Thermodynamic Formalism for Dispersing Billiard Maps and Flows
Lecture 3: Equilibrium States for $-t\tau$

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Lecture 3: Equilibrium States for $-t\tau$

Goal for today: By considering the roof function as a potential, we are able to access some equilibrium states for the flow.

- Discuss sparse recurrence to singularities in the context of a complexity conjecture. This relates to decay of correlations for the MME for the map.
- Generalize our discussion of the case $t = 0$ in Lecture 2 to include the weight $e^{-t\tau}$, $t \geq 0$. Controlling this for t large enough, we prove existence of an MME for the billiard flow.

References: M. Demers and A. Korepanov, *Rates of mixing for the measure of maximal entropy for dispersing billiard maps*, preprint '22.

J. Carrand, *A family of natural equilibrium measures for Sinai billiard flows*, arXiv:2208.14444v2 (March, 2023).

V. Baladi, J. Carrand and M. Demers, *Measure of maximal entropy for finite horizon Sinai billiard flows*, preprint '22.

Sparse Recurrence to Singularities

In Lecture 2, we proved existence and uniqueness of a MME μ_0 for a finite horizon Sinai billiard under an additional assumption of sparse recurrence to singularities.

Recall $h_* = \lim_{n\to\infty} \frac{1}{n} \log(\#\mathcal{M}_0^n)$, $\quad \mathcal{M}_0^n =$ domains of continuity of T^n

- Fix $n_0 \in \mathbb{N}$ and an angle φ_0 close to $\pi/2$.
- Let $s_0 \in (0,1)$ be the smallest number such that any orbit of length n_0 has at most $s_0 n_0$ collisions with $|\varphi| \geq \varphi_0$.

Finite horizon guarantees that we can always choose n_0 and φ_0 so that $s_0 < 1$. (Indeed, no triple tangencies implies that $s_0 \leq \frac{2}{3}$.)

Assumption: $h_* > s_0 \log 2$

We are not aware of any billiard table for which this assumption fails.

Complexity Conjecture

Recall the linear complexity bound of Bunimovich for a finite horizon Sinai billiard:

There exists $K > 0$ depending only on the configuration of scatterers such that $N(\mathcal{S}_n) \leq Kn$ for all $n \geq 1$.

In fact, a much stronger complexity bound is conjectured to hold.

Conjecture [Balint, Toth '08]: For 'typical' finite horizon billiard tables, the complexity is bounded, i.e.

$$\exists K > 0 \text{ s.t. } N(\mathcal{S}_n) \leq K \text{ for all } n \geq 0.$$

Lemma ([D., Korepanov '22])

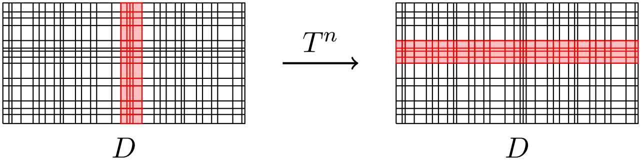
If T has bounded complexity then for any $\varepsilon > 0$, there exists φ_0 and n_0 such that $s_0(\varphi_0, n_0) < \varepsilon$.

This implies that for 'typical' billiard tables, $h_* > s_0 \log 2$ holds.

Rate of Mixing for MME

Main Idea: Construct a recurrence scheme to a Cantor rectangle with hyperbolic product structure.

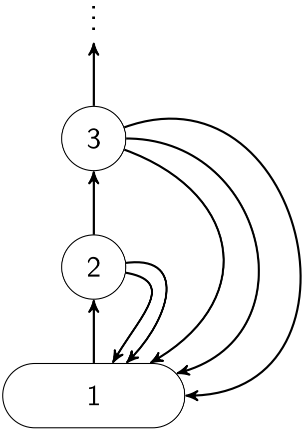
- Key feature is to count the number of first returns to the reference set rather than the measure of the set of points which have not returned.



- $\mathcal{A}_n$ = set of vertical rectangles making a proper return at time n
- $\mathcal{R}_n$ = set of vertical rectangles making a 'prime' proper return at time n

Symbolic Model

Let $r_n = \#\mathcal{R}_n$, $n \geq 1$. Consider a directed graph as on the picture below with r_n arrows going from $\textcircled{n}$ to $\textcircled{1}$.



- Label the edges from $\textcircled{n}$ to $\textcircled{1}$ by elements of $\mathcal{R}_n$, and the edges $\textcircled{n} \to \textcircled{n+1}$ by E_n.
- Δ = set of two-sided admissible sequences in the alphabet $\mathfrak{A} = \{\mathcal{R}_n\}_{n\geq 1} \cup \{E_n\}_{n\geq 1}$ which visit $\textcircled{1}$ infinitely often in the future.
- $\sigma \colon \Delta \to \Delta$, left shift.

Symbolic Model

Proposition

For each $\varepsilon \in \left(0, \frac{h_*}{s_0 \log 2} - 1\right)$, there exists $C \geq 1$ s.t. for all $n \geq 0$,

a) $C^{-1} e^{n h_*} \leq \# \mathcal{A}_n \leq C e^{n h_*}$

b) $\# \mathcal{R}_n \leq C e^{n h_*} n^{-\frac{h_*}{s_0 \log 2} + \varepsilon}$

(a) implies that the inducing scheme sees the full topological entropy of the system.

(b) is a weak bound on the pressure at infinity.

- Prove that $\sum_n r_n e^{-h_* n} = 1$.
- Since $\sum_n n r_n e^{-h_* n} < \infty$, we can define a Markov measure μ_Δ with entropy e^{h_*}.
- $(\sigma, \Delta, \mu_\Delta)$ is a Young tower. Rate of decay of correlations follows from [Young '99]
- The projection of μ_Δ to M has entropy e^{h_*}. By uniqueness, it must be the MME μ_0.

Rate of Mixing for MME

Theorem ([D., Korepanov '22])

Assume $h_* > 2s_0 \log 2$. For each $\alpha > 0$ and $\varepsilon \in \big(0, \frac{h_*}{s_0 \log 2} - 2\big)$, there exists there exists $C > 0$ s.t. for all $f, g \in C^\alpha(M)$,

$$\left| \int f\, g \circ T^n \, d\mu_0 - \int f \, d\mu_0 \int g \, d\mu_0 \right| \le C |f|_{C^\alpha} |g|_{C^\alpha} n^{-\frac{h_*}{s_0 \log 2} + 2 + \varepsilon} .$$

If $h_* > 4s_0 \log 2$, this rate of decay also implies limit theorems such as the Central Limit Theorem and the Almost-Sure Invariance Principle.

Corollary

If T has bounded complexity, then $h_* > 4s_0 \log 2$ holds. μ_0 has super-polynomial decay of correlations for Hölder observables, and enjoys the CLT and Almost-Sure Invariance Principle.

MME for the Billiard Fllow

Goal: Prove the existence and uniqueness of a MME for the billiard flow

- Flow is partially hyperbolic, so the transfer operator is difficult to work with directly.
- Using the roof function as a potential, can access equilibrium states for the flow via the transfer operator for the map.
- Similar situation to $t = 0$ from Lecture 2: the relevant operator will not have a spectral gap.

The Billiard Flow

Let $\mathcal{Q} = \mathbb{T}^2 \setminus \cup_i B_i$ denote the billiard table; scatterers B_i, ∂B_i are C^3 with strictly positive curvature.

The phase space for the flow is

$$\Omega = \{(x, y, \omega) \in \mathbb{T}^3 : (x, y) \in \mathcal{Q}, \ \ \omega \in \mathbb{S}^1\} / \sim$$

where at collisions, $(x, y, \omega^-) \sim (x, y, \omega^+)$.

Between collisions, the **billiard flow** is defined by

$$\Phi_t(x, y, \omega) = (x + t \cos \omega, y + t \sin \omega, \omega),$$

While at collisions,

$$x^+ = x^-, \qquad y^+ = y^-, \qquad \omega^+ = \omega^- + \pi - 2\varphi,$$

where φ is the angle between the post-collision velocity and the outward normal to the boundary at the point of collision.

The flow is continuous, but $D\Phi_t$ blows up at tangential collisions.

Statistical Properties w.r.t. Smooth Invariant Measures

The flow Φ_t preserves Lebesgue measure on Ω.

The map T preserves a smooth measure on M, $\mu_{\text{SRB}} = \cos\varphi \, dr \, d\varphi$

With respect to these measures, many statistical properties are known:

- ergodic and mixing
  - map and flow [Sinai '70]
- Bernoulli
  - map and flow [Gallavotti, Ornstein '74]
- many limit theorems:
  - CLT map [Bunimovich, Sinai '81], flow [Melbourne, Torok '04]
  - ASIP map and flow [Melbourne, Nicol '05],
- exponential decay of correlations
  - map [L.-S. Young '98]
  - flow [Baladi, D., Liverani '18]

Abramov Formula

For $x \in M$, define $\tau(x) =$ distance from x to $T(x)$ in $\mathcal{Q}$.

View the billiard flow Φ_t as a suspension of T with roof function τ.

1-1 correspondence between invariant measures for the map and the flow.

ν invariant probability measure for Φ_1 satisfies, $\nu = \dfrac{\mu}{\int \tau \, d\mu} \otimes$ Leb,

where μ is a T-invariant probability measure.

$$\text{Abramov's formula} \implies h_\nu(\Phi_1) = \frac{h_\mu(T)}{\int \tau \, d\mu} \, .$$

Family of Potentials for the Map, $-t\tau$

Define the **pressure** of the potential, $-t\tau$, $t \in \mathbb{R}$,

$$P(t) = \sup\{h_\mu(T) - t \int_M \tau \, d\mu : \mu \text{ inv. prob. for } T \}$$

μ_t is an **equilibrium state** for $-t\tau$ if μ_t attains the supremum.

- $t = 0$ corresponds to MME for map
- $P(t) = 0 \iff t = h_{\text{top}}(\Phi_1)$ and any corresponding equilibrium state $\mu_{h_{\text{top}}(\Phi_1)}$ lifts to an MME $\nu_{h_{\text{top}}(\Phi_1)}$ for the flow

  Pf: Since $\tau_{\min} \le \tau \le \tau_{\max}$, $\exists!\, t_\star > 0$ s.t. $P(t_\star) = 0$.
  By Abramov, if ν is the lift of μ, then

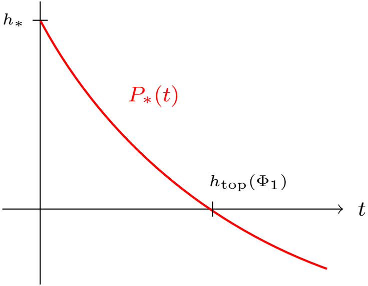
  $$0 \ge \frac{h_\mu(T) - t_\star \int \tau \, d\mu}{\int \tau \, d\mu} = h_\nu(\Phi_1) - t_\star \, .$$

  Moreover, if $\mu_{t_\star}$ is an equilibrium state for $-t_\star\tau$ and $\nu_{t_\star}$ is its lift, then

  $$h_{\nu_{t_\star}}(\Phi_1) = t_\star = \sup\{h_\nu(\Phi_1) : \nu \text{ inv. prob. for } \Phi_t\} = h_{\text{top}}(\Phi_1).$$

Family of Potentials for the Map, $-t\tau$

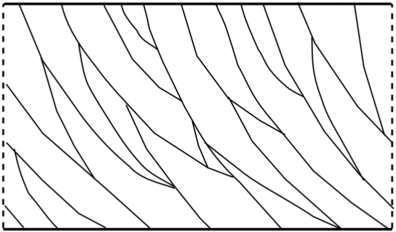
We would like to establish good control of the transfer operator for all $t \in [0, h_{\text{top}}(\Phi_1)]$.



As we saw in Lecture 2, this will require us to obtain uniform control on the growth of stable curves and the size of domains of continuity for T^n, weighted by $e^{-t\tau}$.

Recall: Topological Pressure for the Map

- Let $\mathcal{M}_0^n$ = connected components of $M \setminus \mathcal{S}_n$, $\mathcal{S}_n = \cup_{i=0}^n T^{-i}\mathcal{S}_0$



$M \setminus \mathcal{S}_n$

- Let $\tau_n = \sum_{i=0}^{n-1} \tau \circ T^i$. Define for $t \geq 0$

$$Q_n(t) = \sum_{A \in \mathcal{M}_0^n} |e^{-t\tau_n}|_{C^0(A)}, \quad P_*(t) = \lim_{n\to\infty} \frac{1}{n} \log Q_n(t)$$

- The limit exists since the sequence $\log Q_n(t)$ is subadditive.
- When $t = 0$, $Q_n(t) = \#\mathcal{M}_0^n$ and $P_*(0) =: h_*$ is the topological entropy of the map.

Plan for constructing the equilibrium states

Two steps:

- Prove uniform exponential growth of $Q_n(t)$
- Construct μ_t from eigenfunctions corresponding to maximal eigenvectors (use uniform growth to control spectral radius of associated transfer operators)

Transfer operators: $\mathcal{L}_t f = \left(\frac{f}{J^s T} e^{-t\tau} \right) \circ T^{-1}$

- Due to weight $1/J^s T$, $\mathcal{L}_t$ will not have a spectral gap for any $t \geq 0$. Yet we can follow the program outlined for $t = 0$ to obtain similar results.

Weighted Sums on Stable Curves

The main task in proving the uniform exponential growth of $Q_n(t)$ is controlling the effect of cutting due to singularities.

$\mathcal{W}^s$ set of local stable manifolds, $W \in \mathcal{W}^s$. For $\delta > 0$, define

$\mathcal{G}_n^\delta(W) =$ {connected components of $T^{-n}W$, with pieces longer than δ subdivided to have length between $\delta/2$ and δ}

$S_n^\delta(W) = \{W_i \in \mathcal{G}_n^\delta(W) : |W_i| < \delta/3\}$

Define the weighted sums corresponding to these sets by

$$\mathcal{G}_n^\delta(W,t) = \sum_{W_i \in \mathcal{G}_n^{\delta_t}(W)} |e^{-t\tau_n}|_{C^0(W_i)}$$

$$S_n^\delta(W,t) = \sum_{W_i \in S_n^{\delta_t}(W)} |e^{-t\tau_n}|_{C^0(W_i)}$$

Note: $\mathcal{G}_n^\delta(W,0) = \#\mathcal{G}_n^\delta(W)$, $S_n^\delta(W,0) = \#S_n^\delta(W)$.

Small Singular Pressure

We say **Small Singular Pressure (SSP)** holds at $t \geq 0$ for $\varepsilon \in (0, 1/4]$ if there exist $\delta_t > 0$ and $n_t \in \mathbb{N}$ such that $\forall n \geq n_t$,

$$S_n^{\delta_t}(W,t) \leq \varepsilon \, \mathcal{G}_n^{\delta_t}(W,t), \quad \forall \, W \in \mathcal{W}^s \text{ with } |W| \geq \delta_t/3,$$

and

$$\sum_{n \geq 1} \sup_{\substack{W \in \mathcal{W}^s \\ |W| \geq \delta_t/3}} \frac{e^{-nt\tau_{\min}}}{\mathcal{G}_n^{\delta_t}(W,t)} < \infty$$

A key property for establishing (SSP) is the **linear complexity bound** due to Bunimovich.

$N(\mathcal{S}_n) =$ maximal number of curves in $\mathcal{S}_n$ intersecting at one point

There exists $K > 0$ depending only on the configuration of scatterers such that $N(\mathcal{S}_n) \leq Kn$ for all $n \geq 1$.

Growth Lemmas and Uniform Exponential Growth

The importance of (SSP) is that it implies uniform growth of pressure with respect to both $W \in \mathcal{W}^s$ and elements of $\mathcal{M}_0^n$.

Proposition ([Carrand '22])

Suppose (SSP) holds for some $t \geq 0$. There exists $c_1 > 0$, $C_2, C_3 \geq 1$ s.t. for all $W \in \mathcal{W}^s$ with $|W| \geq \delta_t$ and $n \geq 0$,

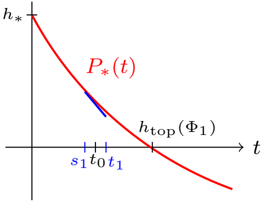
- $c_1 Q_n(t) \leq \mathcal{G}_n^\delta(W,t) \leq C_2 Q_n(t)$
- $e^{nP_*(t)} \leq Q_n(t) \leq C_3 e^{nP_*(t)}$
- $c_1 e^{nP_*(t)} \leq \mathcal{G}_n^\delta(W,t) \leq C_2 C_3 e^{nP_*(t)}$

Pf: (SSP) $\implies$ long elements of $\mathcal{G}_n^\delta(W)$ dominate the weighted sums $\mathcal{G}_n^\delta(W,t)$, and long elements of $\mathcal{M}_0^n$ dominate $Q_n(t)$.

Then a positive fraction of long elements in $\mathcal{G}_n^\delta(W)$ fully cross a positive fraction of long elements of $\mathcal{M}_0^n$ $\implies$ the growth rates are comparable.

Verify (SSP) at $t = h_{\text{top}}(\Phi_1)$ Using a Bootstrapping Argument

(SSP) holds at $t = 0$ [Baladi, D. '20] and for small $t > 0$ [Carrand '22], but we need to extend to $t = h_{top}(\Phi_1)$.



- Suppose (SSP) ends at $t = t_0 < h_{\text{top}}(\Phi_1)$.
- For $\theta \in (e^{-\tau_{\min}}, e^{-\tau_{\min}/2})$, choose $s_1 < t_0$ and $t_1 > t_0$ so that $\theta^{t_1/2} e^{|P_*'(s_1)|(t_1 - s_1)} = 1$.

  Possible since $\tau_{\min} \le |P_*'| \le \tau_{\max}$.
- **Prove**: for $t \in (s_1, t_1)$, $\forall \delta > 0$, $\forall W \in \mathcal{W}^s$, $|W| \ge \delta/3$,

$$\mathcal{G}_n^\delta(W, t) \ge c(t, \delta) e^{n\left(P_*(s_1) - |P_*'(s_1)|(t - s_1)\right)}, \quad \forall\, n \ge 1.$$

Verify (SSP) at $h_{\text{top}}(\Phi_1)$ Via Bootstrapping

Prove: for $t \in (s_1, t_1)$, $\forall \delta > 0$, $\forall W \in \mathcal{W}^s$, $|W| \geq \delta/3$,

$$\mathcal{G}_n^\delta(W, t) \geq c(t, \delta) e^{n\left(P_*(s_1) - |P_*'(s_1)|(t - s_1)\right)}, \quad \forall\, n \geq 1.$$

Idea: Bootstrap via Hölder inequality: For $a_i > 0$, $s < t_0 < t$ and $\eta \in (0,1)$ s.t. $\eta t + (1-\eta)s = t_0$,

$$\sum_i a_i^{t_0} \leq \Big(\sum_i a_i^t\Big)^\eta \Big(\sum_i a_i^s\Big)^{1-\eta}$$

$$\implies \sum_i a_i^t \geq \Big(\sum_i a_i^{t_0}\Big)^{1/\eta} \Big(\sum_i a_i^s\Big)^{1-1/\eta}$$

Apply this to $a_i = |e^{-t\tau_n}|_{C^0(W_i)}$ for $W_i \in \mathcal{G}_n^\delta(W)$. Then good lower bounds on $\mathcal{G}_n^\delta(W, s)$, $s < t_0$ and upper bounds on $\mathcal{G}_n^\delta(W, t_0)$ imply good lower bounds on $\mathcal{G}_n^\delta(W, t)$, $t \in (t_0, t_1)$. □

(1) $\implies$ (SSP) for $t \in (s_1, t_1)$, so $t_0 < h_{\text{top}}(\Phi_1)$ is impossible.

Sparse Recurrence to Singularities

Next we want to use the uniform bounds on $\mathcal{G}_n^\delta(W, t)$ and $Q_n(t)$ to control norm estimates in an appropriate Banach space.

For this we need to modify our assumption of sparse recurrence to singularities. Recall:

- Fix $n_0 \in \mathbb{N}$ and an angle $\varphi_0 < \pi/2$.
- $s_0 := s_0(\varphi_0, n_0) \in (0, 1)$, the smallest number such that any orbit of length n_0 has at most $s_0 n_0$ collisions with $|\varphi| \geq \varphi_0$.

Finite horizon condition $\implies \exists n_0, \varphi_0$ so that $s_0 < 1$.

Assumption for map MME: $P_*(0) = h_* > s_0 \log 2$

Sparse Recurrence to Singularities

To extend our estimates to $t = h_{\mathsf{top}}(\Phi_1)$, we need a slightly stronger assumption.

Assumption for flow MME: $h_{\mathsf{top}}(\Phi_1)\tau_{\min} > s_0 \log 2$

The function $P_*(t) + t\tau_{\min}$ is decreasing for $t \geq 0$, so $h_{\mathbf{top}}(\Phi_1)\tau_{\min} > s_0 \log 2$ implies

$$P_*(t) + t\tau_{\min} > s_0 \log 2, \quad \text{for } t < h_{\mathsf{top}}(\Phi_1).$$

In particular, it implies $h_* > s_0 \log 2$, which is our assumption for the map MME.

If the complexity conjecture holds, this is satisfied for typical finite horizon Sinai billiard tables.

Definition of Norms

Use the same norms as for $t = 0$.

For $f \in C^1(M)$, define the **weak norm** of f by

$$|f|_w = \sup_{W \in \mathcal{W}^s} \sup_{\substack{\psi \in \mathcal{C}^\alpha(W) \\ |\psi|_{\mathcal{C}^\alpha(W)} \le 1}} \int_W f \, \psi \, dm_W \, .$$

Define the **strong stable norm** of f by

$$\|f\|_s = \sup_{W \in \mathcal{W}^s} \sup_{\substack{\psi \in \mathcal{C}^\beta(W) \\ |\psi|_{\mathcal{C}^\beta(W)} \le |\log|W||^\gamma}} \int_W f \, \psi \, dm_W$$

Define the **strong unstable norm** of f by

$$\|f\|_u = \sup_{\varepsilon \le \varepsilon_0} \sup_{\substack{W_1, W_2 \in \mathcal{W}^s \\ d(W_1, W_2) \le \varepsilon}} \sup_{\substack{|\psi_i|_{\mathcal{C}^\alpha(W_i)} \le 1 \\ d_0(\psi_1, \psi_2) = 0}} |\log \varepsilon|^\varsigma \left| \int_{W_1} f \psi_1 - \int_{W_2} f \psi_2 \right|$$

Banach Spaces and Inequalities

Theorem ([Baldi, D. '20], [Carrand '22], [Baladi, Carrand, D. '22])

- $\mathcal{C}^1(M) \subset \mathcal{B} \subset \mathcal{B}_w \subset (\mathcal{C}^\alpha(M))^*$.
- The embedding of the unit ball of $\mathcal{B}$ in $\mathcal{B}_w$ is compact.
- Assume $h_{\mathbf{top}}(\Phi_1)\tau_{\min} > s_0 \log 2$. There exists $C > 0$ such that for all $t \in [0, h_{\mathsf{top}}(\Phi_1)]$, all $f \in \mathcal{B}$ and $n \geq 0$,

$$|\mathcal{L}_t^n f|_w \leq C|f|_w e^{nP_*(t)}$$

$$\|\mathcal{L}_t^n f\|_s \leq C(\sigma^n \|f\|_s + |f|_w) e^{nP_*(t)}, \quad \text{for some } \sigma < 1$$

$$\|\mathcal{L}_t^n f\|_u \leq C(\|f\|_u + \|f\|_s) e^{nP_*(t)}$$

Not true Lasota-Yorke inequalities due to lack of contraction in the strong unstable norm.

Lower bound on $\mathcal{G}_n^\delta(W, t) \implies \|\mathcal{L}_t^n 1\|_s \geq c_0 \delta e^{nP_*(t)}$.
Use this to obtain eigenmeasures for $e^{P_*(t)}$ as limit points.

Construction of μ_t

- The sequence

$$\nu_{t,n} = \frac{1}{n}\sum_{k=0}^{n-1} e^{-kP_*(t)}\mathcal{L}_t^k 1, \text{ is uniformly bounded in } \mathcal{B}.$$

  By compactness, a subsequence converges in $\mathcal{B}_w$.
  Let $\nu_t \in \mathcal{B}_w$ be a limit point of $\nu_{t,n}$. ν_t is a measure.

- Similarly, let $\tilde{\nu}_t \in (B_w)^*$ be a limit point of the sequence

$$\frac{1}{n}\sum_{k=0}^{n-1} e^{-kP_*(t)}(\mathcal{L}_t^*)^k(d\mu_{\mathsf{SRB}}).$$

- Define $\mu_t(\psi) = \dfrac{\tilde{\nu}_t(\psi\nu_t)}{\tilde{\nu}_t(\nu_t)}$, for $\psi \in C^1(M)$.

Since $\mathcal{L}_t\nu_t = e^{P_*(t)}\nu_t$ and $\mathcal{L}_t^*\tilde{\nu}_t = e^{P_*(t)}\tilde{\nu}_t$, we have
$\mu_t(\psi \circ T) = \mu_t(\psi)$, i.e. μ_t is an invariant measure for T.

Hyperbolicity and Ergodicity of μ_*

Key Fact: Although $\nu_t \in \mathcal{B}_w$, it follows from the convergence of $\nu_{t,n}$ to ν_t in the $|\cdot|_w$ norm that $\|\nu_t\|_{\mathcal{B}} < \infty$.

- For any $k \in \mathbb{Z}$, $\exists C_k > 0$ s.t. $\mu_t(\mathcal{N}_\varepsilon(\mathcal{S}_k)) \le C_k(-\log \varepsilon)^{-\gamma}$. $\mathcal{N}_\varepsilon(\mathcal{S}_k) = \varepsilon$-neighborhood of $\mathcal{S}_k$ in M, $\gamma > 1$.
- μ_t-a.e. $x \in M$ has a stable and unstable manifold of positive length. The same is true with respect to ν_t.

Lemma (Absolute continuity of holonomy)

On each Cantor rectangle R, the holonomy map sliding along unstable manifolds in R is absolutely continuous with respect to the conditional measures of μ_t on stable manifolds.

Consequences:

- Each Cantor rectangle R belongs to one ergodic component.
- Since T is topologically mixing, we can force images of rectangles to overlap $\implies$ (T^n, μ_t) is ergodic for all n.

Entropy of μ_t

Define $B(x, n, \varepsilon) = \{y \in M : d(T^{-i}x, T^{-i}y) \le \varepsilon, \forall i \in [0, n]\}$.

Proposition (Measure of Bowen Balls)

There exists $C > 0$ s.t. for all $x \in M$, $n \ge 1$ and $y \in B(x, n, \varepsilon)$,

$$\mu_t(B(x, n, \varepsilon)) \le C e^{-nP_*(t) - t\tau_n(T^{-n}y)}.$$

- [Brin, Katok '81] $\implies$ for μ_t-a.e. $x \in M$,

$$\lim_{\varepsilon \to 0} \limsup_{n \to \infty} -\frac{1}{n} \log \mu_t(B(x, n, \varepsilon)) = h_{\mu_t}(T^{-1}) = h_{\mu_t}(T).$$

- This plus the Proposition implies $h_{\mu_t}(T) \ge P_*(t) + t \int \tau \, d\mu_t$
- But $P_*(t) \ge P(t)$ (easy estimate using classical argument)
- Conclude: $P_*(t) = h_{\mu_t}(T) - t \int \tau \, d\mu_t = P(t)$.

Variational Principle and MME for Map and Flow

Theorem ([Baladi, D. '20], [Carrand '22], [Baladi, Carrand, D. '22])

Let T be a finite horizon Sinai billiard map and let Φ_t be the associated flow.

Assume $h_{\mathbf{top}}(\Phi_1)\tau_{\min} > s_0 \log 2$. Then for all $t \in [0, h_{\mathsf{top}}(\Phi_1)]$,

- $P_*(t) = P(t)$ (variational principle)
- There is a unique equilibrium state μ_t.
- μ_t is T-adapted, Bernoulli and positive on open sets.

Lifting the equilibrium state μ_t for $t = h_{\mathsf{top}}(\Phi_1)$ yields the MME for the flow.

Corollary ([Baladi, Carrand, D. '22])

The measure $\nu_{h_{\mathsf{top}}} := \dfrac{\mu_{h_{\mathsf{top}}}}{\int \tau \, d\mu_{h_{\mathsf{top}}}} \otimes Leb$ is the unique MME of the billiard flow. It is Bernoulli and positive on all open sets.

Some Open Questions

1) What about $t \notin [0, h_{\text{top}}(\Phi_1)]$?
   Is there a phase transition for some $t < 0$ or $t > h_{\text{top}}(\Phi_1)$?

2) Rate of correlation decay for μ_t?
   Can we prove polynomial decay of correlations as for the case $t = 0$, $\lesssim n^{-\frac{h_*}{s_0 \log 2} + 2 + \varepsilon}$?

3) How much of the previous program can be carried out for the infinite horizon Sinai billiard?
   [Chernov, Troubetzkoy '96] $\implies h_{\text{top}}(T) = \infty$.
   But $h_{\text{top}}(\Phi_1) < \infty$ so can one prove similar results for $\mathcal{L}_t$ for t near $h_{\text{top}}(\Phi_1)$?

4) Can similar results (for map or flow) be proved for dispersing billiards with corner points (no cusps)?
   N-step expansion proved in [De Simoi, Toth '14] may not be sufficient. Stronger complexity bound needed.