldsymbol{\varepsilon}}$, the intervals that are cut by the boundary of the hole are such that $z - \varepsilon_L$ and $z + \varepsilon_R$ are β-deep, by assumption. Thus the length of the image under $F^{n_0}_{\boldsymbol{\varepsilon}}$ is determined by the parameter β, together with the distortion constant for $F_{\boldsymbol{\varepsilon}}$. Given our choice of cuts depending on the boundary of the hole, the set of images for $\mathring{F}^{n_0}_{\boldsymbol{\varepsilon}}$ is simply a subset of the set of images for $F^{n_0}_{\boldsymbol{\varepsilon}}$, so the property holds equally for $\mathring{F}^{n_0}_{\boldsymbol{\varepsilon}}$. $\square$

4.1. **A uniform spectral gap for β-allowable holes.** In this section we show that for each fixed $\beta > 0$ and in any set of $\varepsilon_L, \varepsilon_R > 0$ that are β-allowable, the transfer operators associated to $\mathring{F}_{\boldsymbol{\varepsilon}}$ and its punctured counterpart have a uniform spectral gap when acting on functions of bounded variation in Y.

Given a measurable function $\psi : Y \to \mathbb{R}$, define the *variation* of ψ on an interval $J \subset Y$ (or a finite collection of intervals $J \subset Y$) by

$$\bigvee_J \psi = \sup_{x_0 < x_1 < \cdots < x_N} \sum_{k=1}^N |\psi(x_k) - \psi(x_{k-1})|, \tag{4.3}$$

where $\{x_k\}_{k=0}^N$ is the set of endpoints of a partition of J into N intervals, and the supremum ranges over all finite partitions of J. Define the BV norm of ψ by,

$$\|\psi\|_{BV} = \bigvee_Y \psi + |\psi|_{L^1(m)},$$

where m denotes Lebesgue measure on Y. Let[3] $BV(Y)$ denote the set of functions $\{\psi \in L^1(m) : \|\psi\|_{BV} < \infty\}$.

We shall study the action of the transfer operators associated with $F_{\boldsymbol{\varepsilon}}$ and $\mathring{F}_{\boldsymbol{\varepsilon}}$ acting on $BV(Y)$. For $\psi \in L^1(m)$ Define

$$\mathcal{L}_{\boldsymbol{\varepsilon}} \psi(x) = \sum_{y \in F_{\boldsymbol{\varepsilon}}^{-1} x} \frac{\psi(y)}{|DF_{\boldsymbol{\varepsilon}}(y)|}, \quad \text{and for each } n \geqslant 0, \ \mathring{\mathcal{L}}^n_{\boldsymbol{\varepsilon}} \psi = \mathcal{L}^n_{\boldsymbol{\varepsilon}}(1_{\mathring{Y}^n_{\boldsymbol{\varepsilon}}} \psi).$$

---

[3]By the variation of $\psi \in L^1(m)$, we mean the essential variation, i.e. $\bigvee_Y \psi = \inf_g \bigvee_Y g$, where the infimum ranges over all functions g in the equivalence class of ψ.

We do not claim, or need, that $1_{\mathring{Y}_\varepsilon^n} \in BV(Y)$.

Remark 4.4. *Note that for each $x \in Y$, $F_\varepsilon(x) = F_0(x)$. This is easy to see since F_ε simply introduces extra cuts at the boundary of $H_\varepsilon(z)$, but does not change the orbit of x, while F_0 introduces no extra cuts apart from those introduced in the original definition of Y. Thus the 1-cylinders for F_ε and F_0 differ slightly (those for F_ε can only be smaller), but pointwise the definition of the maps is the same.*

Our first result proves a uniform set of Lasota-Yorke inequalities for $\mathring{\mathcal{L}}_\varepsilon^{n_0}$, depending only on β.

Let $\mathring{\mathcal{I}}_\varepsilon$ denote the (countable) collection of one-cylinders for $\mathring{F}_\varepsilon^{n_0}$, and let $\mathring{\mathcal{J}}_\varepsilon$ denote the finite set of images of elements of $\mathring{\mathcal{I}}_\varepsilon$.

Proposition 4.5. *For any $\beta > 0$ and any β-allowable hole $H_\varepsilon(z)$, for all $\psi \in BV(Y)$ and all $k \geqslant 0$,*

$$\bigvee \mathring{\mathcal{L}}_\varepsilon^{kn_0}\psi \;\leqslant\; (\tfrac{2}{3})^k \bigvee_Y \psi + (1 + C_d)(C_d + 2C_\beta^{-1}) \sum_{j=0}^{k-1} (\tfrac{2}{3})^j \int_{\mathring{Y}_\varepsilon^{n_0(k-j)}} |\psi|\, dm\,, \tag{4.4}$$

$$\int_Y |\mathring{\mathcal{L}}_\varepsilon^k \psi|\, dm \;\leqslant\; \int_{\mathring{Y}_\varepsilon^k} |\psi|\, dm\,. \tag{4.5}$$

Proof. The second inequality follows immediately from the definition of $\mathring{\mathcal{L}}_\varepsilon$, so we will prove the first. In fact, we will prove the inequality for $k = 1$, which can then be iterated trivially to produce (4.4).

For $\psi \in BV(Y)$, letting $\{\bar{u}_j, \bar{v}_j\}_j$ denote the endpoints of elements of $\mathring{\mathcal{J}}_\varepsilon$ and $\{u_i, v_i\}_i$ denote the endpoints of elements of $\mathring{\mathcal{I}}_\varepsilon$, we estimate,

$$\begin{aligned}
\bigvee_Y \mathring{\mathcal{L}}_\varepsilon^{n_0}\psi &\leqslant \sum_{J_j \in \mathring{\mathcal{J}}_\varepsilon} \bigvee_{J_j} \mathring{\mathcal{L}}_\varepsilon^{n_0}\psi + \mathring{\mathcal{L}}_\varepsilon^{n_0}\psi(\bar{u}_j) + \mathring{\mathcal{L}}_\varepsilon^{n_0}\psi(\bar{v}_j) \\
&\leqslant \sum_{I_i \in \mathring{\mathcal{I}}_\varepsilon} \bigvee_{I_i} \frac{\psi}{|DF_\varepsilon^{n_0}|} + \sum_{I_i \in \mathring{\mathcal{I}}_\varepsilon} \frac{|\psi(u_i)|}{|DF_\varepsilon^{n_0}(u_i)|} + \frac{|\psi(v_i)|}{|DF_\varepsilon^{n_0}(v_i)|}\,.
\end{aligned} \tag{4.6}$$

For the first term above, given a finite partition $\{x_k\}_{k=0}^N$ of I_i, we split the relevant expression into two terms.

$$\sum_k \left| \frac{\psi(x_k)}{|DF_\varepsilon^{n_0}(x_k)|} - \frac{\psi(x_{k-1})}{|DF_\varepsilon^{n_0}(x_{k-1})|} \right| \leqslant \frac{1}{3} \bigvee_{I_i} \psi + \sum_k |\psi(x_k)| \left| \frac{1}{|DF_\varepsilon^{n_0}(x_k)|} - \frac{1}{|DF_\varepsilon^{n_0}(x_{k-1})|} \right|,$$

where we have used (4.2) in the first term. For the second term, we use bounded distortion, Proposition 2.5(b), to estimate,

$$\left| \frac{1}{|DF_\varepsilon^{n_0}(x_k)|} - \frac{1}{|DF_\varepsilon^{n_0}(x_{k-1})|} \right| \leqslant C_d \frac{|F_\varepsilon^{n_0}(x_k) - F_\varepsilon^{n_0}(x_{k-1})|}{|DF_\varepsilon^{n_0}(x_k)|} \leqslant (1 + C_d)C_d |x_k - x_{k-1}|\,,$$

where we have applied the mean value theorem to $F_\varepsilon^{n_0}$ on $[x_{k-1}, x_k]$. Putting these estimates together yields,

$$\begin{aligned}
\sum_k \left| \frac{\psi(x_k)}{|DF_\varepsilon^{n_0}(x_k)|} - \frac{\psi(x_{k-1})}{|DF_\varepsilon^{n_0}(x_{k-1})|} \right| &\leqslant \frac{1}{3} \bigvee_{I_i} \psi + C_d(1 + C_d) \sum_{k=1}^N |\psi(x_k)|(x_k - x_{k-1}) \\
&\leqslant \frac{1}{3} \bigvee_{I_i} \psi + C_d(1 + C_d) \int_{I_i} |\psi| + \kappa_N(\psi)\,,
\end{aligned}$$

where we have recognised the second term as a Riemann sum, and the error term $\kappa_N(\psi) \to 0$ as $N \to \infty$. Since the variation is attained in the limit of partitions as $N \to \infty$, we have the following bound on the first term from (4.6),

$$\bigvee_{I_i} \frac{\psi}{|DF_{\varepsilon}^{n_0}|} \leqslant \frac{1}{3} \bigvee_{I_i} \psi + C_d(1+C_d) \int_{I_i} |\psi| \, dm \,. \tag{4.7}$$

Next, for the second term in (4.6), we use the bound,

$$|\psi(u_i)| + |\psi(v_i)| \leqslant 2 \inf_{I_i} |\psi| + \bigvee_{I_i} \psi \leqslant \frac{2}{|I_i|} \int_{I_i} |\psi| + \bigvee_{I_i} \psi \,. \tag{4.8}$$

Then using again bounded distortion together with Lemma 4.3, we have

$$|I_i| \inf_{I_i} |DF_{\varepsilon}^{n_0}| \geqslant (1+C_d)^{-1} |F_{\varepsilon}^{n_0}(I_i)| \geqslant (1+C_d)^{-1} C_\beta \,. \tag{4.9}$$

Putting these estimates together with (4.7) into (4.6), and using again (4.2), we conclude,

$$\bigvee_Y \mathring{\mathcal{L}}_{\varepsilon}^{n_0} \psi \leqslant \frac{2}{3} \sum_i \bigvee_{I_i} \psi + \left(C_2(1+C_d) + 2(1+C_d)C_\beta^{-1}\right) \int_{I_i} |\psi| \leqslant \frac{2}{3} \bigvee_Y \psi + (1+C_d)(C_d + 2C_\beta^{-1}) \int_{\mathring{Y}_{\varepsilon}^{n_0}} |\psi|,$$

which is the required inequality for $k = 1$. $\qquad\qquad\square$

Next, in order to show that $\mathring{\mathcal{L}}_{\varepsilon}$ has a uniform spectral gap (depending on β, and for ε sufficiently small), we will apply the perturbative framework of Keller and Liverani [KL1]. To this end, define the norm of an operator $\mathcal{P} : BV(Y) \to L^1(m)$ by,

$$|||\mathcal{P}||| = \sup\{|\mathcal{P}\psi|_{L^1(m)} : \|\psi\|_{BV} \leqslant 1\} \,. \tag{4.10}$$

Our next lemma is standard: $|||\mathcal{L}_0 - \mathring{\mathcal{L}}_{\varepsilon}|||$ is small as a function of $m(H_{\varepsilon}')$.

Lemma 4.6. *For any $\varepsilon > 0$ and $\varepsilon' \in [0, \varepsilon)$, $|||\mathring{\mathcal{L}}_{\varepsilon'} - \mathring{\mathcal{L}}_{\varepsilon}||| \leqslant m(H_{\varepsilon}' \setminus H_{\varepsilon'}')$. This holds in particular for $\varepsilon' = 0$, in which case $\mathring{\mathcal{L}}_0 = \mathcal{L}_0$ is the unpunctured operator.*

Proof. Let $\psi \in BV(Y)$. Then,

$$\int_Y |(\mathring{\mathcal{L}}_{\varepsilon'} - \mathring{\mathcal{L}}_{\varepsilon})\psi| \, dm \leqslant \int_Y |\psi 1_{H_{\varepsilon}' \setminus H_{\varepsilon'}'}| \, dm \leqslant \|\psi\|_{BV} m(H_{\varepsilon}' \setminus H_{\varepsilon'}') \,.$$

$\qquad\qquad\square$

Theorem 4.7. *For any $\beta > 0$, there exists $\varepsilon_\beta > 0$ such that for any β-allowable hole $H_{\varepsilon}(z)$ with $\varepsilon < \varepsilon_\beta$, $\mathring{\mathcal{L}}_{\varepsilon}$ is a continuous perturbation of $\mathcal{L}_0$. Indeed, it is Hölder continuous in $m(H_{\varepsilon}')$.*

As a consequence, $\mathring{\mathcal{L}}_{\varepsilon}$ has a spectral gap on $BV(Y)$. In particular, there exist $\eta_\beta, B_\beta > 0$, such that for all $\varepsilon < \varepsilon_\beta$, the spectral radius of $\mathring{\mathcal{L}}_{\varepsilon}$ is $\Lambda_{\varepsilon} < 1$ and there exist operators $\Pi_{\varepsilon}, \mathcal{R}_{\varepsilon} : BV(Y) \circlearrowleft$ satisfying $\Pi_{\varepsilon}^2 = \Pi_{\varepsilon}$, $\Pi_{\varepsilon}\mathcal{R}_{\varepsilon} = \mathcal{R}_{\varepsilon}\Pi_{\varepsilon} = 0$, and $\|\mathcal{R}_{\varepsilon}^n\|_{BV} \leqslant B_\beta \Lambda_{\varepsilon}^n e^{-\eta_\beta n}$ such that

$$\mathring{\mathcal{L}}_{\varepsilon}\psi = \Lambda_{\varepsilon}\Pi_{\varepsilon}\psi + \mathcal{R}_{\varepsilon}\psi \,. \tag{4.11}$$

Moreover, $\Pi_{\varepsilon} = \mathring{e}_{\varepsilon} \otimes \mathring{g}_{\varepsilon}$ for some $\mathring{e}_{\varepsilon} \in BV(Y)^*$ and $\mathring{g}_{\varepsilon} \in BV(Y)$ satisfying $\mathring{\mathcal{L}}_{\varepsilon}\mathring{g}_{\varepsilon} = \Lambda_{\varepsilon}\mathring{g}_{\varepsilon}$ with $\int_Y \mathring{g}_{\varepsilon} \, dm = 1$.

Lastly, the spectra and spectral projectors vary (Hölder) continuously as functions of ε in the $|||\cdot|||$-norm, i.e. as operators from $BV(Y)$ to $L^1(m)$.

Proof. The Lasota-Yorke inequalities of Proposition 4.5 apply also to the unpunctured operator $\mathcal{L}_\varepsilon = \mathcal{L}_0$ with $\mathring{Y}_\varepsilon^n$ replaced by Y. Thus $\mathcal{L}_0^{n_0}$ is quasi-compact on $BV(Y)$, and since $\mathcal{L}_0$ is also bounded as on operator on $BV(Y)$ (although we do not obtain the same contraction for one iterate of $\mathcal{L}_0$, the norm estimate as in the proof of Proposition 4.5 is finite), then $\mathcal{L}_0$ is also quasi-compact on $BV(Y)$. Since F_0 is mixing by Lemma 2.3(e) and has finite images by Proposition 2.5(a), then F_0 is covering in the sense of [LSV]. It follows that $\mathcal{L}_0$ has a spectral gap. Then so does $\mathcal{L}_0^{n_0}$. Moreover, if g_0 is the unique element of $BV(Y)$ such that $\mathcal{L}_0 g_0 = g_0$ and $\int_Y g_0\, dm = 1$, then [LSV, Theorem 3.1] implies

$$c_g := \inf_Y g_0 > 0. \tag{4.12}$$

Next, due to the uniform Lasota-Yorke inequalities (for fixed $\beta > 0$) of Proposition 4.5 together with Lemma 4.6 for $\varepsilon' = 0$, [KL1, Corollary 1] implies that the spectra and spectral projectors of $\mathring{\mathcal{L}}_\varepsilon^{n_0}$ outside the disk of radius $2/3$ vary continuously in ε for ε sufficiently small (depending on β). The spectral gap and the rest of the spectral decomposition for $\mathring{\mathcal{L}}_\varepsilon$ then follows from the analogous decomposition for $\mathcal{L}_0$.

Lastly, fixing $\varepsilon < \varepsilon_\beta$ and using Lemma 4.6, we apply again [KL1, Corollary 1] to $\mathring{\mathcal{L}}_\varepsilon$ to conclude that the spectra and spectral projectors of $\mathring{\mathcal{L}}_{\varepsilon'}$ vary Hölder continuously as functions of $|\varepsilon' - \varepsilon|$ whenever $\varepsilon' < \varepsilon_\beta$. $\qquad\square$

The above theorem implies in particular that the size of the spectral gap is at least η_β for all β-allowable holes $H_\varepsilon(z)$ with $\varepsilon < \varepsilon_\beta$.

4.2. **Local escape rate.** In this section, we will set up the estimates necessary to prove Theorem 1.1 via the induced map F_ε. The strategy is essentially the same as that carried out in [DT2, Section 7], but of course now we are interested in the case in which z lies in the critical orbit, which was not allowed for geometric potentials in [DT2].

We fix $\beta > 0$ and consider the zero-hole limit as $\varepsilon \to 0$ for β-allowable holes only. As a first step, we use the spectral gap for $\mathring{\mathcal{L}}_\varepsilon$ given by Theorem 4.7 to construct an invariant measure ν_ε for the induced open map $\mathring{F}_\varepsilon$ supported on the survivor set $\mathring{Y}_\varepsilon^\infty := \cap_{n=1}^\infty \mathring{Y}_\varepsilon^n = \cap_{n=0}^\infty \mathring{F}_\varepsilon^{-n}(Y)$.

Define for $\psi \in BV(Y)$,

$$\nu_\varepsilon(\psi) := \lim_{n\to\infty} \Lambda_\varepsilon^{-n} \int_{\mathring{Y}_\varepsilon^n} \psi\, \mathring{g}_\varepsilon\, dm. \tag{4.13}$$

Lemma 4.8. *Fix $\beta > 0$ and let $\varepsilon < \varepsilon_\beta$ be β-allowable. The limit in (4.13) exists and defines a Borel probability measure, supported on $\mathring{Y}_\varepsilon^\infty$, and invariant for $\mathring{F}_\varepsilon$. Moreover, ν_ε varies continuously as a function of ε (for fixed β) and* [4]

$$-\log \Lambda_\varepsilon = \left(\int \tau\, d\nu_\varepsilon \right) \mathfrak{e}(H_\varepsilon(z)), \tag{4.14}$$

where τ is the inducing time for F_ε and $\mathfrak{e}(H_\varepsilon(z))$ is the escape rate for f from (1.1).

Proof. The limit in (4.13) exists due to the spectral decomposition from Theorem 4.7 and the conformality of m:

$$\lim_{n\to\infty} \Lambda_\varepsilon^{-n} \int_{\mathring{Y}_\varepsilon^n} \psi\, \mathring{g}_\varepsilon\, dm = \lim_{n\to\infty} \int_Y \Lambda_\varepsilon^{-n} \mathring{\mathcal{L}}_\varepsilon^n(\psi \mathring{g}_\varepsilon)\, dm = \mathring{e}_\varepsilon(\psi \mathring{g}_\varepsilon),$$

---

[4]Indeed, we show that $\nu_\varepsilon(\tau)$ is continuous in ε although $\tau \notin BV(Y)$.

for any $\psi \in BV(Y)$ since if $\psi \in BV(Y)$, then also $\psi \mathring{g}_{\varepsilon} \in BV(Y)$. From (4.13), $|\nu_{\varepsilon}(\psi)| \leqslant \nu_{\varepsilon}(1)|\psi|_{\infty}$, so that ν_{ε} extends to a bounded linear functional on $C^0(Y)$, i.e. ν_{ε} is a Borel measure, clearly supported on $\mathring{Y}_{\varepsilon}^{\infty}$. Since $\nu_{\varepsilon}(1) = 1$, ν_{ε} is a probability measure.

Next, we prove that ν_{ε} is continuous as an element of $BV(Y)^*$. Remark that by the above calculation, $\nu_{\varepsilon}(\psi) = \mathring{e}_{\varepsilon}(\mathring{g}_{\varepsilon}\psi)$ for $\psi \in BV(Y)$, and when $\varepsilon = 0$, $\mu_Y(\psi) = e_0(g_0\psi) = \int g_0\psi \, dm$, since m is conformal for $\mathcal{L}_0$. Indeed, $\mathring{e}_{\varepsilon}$ defines a conformal measure $\mathring{m}_{\varepsilon}$ for $\mathring{\mathcal{L}}_{\varepsilon}$, so that $\mathring{e}_{\varepsilon}(\psi) = \int_Y \psi \, d\mathring{m}_{\varepsilon}$ and $d\nu_{\varepsilon} = \mathring{g}_{\varepsilon} d\mathring{m}_{\varepsilon}$. Thus, for $\psi \in BV(Y)$ and $\varepsilon, \varepsilon' < \varepsilon_{\beta}$,

$$\begin{aligned}
|\nu_{\varepsilon}(\psi) - \nu_{\varepsilon'}(\psi)| &\leqslant |\mathring{e}_{\varepsilon}(\mathring{g}_{\varepsilon}\psi - \mathring{g}_{\varepsilon'}\psi)| + |\mathring{e}_{\varepsilon}(\mathring{g}_{\varepsilon'}\psi) - e_{\varepsilon'}(\mathring{g}_{\varepsilon'}\psi)| \\
&\leqslant \left| \int (\mathring{g}_{\varepsilon} - \mathring{g}_{\varepsilon'})\psi \, d\mathring{m}_{\varepsilon} \right| + \left| \int \left(\Pi_{\varepsilon}(\mathring{g}_{\varepsilon'}\psi) - \Pi_{\varepsilon'}(\mathring{g}_{\varepsilon'}\psi) \right) dm \right| \qquad (4.15) \\
&\leqslant |\psi|_{\infty}|\mathring{g}_{\varepsilon} - \mathring{g}_{\varepsilon'}|_{L^1(\mathring{m}_{\varepsilon})} + |||\Pi_{\varepsilon} - \Pi_{\varepsilon'}||| \, \|\mathring{g}_{\varepsilon'}\psi\|_{BV} \, .
\end{aligned}$$

Both differences go to 0 as $\varepsilon' \to \varepsilon$ by Theorem 4.7, while $\|\mathring{g}_{\varepsilon'}\|_{BV}$ is uniformly bounded in ε' by Proposition 4.5. We conclude that ν_{ε} is continuous in ε for fixed β when acting on BV functions.

It remains to prove (4.14). Unfortunately, $\tau \notin BV(Y)$ so first we must show that $\nu_{\varepsilon}(\tau)$ is well-defined. Indeed, it is easy to check that $\mathring{\mathcal{L}}_{\varepsilon}\tau \in BV$. This holds since τ is constant on each 1-cylinder $Y_{i,\varepsilon}$ for F_{ε}. Thus $\mathring{\mathcal{L}}_{\varepsilon}\tau$ has discontinuities only at the endpoints of $\{\mathring{F}_{\varepsilon}(Y_{i,\varepsilon})\}_i$, which is a finite collection of intervals.

It follows also that $\mathring{\mathcal{L}}_{\varepsilon}(\tau \mathring{g}_{\varepsilon}) \in BV(Y)$. Thus using (4.11),

$$\begin{aligned}
\lim_{n\to\infty} \Lambda_{\varepsilon}^{-n} \int_{\mathring{Y}_{\varepsilon}^n} \tau \mathring{g}_{\varepsilon} \, dm &= \lim_{n\to\infty} \Lambda_{\varepsilon}^{-n} \int_Y \mathring{\mathcal{L}}_{\varepsilon}^{n-1}(\mathring{\mathcal{L}}_{\varepsilon}(\tau \mathring{g}_{\varepsilon})) \, dm \\
&= \Lambda_{\varepsilon}^{-1} \int \Pi_{\varepsilon}(\mathring{\mathcal{L}}_{\varepsilon}(\tau \mathring{g}_{\varepsilon})) \, dm + \lim_{n\to\infty} \int \Lambda_{\varepsilon}^{-n} \mathcal{R}_{\varepsilon}^{n-1}(\mathring{\mathcal{L}}_{\varepsilon}(\tau \mathring{g}_{\varepsilon})) \, dm \, ,
\end{aligned}$$

and the second term converges to 0 by Theorem 4.7. Thus the limit defining $\nu_{\varepsilon}(\tau)$ exists and is uniformly bounded in ε for fixed β. More than this, the above calculation can be improved to show that τ is uniformly (in ε) integrable with respect to ν_{ε}, as follows. For each $N > 0$, we use the above to estimate,

$$\nu_{\varepsilon}(1_{\tau>N} \cdot \tau) = \lim_{n\to\infty} \Lambda_{\varepsilon}^{-n} \int \mathring{\mathcal{L}}_{\varepsilon}^{n-1}(\mathring{\mathcal{L}}_{\varepsilon}(1_{\tau>N} \cdot \tau \mathring{g}_{\varepsilon})) \, dm \leqslant \Lambda_{\varepsilon}^{-1}|\mathring{\mathcal{L}}_{\varepsilon}(1_{\tau>N} \cdot \tau \mathring{g}_{\varepsilon})|_{\infty} \, .$$

Then using bounded distortion as in (4.19), one has for $x \in Y$,

$$\begin{aligned}
|\mathring{\mathcal{L}}_{\varepsilon}(1_{\tau>N} \cdot \tau \mathring{g}_{\varepsilon})(x)| &\leqslant \frac{(1+C_d)|\mathring{g}_{\varepsilon}|_{\infty}}{C_{\beta}} \sum_{\substack{y \in \mathring{F}_{\varepsilon}^{-1}x \\ \tau(y)>N}} \tau(y)m(Y_{i,\varepsilon}(y)) \\
&\leqslant C \sum_{k>N} k \, m(\tau = k) \leqslant C' \sum_{k>N} k e^{-k(\log \lambda_{per} - \eta)} \, ,
\end{aligned}$$

where we have used Proposition 2.5(c).

It follows that for each $\kappa > 0$, there exists $N > 0$ such that $\sup_{\varepsilon \in [0,\varepsilon_{\beta})}\{\nu_{\varepsilon}(1_{\tau>N} \cdot \tau)\} < \kappa$ where the sup is restricted to β-allowable values of ε. Let $\tau^{(N)} = \min\{\tau, N\}$ and note that $\tau^{(N)} \in BV(Y)$. Then taking a limit along β-allowable ε' yields,

$$\lim_{\varepsilon'\to\varepsilon} |\nu_{\varepsilon}(\tau) - \nu_{\varepsilon'}(\tau)| \leqslant \lim_{\varepsilon'\to\varepsilon} |\nu_{\varepsilon}(\tau^{(N)}) - \nu_{\varepsilon'}(\tau^{(N)})| + |\nu_{\varepsilon}(1_{\tau>N} \cdot \tau)| + |\nu_{\varepsilon'}(1_{\tau>N} \cdot \tau)| \leqslant 2\kappa \, ,$$

since we have shown that $\nu_{\varepsilon'} \to \nu_{\varepsilon}$ as elements of $BV(Y)^*$. Since $\kappa > 0$ was arbitrary, this proves that $\nu_{\varepsilon}(\tau)$ varies continuously in ε.

Recall that $\{Y_i\}_i$ denotes the set of 1-cylinders for F_0 and let $\mathcal{J}_0 = \{F_0(Y_i)\}_i$ denote the finite set of images. The covering property of f implies that the set of preimages of endpoints of elements of $\mathcal{J}_0$ is dense in I. Thus there is a dense set of $\varepsilon < \varepsilon_\beta$ such that F_ε admits a countable Markov partition with finite images. For such ε, [DT1, Section 6.4.1] implies that ν_ε is an equilibrium state for the potential $-\Xi_\varepsilon \cdot \log(DF_\varepsilon) - \log \Lambda_\varepsilon$, where $\Xi_\varepsilon(x) = 1$ if $x \in Y \setminus H'_\varepsilon$ and $\Xi_\varepsilon = \infty$ if $x \in H'_\varepsilon$. Similarly, [BDM, Lemma 5.3] implies that ν_ε is a Gibbs measure for the potential $-\Xi_\varepsilon \cdot \log(DF_\varepsilon) - \tau \mathfrak{e}(H_\varepsilon(z))$ with pressure equal to 0. Putting these together yields (4.14) for such 'Markov holes.'

Finally, we extend the relation to all ε via the continuity of Λ_ε and $\nu_\varepsilon(\tau)$. Since $\mathfrak{e}(H_\varepsilon(z))$ is monotonic in ε and equals $-\log \Lambda_\varepsilon / \nu_\varepsilon(\tau)$ on a dense set of ε, it must also be continuous in ε. Thus the relation (4.14) holds for all $\varepsilon < \varepsilon_\beta$ which are β-allowable. $\qquad\square$

With Lemma 4.8 in hand, we see that the limit we would like to compute to prove Theorem 1.1 can be expressed as follows,

$$\frac{\mathfrak{e}(H_\varepsilon(z))}{\mu(H_\varepsilon(z))} = \frac{-\log \Lambda_\varepsilon}{\mu_Y(H'_\varepsilon)} \cdot \frac{\int \tau \, d\mu_Y}{\int \tau \, d\nu_\varepsilon} \cdot \frac{\mu(H'_\varepsilon)}{\mu(H_\varepsilon(z))} , \qquad (4.16)$$

where as before $\mu_Y = \frac{\mu|_Y}{\mu(Y)}$, and $1/\mu(Y) = \int_Y \tau \, d\mu_Y$ by Kac's Lemma since F_ε is a first-return map to Y. Theorem 1.1 will follow once we show that as $\varepsilon \to 0$,

$$\frac{-\log \Lambda_\varepsilon}{\mu_Y(H'_\varepsilon)} \to 1, \quad \frac{\int \tau \, d\mu_Y}{\int \tau \, d\nu_\varepsilon} \to 1, \quad \frac{\mu(H'_\varepsilon)}{\mu(H_\varepsilon(z))} \to 1 - \lambda_z^{-1/\ell} . \qquad (4.17)$$

The third limit above is precisely (3.4) when $z \in \mathrm{orb}(f(c))$ is periodic (and is simply 1 by Section 3.4 when $z \in \mathrm{orb}(f(c))$ is preperiodic), so we proceed to prove the first two. We first prove these limits for β-allowable holes with a fixed $\beta > 0$ in Lemmas 4.9 and 4.10. Then in Section 5 we show how to obtain the general limit as $\varepsilon \to 0$.

Lemma 4.9. *For fixed $\beta > 0$ and any sequence of β-allowable holes $H_\varepsilon(z)$,*

$$\lim_{\varepsilon \to 0} \frac{-\log \Lambda_\varepsilon}{\mu_Y(H'_\varepsilon)} = 1 .$$

Proof. The lemma could follow using the results of [KL2], yet since $z \notin Y$ in our setting, it is not clear how to verify the aperiodicity of H'_ε without imposing an additional condition on the reentry of points to Y which have spent some time in a neighbourhood of z. To avoid this, we will argue directly, as in [DT2, Proof of Theorem 7.2], yet our argument is simpler since our operators $\mathring{\mathcal{L}}_\varepsilon$ approach a fixed operator $\mathcal{L}_0$ via Lemma 4.6 in contrast to the situation in [DT2].

We assume that $\varepsilon < \varepsilon_\beta$ so that we are in the setting of Theorem 4.7. Then since $g_0 \in BV(Y)$, iterating (4.11) yields for any $n \geqslant 1$,

$$\mathring{\mathcal{L}}_\varepsilon^n g_0 = \Lambda_\varepsilon^n \mathring{e}_\varepsilon(g_0) \mathring{g}_\varepsilon + \mathcal{R}_\varepsilon g_0 \implies \mathring{g}_\varepsilon = \frac{1}{\mathring{e}_\varepsilon(g_0)} (\Lambda_\varepsilon^{-n} \mathring{\mathcal{L}}_\varepsilon^n g_0 - \Lambda_\varepsilon^{-n} \mathcal{R}_\varepsilon^n g_0) .$$

Using this relation and Lemma 4.6 yields,

$$1 - \Lambda_\varepsilon = \int \mathring{g}_\varepsilon \, dm - \int \mathring{\mathcal{L}}_\varepsilon \mathring{g}_\varepsilon \, dm = \int (\mathcal{L}_0 - \mathring{\mathcal{L}}_\varepsilon) \mathring{g}_\varepsilon \, dm = \int_{H'_\varepsilon} \mathring{g}_\varepsilon \, dm$$

$$= \frac{1}{\mathring{e}_\varepsilon(g_0)} \int_{H'_\varepsilon} (\Lambda_\varepsilon^{-n} \mathring{\mathcal{L}}_\varepsilon^n g_0 - \Lambda_\varepsilon^{-n} \mathcal{R}_\varepsilon^n g_0) \, dm \qquad (4.18)$$

$$= \frac{1}{\mathring{e}_\varepsilon(g_0)} \left(\int_{H'_\varepsilon} g_0 \, dm - \int_{H'_\varepsilon} (1 - \Lambda_\varepsilon^{-n} \mathring{\mathcal{L}}_\varepsilon^n) g_0 \, dm - \int_{H'_\varepsilon} \Lambda_\varepsilon^{-n} \mathcal{R}_\varepsilon^n g_0 \, dm \right) .$$

By Theorem 4.7, the third term on the right side of (4.18) is bounded by

$$\int_{H'_\varepsilon} \Lambda_\varepsilon^{-n} \mathcal{R}_\varepsilon^n g_0 \, dm \leqslant B_\beta e^{-\eta_\beta n} \|g_0\|_{BV} c_g^{-1} \int_{H'_\varepsilon} g_0 \, dm \,,$$

where $c_g = \inf_Y g_0 > 0$ according to (4.12).

Next, the second term on the right side of (4.18) can be expressed as,

$$\int_{H'_\varepsilon} (1 - \Lambda_\varepsilon^{-n} \mathring{\mathcal{L}}_\varepsilon) g_0 \, dm = (1 - \Lambda_\varepsilon^{-n}) \int_{H'_\varepsilon} g_0 \, dm + \Lambda_\varepsilon^{-n} \int_{H'_\varepsilon} (\mathcal{L}_0^n - \mathring{\mathcal{L}}_\varepsilon^n) g_0 \, dm \,,$$

using the fact that $\mathcal{L}_0 g_0 = g_0$. We claim that $(\mathcal{L}_0^n - \mathring{\mathcal{L}}_\varepsilon^n) g_0$ can be made small in $L^\infty(Y)$. To see this, choose $n = k n_0$ and write for $x \in Y$,

$$(\mathcal{L}_0^n - \mathring{\mathcal{L}}_\varepsilon^n) g_0(x) = \sum_{y \in F_0^{-n} x} \frac{g_0(y) 1_{Y \setminus \mathring{Y}_\varepsilon^n}(y)}{|DF_0^n(y)|} = \sum_{j=0}^{k-1} \sum_{y \in F_0^{-n} x} \frac{g_0(y) 1_{\mathring{Y}_\varepsilon^{jn_0} \setminus \mathring{Y}_\varepsilon^{(j+1)n_0}}(y)}{|DF_0^n(y)|} \,,$$

where we have used the fact that when $n = k n_0$, $Y \setminus \mathring{Y}_\varepsilon^n = \cup_{j=0}^{k-1} (\mathring{Y}_\varepsilon^{jn_0} \setminus \mathring{Y}_\varepsilon^{(j+1)n_0})$ and the union is disjoint. Next, if $Y_{i,j,\varepsilon}(y)$ is the jn_0-cylinder for $F_\varepsilon^{jn_0}$ containing y, then by bounded distortion and Lemma 4.3,

$$|DF^{jn_0}(y)| \geqslant \frac{m(F^{jn_0}(Y_{i,j,\varepsilon}(y))}{(1+C_d)m(Y_{i,j,\varepsilon}(y))} \geqslant \frac{C_\beta}{(1+C_d)m(Y_{i,j,\varepsilon}(y))} \,. \tag{4.19}$$

Then, applying also (4.2), we have

$$(\mathcal{L}_0^n - \mathring{\mathcal{L}}_\varepsilon^n) g_0(x) \leqslant \frac{(1+C_d)|g_0|_\infty}{C_\beta} \sum_{j=0}^{k-1} m(\mathring{Y}_\varepsilon^{jn_0} \setminus \mathring{Y}_\varepsilon^{(j+1)n_0}) 3^{k-j} =: \rho_n(\varepsilon) \,,$$

and note that $\rho_n(\varepsilon) \to 0$ as $\varepsilon \to 0$ for fixed $n = k n_0$. Thus, the second term on the right side of (4.18) is estimated by,

$$\int_{H'_\varepsilon} (1 - \Lambda_\varepsilon^{-n} \mathring{\mathcal{L}}_\varepsilon) g_0 \, dm \leqslant \left(1 - \Lambda_\varepsilon^{-n} + c_g^{-1} \Lambda_\varepsilon^{-n} \rho_n(\varepsilon) \right) \mu_Y(H'_\varepsilon).$$

Putting these estimates into (4.18) yields,

$$\frac{1 - \Lambda_\varepsilon}{\mu_Y(H'_\varepsilon)} = \frac{1}{\mathring{e}_\varepsilon(g_0)} \left(1 + \mathcal{O}(e^{-\eta_\beta n} + (1 - \Lambda_\varepsilon^{-n}) + \Lambda_\varepsilon^{-n} \rho_n(\varepsilon)) \right).$$

Fixing $\kappa > 0$, first choose $n = k n_0$ so that $e^{-\eta_\beta n} < \kappa$. Next, choose ε so small that by the continuity of the spectral data from the proof of Theorem 4.7, $|1 - \Lambda_\varepsilon^{-n}| < \kappa$, $\Lambda_\varepsilon^{-n} \leqslant 2$, $\rho_n(\varepsilon) < \kappa$ and $|\mathring{e}_\varepsilon(g_0) - 1| < \kappa$. This last bound is possible since $e_0(g_0) = \int g_0 \, dm = 1$. Thus the relevant expression is $1 + \mathcal{O}(\kappa)$ for ε sufficiently small and β-allowable. Since κ is arbitrary, the lemma is proved. $\qquad\square$

Recall that the inducing time τ for F_ε does not depend on ε.

Lemma 4.10. *For fixed $\beta > 0$ and any sequence of β-allowable holes $H_\varepsilon(z)$,*

$$\lim_{\varepsilon \to 0} \frac{\int \tau \, d\mu_Y}{\int \tau \, d\nu_\varepsilon} = 1 \,.$$

Proof. As above, we assume $\varepsilon < \varepsilon_\beta$. Recall that $\nu_0 = \mu_Y$ and $e_0(\psi) = \int \psi \, dm$ since m is conformal for $\mathcal{L}_0$. Thus the estimates (4.15) and following in the proof of Lemma 4.8 hold equally well with $\varepsilon' = 0$ throughout. Thus the continuity of $\nu_\varepsilon(\tau)$ extends to $\varepsilon = 0$. This, plus the fact that $\nu_\varepsilon(\tau) \geqslant 1$ since $\tau \geqslant 1$ implies the required limit. $\qquad\square$

5. Completion of the proof of Theorem 1.1 via approximation

Putting together Section 3.3 and Lemmas 4.9 and 4.10, we have proved Theorem 1.1 for each $\beta > 0$ along sequences $(\varepsilon_n)_n$ where each ε_n is β-allowable. It remains to consider the alternative case where we have to approximate a given sequence ε_n by β-allowable ε'_n. We will focus our estimates on approximating on the left, with the right-hand side following similarly. We also start by assuming $z \in \mathrm{orb}(f(c))$ is periodic.

We remark that if H_ε is β-allowable, then it is also β'-allowable for any $\beta' < \beta$, so we will take our approximating sequence with β tending to 0. Without loss of generality and for convenience, we assume $\beta < (2\lambda_z)^{-1}$.

Recall the notation $(a_i)_i, (b_i)_i$ from Section 2.3, i.e. (a_i, a_{i+1}) are the f^s-images of subchains of intervals accumulating on z from the left, while (b_{i+1}, b_i) accumulate on z from the right. Recall that $\frac{z-a_i}{b_i-z}$ are uniformly bounded away from 0 and ∞. As in Lemma 2.9, we have $f^p(a_i, a_{i+1}) = (a_{i-1}, a_i)$ if f^p is orientation preserving at z, and $f^{2p}(a_i, a_{i+1}) = (a_{i-2}, a_{i-1})$ if f^p is orientation reversing. We will assume the orientation preserving case in the following, the orientation reversing case being similar.

Each interval (a_i, a_{i+1}) is $f^s(I_j)$ for some one-cylinder I_j for F_ε. Each I_j in turn is subdivided into a countable union of n_0-cylinders for F_ε and so (a_i, a_{i+1}) is subdivided into a countable union of f^s images of these n_0-cylinders. Indeed, given the chain structure, the arrangement of f^s images of the n_0-cylinders in (a_i, a_{i+1}) maps precisely under f^p onto the f^s images of the n_0-cylinders in (a_{i-1}, a_i).

We describe this structure as follows: each (a_i, a_{i+1}) is subdivided into finitely many intervals (depending on n_0), which we index by j, $j = 1, \ldots, J_i$. The end points of these intervals are preimages of the boundaries of one-cylinders in Y, some of which may map onto c before n_0 iterates of F_ε, in which case they are in fact accumulation points of preimages of the chains recalled above. We label these chains of intervals $(c_{i,j,k}, c_{i,j,k+1})$ which accumulate on the jth point in (a_i, a_{i+1}) from the left, and similarly $(d_{i,j,k+1}, d_{i,j,k})$ accumulate from the right.

Set $v_{i,j,k} = c_{i,j,k+1} - c_{i,j,k}$. According to our definition, if ε is β-left-allowable, then

$$z - \varepsilon \in A_{L,\beta} := \cup_{i,j,k}[c_{i,j,k} + \beta v_i, c_{i,j,k+1} - \beta v_i].$$

So if ε is not β-left-allowable, then $z - \varepsilon \in (c_{i,j,k+1} - \beta v_{i,j,k}, c_{i,j,k+1} + \beta v_{i,j,k})$ for some i, j, k. Remark that if there is no accumulation of chains at the jth point, then the index k is redundant. Also, there is some duplication since $c_{i,0} = a_i$, while $c_{i,J_i} = a_{i+1}$, yet for uniformity of notation we denote each range of non-β-left-allowable values of $z - \varepsilon$ as above.

We approximate ε from above by $\varepsilon_o^L := z - (c_{i,j,k+1} - \beta v_{i,j,k})$ and from below by $\varepsilon_u^L := z - (c_{i,j,k+1} + \beta v_{i,j,k})$. Both ε_o^L and ε_u^L are β-left-allowable and $\varepsilon \in (\varepsilon_u^L, \varepsilon_o^L)$. Thus,

$$\frac{\varepsilon_u^L}{\varepsilon_o^L} \leqslant \frac{\varepsilon_u^L}{\varepsilon} \leqslant \frac{\varepsilon_o^L}{\varepsilon} \leqslant \frac{\varepsilon_o^L}{\varepsilon_u^L},$$

so we focus on estimating the outer two quantities. Notice also that the above implies $\frac{\varepsilon_u^L}{\varepsilon_u^R}, \frac{\varepsilon_o^L}{\varepsilon_o^R}$ are uniformly bounded away from 0 and ∞, due to the uniform bound on $\frac{z-a_i}{b_i-z}$, so we may apply (3.4).

Now,

$$\frac{\varepsilon_u^L}{\varepsilon_o^L} = \frac{z - (c_{i,j,k+1} + \beta v_{i,j,k})}{z - (c_{i,j,k+1} - \beta v_{i,j,k})} = 1 - \frac{2\beta v_{i,j,k}}{z - (c_{i,j,k} - \beta v_{i,j,k})}\,.$$

Note that $z - (c_{i,j,k+1} - \beta v_{i,j,k}) \geqslant z - a_{i+1}$ while $v_{i,j,k} \leqslant e_i := a_{i+1} - a_i$. But due to the chain structure around z, we have $z - a_{i+1} > e_{i+1} \sim \lambda_z^{-1} e_i$, so that

$$\frac{\varepsilon_u^L}{\varepsilon_o^L} \geqslant 1 - \frac{2\beta e_i}{e_{i+1}} \gtrsim 1 - 2\beta\lambda_z\,. \tag{5.1}$$

In the same way, $\frac{\varepsilon_o^L}{\varepsilon_u^L} \lesssim 1 + 2\beta\lambda_z$. The right hand estimates for the analogous ε_u^R and ε_o^R are similar.

To complete the proof of Theorem 1.1, we must evaluate the following limit as $\varepsilon \to 0$,

$$\frac{1}{\mu(z - \varepsilon, z + \varepsilon)} \frac{-1}{n} \log\mu\left(\{x \in I : f^i(x) \notin (z - \varepsilon, z + \varepsilon),\ i = 1, \ldots, n\}\right)\,. \tag{5.2}$$

We estimate this from below by

$$\frac{\mu(z - \varepsilon_u^L, z + \varepsilon_u^R)}{\mu(z - \varepsilon, z + \varepsilon)} \frac{1}{\mu(z - \varepsilon_u^L, z + \varepsilon_u^R)} \frac{-1}{n} \log\mu\left(\{x \in I : f^i(x) \notin (z - \varepsilon_u^L, z + \varepsilon_u^L),\ i = 1, \ldots, n\}\right)\,.$$

By the above estimates, $\frac{\mu(z - \varepsilon_u^L, z + \varepsilon_u^R)}{\mu(z - \varepsilon, z + \varepsilon)} \sim (1 - 2\beta\lambda_z)^{\frac{1}{\ell}}$, so taking the limit as $\varepsilon \to 0$ yields an lower bound of

$$(1 - 2\beta\lambda_z)^{\frac{1}{\ell}} \cdot \left(1 - \lambda_z^{-1/\ell}\right)\,,$$

where the second factor comes from the application of Theorem 1.1 to β-allowable holes via (4.17). Similarly, one obtains an upper bound for (5.2) of $(1 + 2\beta\lambda_z)^{\frac{1}{\ell}} \cdot \left(1 - \lambda_z^{-1/\ell}\right)$. Since these bounds hold for all sufficiently small β, we take $\beta \to 0$ to obtain the required limit for Theorem 1.1 along an arbitrary sequence $(\varepsilon_n)_n$ in the case that $z \in \mathrm{orb}(f(c))$ is periodic.

Finally notice that if $z \in \mathrm{orb}(f(c))$ is preperiodic then the above calculations all go through similarly: from the construction of (Y, F) in Section 3.4, the periodic structure of the postcritical orbit can be pulled back to z to generate the $(a_i)_i$, $(b_i)_i$ required, but our bounds are of the form $(1 \pm 2\beta\lambda)$ where $\lambda = |Df^p(f^{k_0}(z))|$. These tend to 1 as $\beta \to 0$, yielding the required limit in the case that z is preperiodic, but not periodic.

6. Proof of Theorem 1.2

In this section, we prove the cases of Theorem 1.2 in several steps. First, we address the case where $z \in \mathrm{orb}(f(c))$. As in the proof of Theorem 1.1, we first fix $\beta > 0$ and consider sequences of holes H_ε that are β-allowable. Leveraging the existence of the induced map F_0 and the local escape rate proved in Theorem 1.1, we prove Theorem 1.2 for sequences of such holes in Section 6.2. Then in Section 6.3, we will let $\beta \to 0$ to prove the required hitting time statistics for all sequences $\varepsilon \to 0$.

Finally, in Section 6.4 we show how to adapt the results of [BDT] to prove the hitting time statistics when $z \notin \mathrm{orb}(f(c))$.

6.1. **A non-Markovian tower with a hole.** Throughout Sections 6.1 – 6.3 we will assume $z \in \mathrm{orb}(f(c))$.

Recall the induced map $F_0^{n_0}$, where n_0 is chosen so that (4.2) holds. Indeed, by the proof of Proposition 2.5 (see also Lemma 4.1), there exists $\gamma > 0$ such that

$$\inf_{x \in Y} |Df^{\tau^{n_0}(x)}(x)| \, e^{-\gamma \tau^{n_0}(x)} > 2.5 \,, \tag{6.1}$$

where $\tau^{n_0} = \sum_{k=0}^{n_0-1} \tau \circ F_0^k$.

Using $F_0^{n_0}$, we define an extended system that resembles a Young tower as follows. Define

$$\Delta = \{(y, k) \in Y \times \mathbb{N} : k \leqslant \tau^{n_0} - 1\}.$$

We refer to the kth level of the tower as $\Delta_k = \{(y, n) \in \Delta : n = k\}$. Sometimes we refer to $(x, k) \in \Delta_k$ as x if k is clear from context.

The dynamics on the tower is defined by $f_\Delta(y, k) = (y, k+1)$ if $k < \tau^{n_0} - 1$ and $f_\Delta(y, \tau^{n_0}(y) - 1) = (F_0^{n_0}(y), 0)$. There is a natural projection $\pi_\Delta : \Delta \to I$ which satisfies $f \circ \pi_\Delta = \pi_\Delta \circ f_\Delta$. Clearly, $\pi_\Delta(\Delta_0) = Y$. Remark that F_0, f_Δ and Δ depend on z, but not on ε.

It follows from Lemma 2.3(c) and the fact that f is locally eventually onto that f_Δ is mixing. Indeed, $f(Y) \supset Y$ and the chain structure around z implies $f^{k_0+p}(Y) \supset I$. Then since f is locally eventually onto, for any interval $A \subset Y$, there exists $n_A \in \mathbb{N}$ such that $f^{n_A}(A) \supset Y$, and again by Lemma 2.3(c), $f^{n_A+k}(A) \supset Y$ for all $k \geqslant 0$. Setting $A_0 = (\pi_\Delta|_{\Delta_0})^{-1}(A)$, this implies $f_\Delta^{n_A+k}(A_0) \supset \cup_{j=0}^k \Delta_j$ for each $k \geqslant 0$. Thus f_Δ is topological mixing.[5]

We lift Lebesgue measure to Δ simply by defining $m_\Delta|_{\Delta_0} = m|_Y$ and $m_\Delta|_{\Delta_k} = (f_\Delta^k)_*(m_\Delta|_{\Delta_0})$. Note that by Proposition 2.5, $m_\Delta(\tau^{n_0} > n) \leqslant Ce^{-\zeta n}$, where $\zeta = \log \lambda_{\mathrm{per}} - \eta > 0$ and η is chosen before Remark 2.6. Thus our tower has exponential tails.

Now recall $\alpha \in (0, \infty)$ from the statement of Theorem 1.2. This will determine the scaling at which we consider the hitting time to $H_\varepsilon(z)$. We reduce $\gamma > 0$ in (6.1) if necessary so that

$$\gamma < \zeta \quad \text{and} \quad \gamma/\zeta < \alpha \,. \tag{6.2}$$

Note that the second condition is relevant only if we consider $\alpha < 1$.

Given a hole $H_\varepsilon(z)$, define $H_{\Delta_k}(\varepsilon) = (\pi_\Delta|_{\Delta_k})^{-1}(H_\varepsilon(z)$ and $H_\Delta(\varepsilon) = \cup_{k \geqslant 1} H_{\Delta_k}$. The corresponding punctured tower is defined analogously to (4.1) for $n \geqslant 1$,

$$\mathring{f}_{\Delta,\varepsilon}^n = f_\Delta^n|_{\mathring{\Delta}_\varepsilon^n}, \quad \text{where} \quad \mathring{\Delta}_\varepsilon^n := \cap_{i=0}^{n-1} f_\Delta^{-i}(\Delta \setminus H_\Delta(\varepsilon)) \,.$$

The main difference between $f_\Delta : \Delta \circlearrowleft$ and the usual notion of a Young tower is that we do not define a Markov structure on Δ. Yet as demonstrated above, the tower has the following key properties:

- Exponential tails: $m_\Delta(\tau^{n_0} > n) \leqslant Ce^{-\zeta n}$;
- f_Δ is topologically mixing;
- Large images at returns to Δ_0: if $H_\varepsilon(z)$ is β-allowable, then $C_\beta > 0$ from Lemma 4.3 provides a lower bound on the length of images in returns to Δ_0 under both f_Δ and $\mathring{f}_{\Delta,\varepsilon}$.

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[5]Indeed, the above argument also implies that there exist $x, y \in A_0$ such that $\tau^{n_0}(x) = n_A$ and $\tau^{n_0}(y) = n_A + 1$ so g.c.d.$(\tau^{n_0}) = 1$, yet this is not a sufficient condition for f_Δ to be mixing since our tower has multiple bases.

We will use these properties to prove the existence of a uniform spectral gap for the punctured transfer operators on the tower acting on a space of functions of weighted bounded variation, as follows.

The inducing domain Y is comprised of finitely many maximal connected components that are intervals in I. Let $\widetilde{\mathcal{D}}_0$ denote this collection of intervals. Similarly, for each $k \geqslant 1$, $f^k(Y \cap \{\tau^{n_0} \geqslant k+1\})$ can be decomposed into a set of maximal connected components. We further subdivide these intervals at the first time $j \leqslant k$ that they contain a point in $\mathrm{orb}(f(c))$, $z - \varepsilon_L$ or $z + \varepsilon_R$. These are the cuts introduced in the definition of F_ε in Section 2.3. Let $\widetilde{\mathcal{D}}_k$ denote this collection of intervals and define the collection of lifts by,

$$\mathcal{D}_k = \{J = (\pi|_{\Delta_k})^{-1}(\tilde{J}) : \tilde{J} \in \widetilde{\mathcal{D}}_k\}, \text{ for all } k \geqslant 0.$$

For an interval $J \in \mathcal{D}_k$ and a measurable function $\psi : \Delta \to \mathbb{R}$, define $\bigvee_J \psi$, the variation of ψ on J as in (4.3). On each level Δ_k, $k \geqslant 1$, define

$$\|\psi\|_{\Delta_k} = e^{-\gamma k} \sum_{J \in \mathcal{D}_k} \bigvee_J \psi + e^{-\gamma k} \sup_{\Delta_k} |\psi|.$$

Finally, define the weighted variation norm on Δ by,

$$\|\psi\|_{WV} = \sum_{k \geqslant 0} \|\psi\|_{\Delta_k}.$$

If $\|\psi\|_{WV} < \infty$ then $|\psi|_{L^1(m_\Delta)} < \infty$ since $\gamma < \zeta$ by (6.2). So we denote by $WV(\Delta)$ the set of functions $\psi \in L^1(m_\Delta)$ such that $\|\psi\|_{WV} < \infty$. Since we consider $WV(\Delta)$ as a subset of $L^1(m_\Delta)$, we define the variation norm of the equivalence class to be the infimum of variation norms of functions in the equivalence class.

We define the transfer operator $\mathcal{L}_\Delta$ corresponding to f_Δ acting on $L^1(m_\Delta)$ in the natural way,

$$\mathcal{L}_\Delta \psi(x) = \sum_{y \in f_\Delta^{-1}(x)} \frac{f(y)}{J f_\Delta(y)},$$

where $J f_\Delta$ is the Jacobian of f_Δ with respect to m_Δ. Then the transfer operator for the punctured tower is defined for each $n \geqslant 1$ by,

$$\mathring{\mathcal{L}}_{\Delta,\varepsilon}^n \psi = \mathcal{L}_\Delta^n (1_{\mathring{\Delta}_\varepsilon^n} \psi).$$

Our goal is to prove the following proposition, which is the analogue of Theorem 4.7, but for the tower map rather than the induced map.

Proposition 6.1. *For any $\beta > 0$, there exists $\varepsilon_\beta(\Delta)$ such that for any β-allowable hole $H_\varepsilon(z)$ with $\varepsilon < \varepsilon_\beta(\Delta)$, both $\mathcal{L}_\Delta$ and $\mathring{\mathcal{L}}_{\Delta,\varepsilon}$ have a spectral gap on $WV(\Delta)$.*

Specifically, there exist $\sigma_\beta, A_\beta > 0$ such that for all β-admissible $\varepsilon < \varepsilon_\beta(\Delta)$, the spectral radius of $\mathring{\mathcal{L}}_{\Delta,\varepsilon}$ is $\lambda_{\Delta,\varepsilon} < 1$ and there exist operators $\Pi_{\Delta,\varepsilon}, \mathcal{R}_{\Delta,\varepsilon} : WV(\Delta) \circlearrowleft$ satisfying $\Pi_{\Delta,\varepsilon}^2 = \Pi_{\Delta,\varepsilon}$, $\Pi_{\Delta,\varepsilon} \mathcal{R}_{\Delta,\varepsilon} = \mathcal{R}_{\Delta,\varepsilon} \Pi_{\Delta,\varepsilon} = 0$, and $\|\mathcal{R}_{\Delta,\varepsilon}^n\|_{WV} \leqslant A_\beta \lambda_{\Delta,\varepsilon}^n e^{-\sigma_\beta n}$ such that

$$\mathring{\mathcal{L}}_{\Delta,\varepsilon} \psi = \lambda_{\Delta,\varepsilon} \Pi_{\Delta,\varepsilon} \psi + \mathcal{R}_{\Delta,\varepsilon} \psi, \text{ for all } \psi \in WV(\Delta).$$

Indeed, $\Pi_{\Delta,\varepsilon} = \mathring{e}_{\Delta,\varepsilon} \otimes \mathring{g}_{\Delta,\varepsilon}$ for some $\mathring{e}_{\Delta,\varepsilon} \in WV(\Delta)^*$ and $\mathring{g}_{\Delta,\varepsilon} \in WV(\Delta)$ satisfying $\mathring{\mathcal{L}}_{\Delta,\varepsilon} \mathring{g}_{\Delta,\varepsilon} = \lambda_{\Delta,\varepsilon} \mathring{g}_{\Delta,\varepsilon}$ and $\int_\Delta \mathring{g}_{\Delta,\varepsilon} \, dm_\Delta = 1$.

The proof of this proposition is based on the following sequence of lemmas, which prove the compactness of $WV(\Delta)$ in $L^1(m_\Delta)$, uniform Lasota-Yorke inequalities for $\mathring{\mathcal{L}}_{\Delta,\varepsilon}$, and the smallness of the perturbation when viewing $\mathcal{L}_\Delta - \mathring{\mathcal{L}}_{\Delta,\varepsilon}$ as an operator from $WV(\Delta)$ to $L^1(m_\Delta)$.

Lemma 6.2. *The unit ball of $WV(\Delta)$ is compactly embedded in $L^1(m_\Delta)$.*

Proof. If $\|\psi\|_{WV} \leqslant 1$, then restricted to each $J \in \mathcal{D}_k$, the variation of ψ is at most $e^{\gamma k}$ and $|\psi|_J|_\infty \leqslant e^{\gamma k}$. Thus if B_1 is the ball of radius 1 in $WV(\Delta)$, then $B_1|_J$ is compactly embedded in $L^1(m_\Delta|_J)$.

Taking a sequence $(\psi_n)_n \subset B_1$, we first enumerate the elements of $\cup_{k \geqslant 0} \mathcal{D}_k$ and then use compactness on each J and a diagonalization procedure to extract a subsequence $(\psi_{n_k})_k$ which converges on every $J \in \cup_{k \geqslant 0} \mathcal{D}_k$ to a function ψ. ψ necessarily belongs to $L^1(m_\Delta)$ due to dominated convergence since $|\psi_n|_{\Delta_k}|_\infty \leqslant e^{\gamma k}$ and the function which is equal to $e^{\gamma k}$ on Δ_k for each k is integrable since $\gamma < \zeta$. $\square$

Lemma 6.3. *Assume $H_\varepsilon(z)$ is β-allowable. Let $C = (1 + C_d)(C_d + 2C_\beta^{-1})$. For $k \geqslant 1$, and all $\psi \in WV(Y)$,*

$$\|\mathring{\mathcal{L}}_{\Delta,\varepsilon}\psi\|_{\Delta_k} \leqslant e^{-\gamma}\|\psi\|_{\Delta_{k-1}} \tag{6.3}$$

$$\bigvee_{\Delta_0} \mathring{\mathcal{L}}_{\Delta,\varepsilon}\psi \leqslant \tfrac{4}{5}\|\psi\|_{WV} + C \int_{\mathring{\Delta}_\varepsilon^1} |\psi| dm_\Delta \tag{6.4}$$

$$|\mathring{\mathcal{L}}_{\Delta,\varepsilon}\psi|_{L^\infty(\Delta_0)} \leqslant \tfrac{2}{5}\|\psi\|_{WV} + C_\beta^{-1}(1 + C_d) \int_{\mathring{\Delta}_\varepsilon^1} |\psi| \tag{6.5}$$

$$\int_\Delta |\mathring{\mathcal{L}}_{\Delta,\varepsilon}^n\psi|| dm_\Delta \leqslant \int_{\mathring{\Delta}_\varepsilon^n} |\psi| dm_\Delta \ \text{ for all } n \geqslant 1. \tag{6.6}$$

The same estimates hold for $\mathcal{L}_\Delta$ with $\mathring{\Delta}_\varepsilon^n$ replaced by Δ.

Proof. Note that $Jf_\Delta(x,k) = 1$ if $k < \tau^{n_0}(x) - 1$. Thus if $k \geqslant 1$ and $x \in \Delta_k$, then $\mathring{\mathcal{L}}_{\Delta,\varepsilon}\psi(x) = (1_{\mathring{\Delta}_\varepsilon^1}\psi) \circ f_\Delta^{-1}(x)$.

Moreover, for $J \in \mathcal{D}_k$ and $k \geqslant 1$, we have $f_\Delta^{-1}(J) \subset J' \in \mathcal{D}_{k-1}$, and by definition of $\mathcal{D}_{k-1}$, J' is either disjoint from H_Δ or contained in it. Thus $1_{\mathring{\Delta}_\varepsilon^1}$ is either identically 0 or 1 on J' and so does not affect the variation. With these points noted, (6.3) holds trivially. Similarly, (6.6) follows immediately from the definition of $\mathring{\mathcal{L}}_{\Delta,\varepsilon}$.

We proceed to prove (6.4). The proof follows closely the proof of Proposition 4.5. As in Section 4.1, denote by $\mathcal{J}_\varepsilon$ the finite set of intervals in Y that are the images of the one-cylinders for $F_\varepsilon^{n_0}$. Since we identify Δ_0 with Y, we will abuse notation and again denote the finite set of images $(\pi_\Delta|_{\Delta_0})^{-1}(\mathcal{J}_\varepsilon)$ in Δ_0 by simply $\mathcal{J}_\varepsilon$.

Let $\mathcal{I}_\varepsilon$ denote the countable collection of intervals in $f_\Delta^{-1}(\Delta_0)$ so that for each $I_i \in \mathcal{I}_\varepsilon$, $f_\Delta(I_i) = J$ for some $J \in \mathcal{J}_\varepsilon$. We use the notation $I_i = (u_i, v_i)$, and $I_{i,k}$ denotes the intervals in $\mathcal{I}_\varepsilon$ that lie in Δ_k. Remark that each $I_{i,k} \subset J' \in \mathcal{D}_k$ so that $I_{i,k}$ is either in H_Δ or is disjoint from it.[6] Also, for $(x,k) \in I_{i,k}$, by definition $\tau^{n_0}(\pi_\Delta(x,0)) = k$, so that

$$Jf_\Delta(x,k) = |DF^{n_0}(\pi_\Delta(x,0))| = |Df^{\tau^{n_0}(x)}(x)| = |Df^k(x)|, \tag{6.7}$$

where for brevity, we denote $\pi_\Delta(x,0) = x$.

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[6]Indeed, since a neighbourhood of z cannot enter Y in one step, every $I_i \in \mathcal{I}_\varepsilon$ is disjoint from H_Δ in this particular tower.

Now for $\psi \in WV(\Delta)$, following (4.6), we obtain,

$$\bigvee_{\Delta_0} \mathring{\mathcal{L}}_{\Delta,\varepsilon} \psi \leqslant \sum_{I_i \in \mathcal{I}_\varepsilon} \bigvee_{I_i} \frac{\psi}{Jf_\Delta} + \sum_{I_i \in \mathcal{I}_\varepsilon} \frac{|\psi(u_i)|}{Jf_\Delta(u_i)} + \frac{|\psi(v_i)|}{Jf_\Delta(v_i)} . \tag{6.8}$$

For $I_i = I_{i,k} \subset \Delta_k$, using (6.1) and (6.7), we estimate the first term above as in (4.7),

$$\bigvee_{I_{i,k}} \frac{\psi}{Jf_\Delta} \leqslant \frac{1}{2.5} \bigvee_{I_{i,k}} \psi e^{-\gamma k} + C_d(1 + C_d) \int_{I_{i,k}} |\psi| .$$

We estimate the second term in (6.8) using (4.8) and the large images property as in (4.9),

$$\frac{|\psi(u_i)|}{Jf_\Delta(u_i)} + \frac{|\psi(v_i)|}{Jf_\Delta(v_i)} \leqslant \frac{1}{2.5} \bigvee_{I_{i,k}} \psi e^{-\gamma k} + 2C_\beta^{-1}(1 + C_d) \int_{I_{i,k}} |\psi| \, dm_\Delta . \tag{6.9}$$

Putting these estimates together in (6.8) completes the proof of (6.4),

$$\bigvee_{\Delta_0} \mathring{\mathcal{L}}_{\Delta,\varepsilon} \psi \leqslant \frac{2}{2.5} \sum_{k \geqslant 1} \sum_{I_{i,k}} e^{-\gamma k} \bigvee_{I_{i,k}} \psi + (1 + C_d)(C_d + 2C_\beta^{-1}) \int_{I_{i,k}} |\psi|$$

$$\leqslant \frac{4}{5} \|\psi\|_{WV} + (1 + C_d)(C_d + 2C_\beta^{-1}) \int_{\mathring{\Delta}_\varepsilon^1} |\psi| .$$

Finally, (6.5) follows immediately from (6.9) since for $x \in \Delta_0$,

$$|\mathring{\mathcal{L}}_{\Delta,\varepsilon} \psi(x)| \leqslant \sum_{I_i \in \mathcal{I}_\varepsilon, y \in I_i} \frac{|\psi(y)|}{Jf_\Delta(y)} \leqslant \frac{1}{2.5} \sum_{i,k} \bigvee_{I_{i,k}} \psi e^{-\gamma k} + C_\beta^{-1}(1 + C_d) \int_{I_{i,k}} |\psi| \, dm_\Delta$$

$$\leqslant \frac{2}{5} \|\psi\|_{WV} + C_\beta^{-1}(1 + C_d) \int_{\mathring{\Delta}_\varepsilon^1} |\psi| \, dm_\Delta .$$

$\square$

Lemma 6.4. *The operator $\mathcal{L}_\Delta$ has a spectral gap on $WV(\Delta)$.*

Proof. Lemma 6.3 applied to $\mathcal{L}_\Delta$ implies that as an operator on $WV(\Delta)$, $\mathcal{L}_\Delta$ has spectral radius at most 1 and essential spectral radius at most $\max\{e^{-\gamma}, 4/5\}$. Since $\mathcal{L}_\Delta^* m_\Delta = m_\Delta$, 1 is in the spectrum of $\mathcal{L}_\Delta$ so that $\mathcal{L}_\Delta$ is quasi-compact. Indeed, (4.5) implies that the peripheral spectrum has no Jordan blocks. Thus by [Ka, III.6.5], $\mathcal{L}_\Delta$ admits the following representation: there exist finitely many eigenvalues $e^{i\theta_j}$, $j = 1, \ldots, N$ and finite-dimensional eigenprojectors Π_j corresponding to θ_j such that

$$\mathcal{L}_\Delta = \sum_{j=1}^N e^{i\theta_j} \Pi_j + \mathcal{R} , \tag{6.10}$$

where $\mathcal{R}$ has spectral radius strictly less than 1 and $\Pi_j \Pi_k = \Pi_j \mathcal{R} = \mathcal{R}\Pi_j = 0$ for all $j \neq k$.

Note that if $g \in WV(\Delta)$ satisfies $\mathcal{L}_\Delta g = g$, then $g_0 := g \circ (\pi_\Delta|_{\Delta_0})^{-1}$ is in $BV(Y)$ and $\mathcal{L}_0 g_0 = g_0$. Since $\mathcal{L}_0$ has a spectral gap (see the proof of Theorem 4.7), this implies that there can be at most one (normalised) fixed point for $\mathcal{L}_\Delta$. Thus the eigenvalue 1 is simple.

Conversely, if g_0 denotes the unique element of $BV(Y)$ with $\int g_0 \, dm = 1$ such that $\mathcal{L}_0 g_0 = g_0$, then we may define g_Δ such that $\mathcal{L}_\Delta g_\Delta = g_\Delta$ by

$$g_\Delta(x, k) = c_0 g_0 \circ \pi_\Delta(x, 0), \quad \text{for each } (x, k) \in \Delta_k \text{ and } k \geqslant 0,$$

where c_0 is chosen so that $\int_\Delta g_0 \, dm_\Delta = 1$. In particular, for each (x, k) and $k \geqslant 0$,

$$c_0 c_g \leqslant g_\Delta(x, k) = g_\Delta(x, 0) \leqslant c_0 C_g , \tag{6.11}$$

where $C_g = |g_0|_\infty$ and $c_g > 0$ is from (4.12). We will use this fact to eliminate the possibility of other eigenvalues of modulus 1.

It is convenient to first establish the following claim.

Claim. *The peripheral spectrum of $\mathcal{L}_\Delta$ on $WV(\Delta)$ is cyclic: if $e^{i\theta}$ is an eigenvalue, then so is $e^{i\theta n}$ for each $n \in \mathbb{N}$.*

Proof of Claim. Since $\mathcal{L}_\Delta$ is a positive operator and $\mathcal{L}_\Delta^* m_\Delta = m_\Delta$, it follows from Rota's Theorem [BG, Theorem 2.2.9] (see also [S, Theorem 1]) that the peripheral spectrum of $\mathcal{L}_\Delta$ on $L^1(m_\Delta)$ is cyclic. It remains to show that this property holds as well in $WV(\Delta)$. We follow the strategy in [K, proof of Theorem 6.1]. For $\theta \in [0, 2\pi)$, define

$$S_{n,\theta} = \frac{1}{n} \sum_{k=0}^{n-1} e^{-\theta k} \mathcal{L}_\Delta^k.$$

It follows from (6.10) that $\lim_{n\to\infty} S_{n,\theta} = \Pi_j$ if $\theta = \theta_j$ and $\lim_{n\to\infty} S_{n,\theta} = 0$ otherwise. Indeed, the convergence in both cases is pointwise uniformly since $\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} e^{(\theta_j - \theta)k} = 0$ whenever $\theta \neq \theta_j$. Then $\bigvee c\psi = |c| \bigvee \psi$ for any constant c implies the sequence converges in $\|\cdot\|_{WV}$.

Since $\mathcal{L}_\Delta$ is an $L^1(m_\Delta)$ contraction (i.e. $\int |\mathcal{L}_\Delta \psi| \, dm_\Delta \leqslant \int |\psi| \, dm_\Delta$), then so is $S_{n,\theta}$, and since $WV(\Delta)$ is dense in $L^1(m_\Delta)$, $\lim_{n\to\infty} S_{n,\theta}$ extends to a bounded linear operator on $L^1(m_\Delta)$, with convergence in $\|\cdot\|_{L^1(m_\Delta)}$. Taking $\theta = \theta_j$, we may view $\Pi_j : L^1(m_\Delta) \to \Pi_j(WV(\Delta))$.

Now if $0 \neq f \in L^1(m_\Delta)$ satisfies $\mathcal{L}_\Delta f = e^{i\theta} f$, then $0 \neq f = \lim_{n\to\infty} S_{n,\theta}$, which implies that $\theta = \theta_j$ for some j and $f \in \Pi_j(WV(\Delta))$ is an element of $WV(\Delta)$. Thus the peripheral spectra of $\mathcal{L}_\Delta$ on $WV(\Delta)$ and $L^1(\Delta)$ coincide. $\square$

Returning to the proof of the lemma, suppose for the sake of contradiction that 1 is not the only eigenvalue of modulus 1. Then according to the Claim, there exists $h \in WY(\Delta)$ and $p, q \in \mathbb{N} \setminus \{0\}$ such that $\mathcal{L}_\Delta h = e^{i\pi p/q} h$. It follows that h is complex-valued and that $\int_\Delta \mathrm{Re}(h) \, dm_\Delta = \int_\Delta \mathrm{Im}(h) \, dm_\Delta = 0$.

Since $\mathcal{L}_\Delta^q h = h$, h takes on all its possible values in the first q levels, $\cup_{k=0}^{q-1} \Delta_k$. In particular, $\sup_\Delta |\mathrm{Re}(h)| = \sup_{\cup_{k=0}^{q-1} \Delta_k} |\mathrm{Re}(h)|$, and similarly for $\mathrm{Im}(h)$. Thus by (6.11), we may choose $\kappa > 0$ such that

$$\psi := \kappa \mathrm{Re}(h) + g_\Delta \quad \text{satisfies} \quad \inf_\Delta \psi > 0.$$

Note $\int_\Delta \psi \, dm_\Delta = 1$. Next, for $s \in \mathbb{R}$, define

$$\psi_s = s\psi + (1-s)g_\Delta = s\kappa \mathrm{Re}(h) + g_\Delta. \tag{6.12}$$

Note that ψ_s also takes on all its possible values in $\cup_{k=0}^{q-1} \Delta_k$ and $\mathcal{L}_\Delta^q \psi_s = \psi_s$.

Let $\mathcal{S} = \{s \in \mathbb{R} : \mathrm{ess\,inf}_\Delta \psi_s > 0\}$. By construction of ψ, and the compactness of $\cup_{k=0}^{q-1} \Delta_k$, $\mathcal{S}$ contains $[0, 1]$ and is open.

We will show that $\mathcal{S}$ contains $\mathbb{R}^+$. Suppose not. Let $t > 1$ be an endpoint of $\mathcal{S}$ that is not in $\mathcal{S}$. Then $\mathrm{ess\,inf}_\Delta \psi_t = \mathrm{ess\,inf}_{\cup_{k=0}^{q-1} \Delta_k} \psi_t = 0$. Without loss of generality, we may work with a representative of ψ_t that is lower semicontinuous.[7] Since $\int_\Delta \psi_t \, dm_\Delta = 1$, there must exist $(y, j) \in \cup_{k=0}^{q-1} \Delta_k$ such that $\psi_t(y) > 0$. By lower semicontinuity, there exists an interval $A \subset \Delta_j$ such that $\inf_A \psi_t := a > 0$.

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[7] We use here that any function of bounded variation can be written as the difference of two monotonic functions so that one-sided limits exist at each point [R, Theorem 5, Section 5.2].

Since f_Δ is topologically mixing, we can find $N \in \mathbb{N}$ such that $f_\Delta^{N+i}(A) \supseteq \cup_{k=0}^{q-1}\Delta_k$, for $i = 0, \ldots, q-1$. One of these iterates must equal nq for some $n \in \mathbb{N}$. Thus for any $x \in \cup_{k=0}^{q-1}\Delta_k$,

$$\psi_t(x) = \mathcal{L}_\Delta^{nq}\psi_t(x) \geqslant \frac{a}{\sup_A |Df^{nq} \circ \pi_\Delta|} > 0 \,,$$

since $\sup |Df| < \infty$ and n is fixed. This proves that $t \in \mathcal{S}$ so in fact $\mathbb{R}^+ \subset \mathcal{S}$. By (6.12), this implies that $\text{Re}(h) \geqslant 0$, but since $\int_\Delta \text{Re}(h)\, dm_\Delta = 0$, it must be that $\text{Re}(h) \equiv 0$. A similar argument forces $\text{Im}(h) \equiv 0$, providing the needed contradiction. $\qquad\square$

As in (4.10), denote by $||| \cdot |||$ the norm which views $\mathcal{L}_\Delta$ as an operator from $WV(\Delta)$ to $L^1(m_\Delta)$.

Lemma 6.5. *There exists $C > 0$ such that for any $\varepsilon > 0$, $|||\mathcal{L}_\Delta - \mathring{\mathcal{L}}_{\Delta,\varepsilon}||| \leqslant C\mu(H_\varepsilon(z))^{1-\gamma/\zeta}$.*

Proof. Let $\psi \in WV(\Delta)$. Then,

$$\int_\Delta |(\mathcal{L}_\Delta - \mathring{\mathcal{L}}_{\Delta,\varepsilon})\psi|\, dm_\Delta \leqslant \int_{H_\Delta} |\psi|\, dm_\Delta \leqslant \|\psi\|_{WV} \sum_{k\geqslant 1} e^{\gamma k} m_\Delta(H_\Delta \cap \Delta_k) \,.$$

This expression can be made small in $\mu(H_\varepsilon(z))$ as follows. Let $d\mu_\Delta = g_\Delta dm_\Delta$. Then $(\pi_\Delta)_*\mu_\Delta = \mu$, so that using (6.11),

$$\sum_{k\geqslant 1} e^{\gamma k} m_\Delta(H_\Delta \cap \Delta_k) = \sum_{k=1}^{-\zeta^{-1}\log\mu(H_\varepsilon(z))} e^{\gamma k} m_\Delta(H_\Delta \cap \Delta_k) + \sum_{k\geqslant -\zeta^{-1}\log\mu(H_\varepsilon(z))} e^{\gamma k} m_\Delta(H_\Delta \cap \Delta_k)$$

$$\leqslant \mu(H_\varepsilon(z))^{-\gamma/\zeta}(c_0 c_g)^{-1} \sum_{k=1}^{-\zeta^{-1}\log\mu(H_\varepsilon(z))} \mu_\Delta(H_\Delta \cap \Delta_k) + \sum_{k\geqslant -\zeta^{-1}\log\mu(H_\varepsilon(z))} C e^{(\gamma-\zeta)k}$$

$$\leqslant C\mu(H_\varepsilon(z))^{1-\gamma/\zeta} \,.$$

$\qquad\square$

With these elements in place, we are ready to prove Proposition 6.1.

Proof of Proposition 6.1. For fixed $\beta > 0$, the Lasota-Yorke inequalities in Lemma 6.3 have uniform constants. Thus the spectral radius of $\mathring{\mathcal{L}}_{\Delta,\varepsilon}$ has essential spectral radius at most $\max\{e^{-\gamma}, \frac{4}{5}\}$ for all β-allowable holes.

This, together with Lemma 6.5 implies by [KL1, Corollary 1] that the spectra and spectral projectors of $\mathring{\mathcal{L}}_{\Delta,\varepsilon}$ outside the disk of radius $\max\{e^{-\gamma}, \frac{4}{5}\}$ vary Hölder continuously in $\mu(H_\varepsilon(z))$. Thus there exists $\varepsilon_\beta(\Delta) > 0$ such that for all β-admissible holes with $\varepsilon < \varepsilon_\beta(\Delta)$, the operators $\mathring{\mathcal{L}}_{\Delta,\varepsilon}$ enjoy a uniform spectral gap and can be decomposed as in the statement of the proposition. $\qquad\square$

Our final lemma of this section demonstrates that the spectral radius of $\mathring{\mathcal{L}}_{\Delta,\varepsilon}$ yields the escape rate from both Δ and I.

Lemma 6.6. *Under the hypotheses of Proposition 6.1, $-\log\lambda_{\Delta,\varepsilon} = \mathfrak{e}(H_\varepsilon(z))$, where $\mathfrak{e}(H_\varepsilon(z))$ is from (1.1).*

Proof. Using Proposition 6.1, we compute

$$-\mathfrak{e}(H_\varepsilon(z)) = \lim_{n\to\infty} \frac{1}{n} \log \mu(\cap_{i=0}^{n-1} f^{-i}(I \setminus H_\varepsilon)) = \lim_{n\to\infty} \frac{1}{n} \log \mu_\Delta(\mathring{\Delta}_\varepsilon^n)$$

$$= \lim_{n\to\infty} \frac{1}{n} \log \int_\Delta \mathring{\mathcal{L}}_{\Delta,\varepsilon}^n(g_\Delta)\, dm_\Delta = \lim_{n\to\infty} \frac{1}{n} \log \left(\lambda_{\Delta,\varepsilon}^n \mathring{e}_{\Delta,\varepsilon}(g_\Delta) + \int_\Delta \mathcal{R}_{\Delta,\varepsilon}^n(g_\Delta)\, dm_\Delta \right)$$

$$= \log \lambda_{\Delta,\varepsilon}\,,$$

since $\mathring{e}_{\Delta,\varepsilon}(g_\Delta) > 0$ due to (6.11). $\qquad\square$

6.2. **Hitting time statistics for β-allowable holes.** To prove Theorem 1.2, we will compute the following limit for fixed $\alpha > 0$ amd $t > 0$,

$$\lim_{\varepsilon\to0} \frac{-1}{t\mu(H_\varepsilon(z))^{1-\alpha}} \log \mu \left(r_{H_\varepsilon(z)} > \frac{t}{\mu(H_\varepsilon(z))^\alpha} \right)\,.$$

Recall that $\alpha > 0$ was fixed at the beginning of Section 6.1 and affected the chosen value of γ via (6.2).

Since $\pi_\Delta \circ f_\Delta = f \circ \pi_\Delta$, $\pi_\Delta(H_\Delta) = H_\varepsilon(z)$ then $r_\Delta := r_{H_\varepsilon(z)} \circ \pi_\Delta$ defines the first hitting time to H_Δ. Then since $(\pi_\Delta)_* \mu_\Delta = \mu$, it is equivalent to estimate,

$$\lim_{\varepsilon\to0} \frac{-1}{t\mu_\Delta(H_\Delta)^{1-\alpha}} \log \mu_\Delta \left(r_\Delta > \frac{t}{\mu_\Delta(H_\Delta)^\alpha} \right)\,.$$

Setting $n_\varepsilon = \lfloor t\mu_\Delta(H_\Delta)^{-\alpha} \rfloor = \lfloor t\mu(H_\varepsilon(z))^{-\alpha} \rfloor$, we estimate as in [BDT, Section 2.5],

$$\mu_\Delta(r_\Delta > n_\varepsilon) = \int_{\mathring{\Delta}_\varepsilon^{n_\varepsilon}} g_\Delta\, dm_\Delta = \int_\Delta \mathring{\mathcal{L}}_{\Delta,\varepsilon}^{n_\varepsilon+1} g_\Delta\, dm_\Delta$$

$$= \lambda_{\Delta,\varepsilon}^{n_\varepsilon+1} \int_\Delta \lambda_{\Delta,\varepsilon}^{-n_\varepsilon-1} \mathring{\mathcal{L}}_{\Delta,\varepsilon}^{n_\varepsilon+1}(g_\Delta - \mathring{g}_{\Delta,\varepsilon})\, dm_\Delta + \lambda_{\Delta,\varepsilon}^{n_\varepsilon+1} \int_\Delta \mathring{g}_{\Delta,\varepsilon}\, dm_\Delta\,,$$

where $\mathring{g}_{\Delta,\varepsilon}$ is from Proposition 6.1. Thus,

$$\log \mu_\Delta(r_\Delta > n_\varepsilon) = (n_\varepsilon + 1)\lambda_{\Delta,\varepsilon} + \log \left(1 + \int_\Delta \lambda_{\Delta,\varepsilon}^{-n_\varepsilon-1} \mathring{\mathcal{L}}_{\Delta,\varepsilon}^{n_\varepsilon+1}(g_\Delta - \mathring{g}_{\Delta,\varepsilon})\, dm_\Delta \right)\,.$$

Dividing by $-t\mu_\Delta(H_\Delta)^{1-\alpha}$, we see that the first term becomes simply $-\frac{\log \lambda_{\Delta,\varepsilon}}{\mu_\Delta(H_\Delta)}$. Since $-\log \lambda_{\Delta,\varepsilon} = \mathfrak{e}(H_\varepsilon(z))$ by Lemma 6.6, the first term yields $\mathrm{esc}(z)$, which is either 1 or $1 - \lambda_z^{-1/\ell}$ as needed, as $\varepsilon \to 0$ according to Theorem 1.1. It remains to show that the second term tends to 0 as $\varepsilon \to 0$.

Using the spectral decomposition in Proposition 6.1, we define $c_\varepsilon = \mathring{e}_{\Delta,\varepsilon}(g_\Delta)$ and write

$$\lambda_{\Delta,\varepsilon}^{-n_\varepsilon-1} \mathring{\mathcal{L}}_{\Delta,\varepsilon}^{n_\varepsilon+1}(g_\Delta - \mathring{g}_{\Delta,\varepsilon}) = (c_\varepsilon - 1)\mathring{g}_{\Delta,\varepsilon} + \lambda_{\Delta,\varepsilon}^{-n_\varepsilon-1}\mathcal{R}_{\Delta,\varepsilon}^{n_\varepsilon+1} g_\Delta\,.$$

Integrating this equation, we see that we must estimate,

$$\log \left(1 + \int_\Delta \lambda_{\Delta,\varepsilon}^{-n_\varepsilon-1} \mathring{\mathcal{L}}_{\Delta,\varepsilon}^{n_\varepsilon+1}(g_\Delta - \mathring{g}_{\Delta,\varepsilon})\, dm_\Delta \right) = \log \left(c_\varepsilon + \int_\Delta \lambda_{\Delta,\varepsilon}^{-n_\varepsilon-1}\mathcal{R}_{\Delta,\varepsilon}^{n_\varepsilon+1} g_\Delta\, dm_\Delta \right)\,. \qquad (6.13)$$

Again using Proposition 6.1, we bound the integral by,

$$\left| \int_\Delta \lambda_{\Delta,\varepsilon}^{-n_\varepsilon-1}\mathcal{R}_{\Delta,\varepsilon}^{n_\varepsilon+1} g_\Delta\, dm_\Delta \right| \leqslant A_\beta e^{-\sigma_\beta(n_\varepsilon+1)}\|g_\Delta\|_{WV} \leqslant C e^{-\sigma_\beta t\mu(H_\varepsilon(z))^{-\alpha}}\,,$$

and this quantity is super-exponentially small in $\mu(H_\varepsilon(z))$. By Lemma 6.5 and [KL1, Corollary 1],

$$|c_\varepsilon - 1| = |\mathring{e}_{\Delta,\varepsilon}(g_\Delta) - e_\Delta(g_\Delta)| \leqslant C\mu(H_\varepsilon(z))^{1-\gamma/\zeta} \log \mu(H_\varepsilon(z))^{-1}\,.$$

Putting these estimates together in (6.13) and dividing by $t\mu(H_\varepsilon(z))^{1-\alpha}$, we obtain

$$\lim_{\varepsilon \to 0} \frac{1}{t\mu(H_\varepsilon(z))^{1-\alpha}} \log\left(1 + \mathcal{O}(-\mu(H_\varepsilon(z))^{1-\gamma/\zeta} \log\mu(H_\varepsilon(z)))\right)$$
$$= \lim_{\varepsilon \to 0} \frac{1}{t} \mathcal{O}\left(-\mu(H_\varepsilon(z))^{\alpha-\gamma/\zeta} \log\mu(H_\varepsilon(z))\right),$$

which tends to 0 since $\alpha > \gamma/\zeta$ by (6.2). The above limit $\varepsilon \to 0$ is understood to be taken along sequences of β-allowable holes.

6.3. **Proof of Theorem 1.2 via approximation when** $z \in \mathrm{orb}(f(c))$. Section 6.2 proves Theorem 1.2 when $z \in \mathrm{orb}(f(c))$ for each $\alpha > 0$ and $\beta > 0$ along sequences $(\varepsilon_n)_n$ where each ε_n is β-allowable. It remains to consider the alternative case when $\alpha > 0$ is still fixed and we have to approximate a given sequence ε_n by β-allowable ε_n'. The approximation follows closely the strategy in Section 5. As in that section, we first present the argument in the case that $z \in \mathrm{orb}(f(c))$ is periodic.

Recall that if H_ε is β-allowable, then it is also β'-allowable for any $\beta' < \beta$, so as in Section 5, we take our approximating sequence with β tending to 0. As before, we assume $\beta < (2\lambda_z)^{-1}$.

Using precisely the same discussion and notation as in Section 5, we suppose that each non-β-left-allowable value of ε satisfies $z-\varepsilon \in (c_{i,j,k+1}-\beta v_{i,j,k}, c_{i,j,k+1}+\beta v_{i,j,k})$ for some i, j, k. We approximate ε from above by $\varepsilon_o^L := z - (c_{i,j,k+1} - \beta v_{i,j,k})$ and from below by $\varepsilon_u^L := z - (c_{i,j,k+1} + \beta v_{i,j,k})$. Both ε_o^L and ε_u^L are β-left-allowable and $\varepsilon \in (\varepsilon_u^L, \varepsilon_o^L)$. Applying (5.1), we have

$$\frac{\varepsilon_u^L}{\varepsilon_o^L} \gtrsim 1 - 2\beta\lambda_z \quad \text{and} \quad \frac{\varepsilon_o^L}{\varepsilon_u^L} \lesssim 1 + 2\beta\lambda_z\,.$$

The right hand estimates for the analogous ε_u^R and ε_o^R enjoy similar bounds.

To complete the proof of Theorem 1.2, we consider the following limit as $\varepsilon \to 0$ for fixed $t, \alpha > 0$,

$$\frac{-1}{t\mu(H_\varepsilon(z))^{1-\alpha}} \log\mu\left(r_{H_\varepsilon(z)} > \frac{t}{\mu(H_\varepsilon(z))^\alpha}\right)\,. \tag{6.14}$$

We first estimate this from below. Let r_u denote the first hitting time to the smaller set $(z-\varepsilon_u^L, z+\varepsilon_u^R) \subset H_\varepsilon(z)$. Note that $r_u > r_{H_\varepsilon(z)}$, and $\frac{\mu(z-\varepsilon_u^L, z+\varepsilon_u^R)}{\mu(z-\varepsilon, z+\varepsilon)} \geqslant C_\varepsilon(1 - 2\beta\lambda_z)^{\frac{1}{\ell}}$ where $C_\varepsilon \to 1$ as $\varepsilon \to 0$.

Setting $s = t(C_\varepsilon(1 - 2\beta\lambda_z)^{\frac{1}{\ell}})^\alpha$ we estimate (6.14) from below by,

$$\frac{\mu(z - \varepsilon_u^L, z + \varepsilon_u^R)^{1-\alpha}}{\mu(z - \varepsilon, z + \varepsilon)^{1-\alpha}} \frac{-(C_\varepsilon(1 - 2\beta\Lambda_z)^{1/\ell})^\alpha}{s\mu(z - \varepsilon_u^L, z + \varepsilon_u^R)^{1-\alpha}} \log\mu\left(r_u > \frac{t}{\mu(H_\varepsilon(z))^\alpha}\right)$$
$$\geqslant (C_\varepsilon(1 - 2\beta\Lambda_z)^{1/\ell}) \frac{-1}{s\mu(z - \varepsilon_u^L, z + \varepsilon_u^R)^{1-\alpha}} \log\mu\left(r_u > \frac{s}{\mu(z - \varepsilon_u^L, z + \varepsilon^R)^\alpha}\right)\,.$$

Taking the limit as $\varepsilon \to 0$ yields a lower bound of

$$(1 - 2\beta\lambda_z)^{\frac{1}{\ell}} \cdot \left(1 - \lambda_z^{-1/\ell}\right),$$

where the second factor comes from the application of Theorem 1.2 to β-allowable holes in the case that $z \in \mathrm{orb}(f(c))$ is periodic.

Similarly, one obtains an upper bound for (6.14) of $(1 + 2\beta\lambda_z)^{\frac{1}{\ell}} \cdot \left(1 - \lambda_z^{-1/\ell}\right)$. Since these bounds hold for all sufficiently small β, we take $\beta \to 0$ to obtain the required limit for Theorem 1.2 along an arbitrary sequence $(\varepsilon_n)_n$.

Finally, if $z \in \operatorname{orb}(f(c))$ is preperiodic then the above calculations all go through similarly. As in Section 5, from the construction of (Y, F) in Section 3.4, the periodic structure of the postcritical orbit can be pulled back to z to generate the $(a_i)_i$, $(b_i)_i$ required, but the resulting bounds are of the form $(1 \pm 2\beta\lambda)$ where $\lambda = |Df^p(f^{k_0}(z))|$. Thus they tend to 1 as $\beta \to 0$, as required.

6.4. **Proof of Theorem 1.2 when $z \notin \operatorname{orb}(f(c))$.** We explain here how to adapt the results of [BDT, Section 4.2.1] to achieve the required limit in Theorem 1.2 for any $z \notin \operatorname{orb}(f(c))$. Since f is Misiurewicz, one can choose an interval Y containing z whose endpoints are two points of a periodic orbit $\operatorname{orb}(p)$ where $\operatorname{orb}(p)$ is disjoint from $\operatorname{orb}(z)$ and the interior of Y. If we define F to be the induced map with first return time τ, then F is a full-branched Gibbs-Markov map, which satisfies the conditions of [BDT, Theorem 2.1]. In particular, we consider the parameter n_1 from [BDT, eq. (2.1)] to be chosen: this is chosen so that F^{n_1} has sufficient expansion.

At this point, we find it convenient to consider separately two cases: z is a recurrent point (every ε-neighbourhood of z contains a point in $\operatorname{orb}(f(z))$); or z is a nonrecurrent point.

Case 1: z is a recurrent point. By choice of ∂Y, z is necessarily contained in the interior of a domain Y_i^k of F^k for each $k \geqslant 1$. Thus $z \in Y_{\mathrm{cont}} := \{y \in Y : F^k \text{ is continuous at } y \text{ for all } k \geqslant 1\}$. Moreover, for any sufficiently small ε, $(z - \varepsilon, z + \varepsilon) \subset Y_i^{n_1}$ and so the lengths of images of intervals of monotonicity for $\mathring{F}_{\varepsilon'}^{n_1}$, where $\mathring{F}_{\varepsilon'} := F|_{Y \setminus (z-\varepsilon', z+\varepsilon')}$, have a positive uniform lower bound for all $\varepsilon' < \varepsilon$. This ensures that condition (**U**) of [BDT] is satisfied. Thus we may apply [BDT, Theorem 3.2] to conclude that $L_{\alpha,t}(z) = 1$.

Case 2: z is not a recurrent point. If an accumulation point of $\operatorname{orb}(z)$ lies in Y, then z lies in the interior of a domain of F^k for each k and by the above argument, [BDT, Theorem 3.2] applies so that Theorem 1.2 follows.

If, on the other hand, no accumulation points of $\operatorname{orb}(z)$ lie in Y, then since ∂Y is periodic, z is necessarily an accumulation point of domains $\{Y_i\}_i$ of F. In this case, a modification of the approach of [BDT] is needed on two points.

First, fixing $\beta > 0$, we only consider values of ε_L and ε_R so that $z - \varepsilon_L$ and $z + \varepsilon_R$ are β-deep in intervals of monotonicity for F^{n_1} around z. These constitute β-allowable holes $(z - \varepsilon_L, z + \varepsilon_R)$ so that the uniformly large images property (**U**) of [BDT] applies to the punctured induced map, $\mathring{F}_\varepsilon^{n_1}$. In particular, under these conditions, the associated punctured transfer operators enjoy a uniform spectral gap in $BV(Y)$ for all sufficiently small β-allowable holes.

Applying the results of [KL2] as in [BDT, Section 2.3], we see that $\operatorname{esc}(z) = 1$ as long as

$$\lim_{\varepsilon \to 0} \frac{\mu(E_\varepsilon^k)}{\mu(H_\varepsilon(z))} = 0 \text{ for each } k \geqslant 0,$$

where

$$E_\varepsilon^k = \{y \in H_\varepsilon(z) : F^i(y) \notin H_\varepsilon(z), i = 1, \ldots, k, \text{ and } F^{k+1}(y) \in H_\varepsilon(z)\}.$$

Since F is full branched, each domain Y_i^k of F^k has an interval which maps onto $H_\varepsilon(z)$. However, if $Y_i^k \subset H_\varepsilon(z)$, then $\tau(Y_i^k) \geqslant \log(|Y|\varepsilon^{-1})/\log|Df|_\infty \geqslant C_0 \log \varepsilon^{-1}$, where $\varepsilon = \max\{\varepsilon_L, \varepsilon_R\}$. This implies that the contribution to E_ε^k from the collection of such intervals is dominated by

$$\sum_{j \geqslant C_0 \log \varepsilon^{-1}} C_k |H_\varepsilon(z)| \lambda_{\mathrm{per}}^{-j} \leqslant C_k' \varepsilon^{C_0 \lambda_{\mathrm{per}}} |H_\varepsilon(z)|,$$

where we have applied Proposition 2.1 since F is full branched. Since the invariant measure μ has density at z bounded away from 0 and ∞, we estimate,

$$\frac{\mu(E_\varepsilon^k)}{\mu(H_\varepsilon(z))} \leqslant C_k'' \varepsilon^{C_0 \lambda_{\mathrm{per}}} \to 0 \ \text{ as } \varepsilon \to 0 \text{ for each } k.$$

With these modifications, $\mathrm{esc}(z) = 1$ and [BDT, Theorem 3.2] implies the desired limit $L_{\alpha,t}(z) = 1$ as well along sequences of β-allowable holes.

The approximation of more general holes $(z - \varepsilon, z + \varepsilon)$ by β-allowable holes in order to prove the required limit 1 for Theorem 1.2 proceeds as in Section 6.3. The case here is simpler since there is no density spike so we do not need to maintain bounded ratios $\frac{\varepsilon_u^L}{\varepsilon_u^R}$, $\frac{\varepsilon_o^L}{\varepsilon_o^R}$ for the approximating holes.

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