Thermodynamic Formalism for Dispersing Billiard Maps and Flows
Lecture 2: Geometric Potentials, Topological Pressure and Entropy

Mark Demers
Fairfield University
Research supported in part by NSF grant DMS 2055070

Summer School: Partial Hyperbolicity
Brin Mathematics Research Center
University of Maryland
May 30 - June 9, 2023

Lecture 2: Geometric Potentials and Pressure

Goal for today: Introduce geometric potentials and formulate definition of associated topological pressure for finite horizon Lorentz gas.

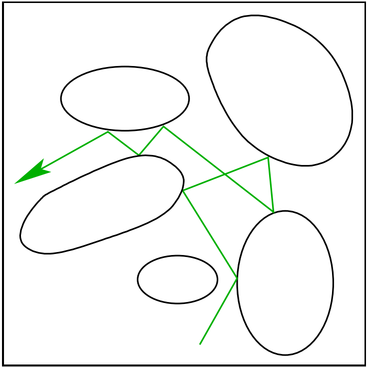
- $t > 0$: the standard picture holds and the transfer operator has a spectral gap. Associated equilibrium state is unique and exponentially mixing.
- $t = 0$: the standard setup fails and the transfer operator has no spectral gap. We are still able to use the Banach spaces to construct a unique measure of maximal entropy.

References: V. Baladi and M. Demers, *On the measure of maximal entropy for finite horizon Sinai billiard maps*, Journal Amer. Math. Soc. (2020).

V. Baladi and M. Demers, *Thermodynamic formalism for dispersing billiards*, Journal of Modern Dynamics (2022).

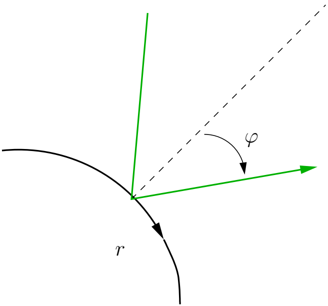
Periodic Lorentz gas (Sinai Billiard) [Sinai '68]

- Billiard table $\mathcal{Q} = \mathbb{T}^2 \setminus \cup_i B_i$; scatterers B_i.
- Boundaries of scatterers are $\mathcal{C}^3$ and have strictly positive curvature.
- Billiard flow is given by a point particle moving at unit speed with elastic collisions at the boundary

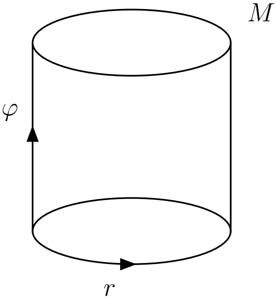


Assume **Finite Horizon** condition: there is an upper bound on the free flight time between collisions.

The Associated Billiard Map



- $r =$ position coordinate oriented clockwise on boundary of scatterer ∂B_i
- $\varphi =$ angle outgoing trajectory makes with normal to scatterer



$M = \big(\cup_i \, \partial B_i \big) \times [-\frac{\pi}{2}, \frac{\pi}{2}]$, the natural "collision" cross-section for the billiard flow.

$T : (r, \varphi) \to (r', \varphi')$ is the first return map: the **billiard map**.

- a hyperbolic map with singularities

Pressure and Equilibrium States

Given a function ϕ, define the **pressure** of ϕ by,

$$P(\phi) := \sup\left\{h_\nu(T) + \int \phi\, d\nu : \nu \text{ invariant prob. measure}\right\}$$

If μ is an invariant probability for T satisfying $h_\mu(T) + \int \phi\, d\mu = P(\phi)$, then μ is an **equilibrium state** for ϕ.

For Hölder continuous ϕ, the existence and uniqueness of equilibrium states has been established for many systems.

- uniformly hyperbolic systems (Anosov and Axiom A) [Sinai '72], [Bowen '74], [Ruelle '78]
- nonuniformly hyperbolic maps and flows
  - Markov partitions [Sarig '11], [Lima, Matheus '18], [Buzzi, Crovisier, Sarig '19]
  - Young towers [Pesin, Senti, Zhang '16]
  - non-uniform specification [Climenhaga, Thompson '13], [Burns, Climenhaga, Fisher, Thompson '18]

Geometric Potentials

Important family of potentials: geometric potentials,

$$t\phi = -t \log J^u T, \quad t \in \mathbb{R}\,.$$

- $t = 1$ gives the smooth invariant measure $\mu_{\mathsf{SRB}} = \cos\varphi dr d\varphi$. This is an equilibrium state for ϕ and uniqueness is proved in a class of measures whose support decays sufficiently near singularities [Katok, Strelcyn '86].
- $t = 0$ yields the measure of maximal entropy [Baladi, D. '20]. This is Bernoulli (and hence mixing) and globally unique, but its rate of mixing is not known.
- $t < 0$ implies $P(t) = \infty$ when there is a periodic orbit with grazing collisions. Today restrict to $t \geq 0$.
- [Chen, Wang, Zhang '20] proves existence (but not uniqueness) of equilibrium state for t near 1 using Young towers.

Associated Transfer Operator

The main tool we will use is the transfer operator associated to the potential $t\phi = -t \log J^u T$.

For a smooth hyperbolic system, the transfer operator with spectral radius $e^{P(t\phi)}$ is

$$\widetilde{\mathcal{L}}_t f = \frac{f \circ T^{-1}}{((J^u T)^t J^s T) \circ T^{-1}}$$

For a billiard, setting $E(x) = \sin(\angle(E^s(x), E^u(x)))$,

$$\frac{\cos\varphi(x)}{\cos\varphi(Tx)} = J_{\mathsf{Leb}} T(x) = J^s T(x) J^u T(x) \frac{E(Tx)}{E(x)},$$

$$\implies (J^u T)^t J^s T = \left(\frac{E \cos\varphi}{(E\cos\varphi) \circ T} \right)^t (J^s T)^{1-t}$$

So $\widetilde{\mathcal{L}}_t$ has the same spectrum as

$$\mathcal{L}_t f = \frac{f \circ T^{-1}}{(J^s T)^{1-t} \circ T^{-1}}$$

Associated Transfer Operator

$$\mathcal{L}_t f = \frac{f \circ T^{-1}}{(J^s T)^{1-t} \circ T^{-1}}$$

For $t = 1$, this corresponds to using μ_{SRB} as the conformal measure. We will identify a function f with the measure $d\mu = f d\mu_{\mathsf{SRB}}$. Then acting on distributions,

$$\mathcal{L}_t \mu(\psi) = \mu \left(\frac{\psi \circ T}{(J^s T)^{1-t}} \right), \quad \text{test function } \psi$$

Construct equilibrium state μ_t out of left and right eigenvectors of $\mathcal{L}_t$ corresponding to the eigenvalue of maximum modulus.

Sources of difficulty:

- T has discontinuities so a topological definition of pressure must overcome the effect of this cutting.
- The potential is not Hölder continuous
  - $J^s T \approx \cos \varphi$ so the potential is unbounded
  - $J^s T$ is not continuous on any open set

Weight Function for Topological Pressure

To control the evolution of $\mathcal{L}_t^n f$, must control integrals of the type,

$$\int_W \mathcal{L}_t^n f \, \psi \, dm_W = \int_{T^{-n}W} f \, \psi \circ T^n \, |J^s T^n|^t \, dm_{T^{-n}W} \, .$$

- $W \in \mathcal{W}_H^s$, the set of (weakly) homogeneous local stable manifolds with uniformly bounded curvature
- m_W is arclength measure on W
- $\psi \in C^\alpha(W)$ is a Hölder continuous test function
- f is an element of our Banach space (closure of $C^1(M)$ in some norm)

$T^{-n}W = \cup_i W_i$, $W_i \in \mathcal{G}_n(W)$, homogeneous components.

We need to estimate precisely how $\sum_{W_i} |J^s T^n|^t_{C^0(W_i)}$ grows as a function of n and W. This resembles the expression from our growth lemma in Lecture 1.

Homogeneity Strips and Modified One-step Expansion

$$H_{\pm k} = \{(r, \varphi) \in M : (k+1)^{-q} \le |\varphi \mp \frac{\pi}{2}| \le k^{-q}\}, \quad k \ge k_0$$

For $V \in \widehat{\mathcal{W}}^s$, let V_i denote the homogeneous connected components of $T^{-1}V$.

Lemma (Modified One-step Expansion)

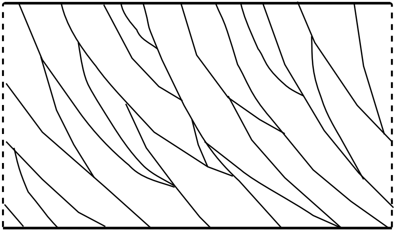
Fix $t_0 > 0$ and $q \ge 2/t_0$. There exists $\theta(t_0) < 1$, $k_0(t_0), \delta_0(t_0) > 0$ such that for all $V \in \widehat{\mathcal{W}}^s$,

$$\sup_{|V| \le \delta_0} \sum_{V_i} |J_{V_i} T|_*^t < \theta^t, \quad \text{for all } t \ge t_0.$$

The proof is similar to the standard estimate: near a tangential collision, $\sum_{k \ge k_0} |J_{V_i} T|_*^t \sim \sum_{k \ge k_0} k^{-qt} \le C k_0^{-1}$. Then k_0 can be chosen large enough (and δ_0 small enough) to make θ^t arbitrarily close to Λ^{-t}, where $\Lambda = 1 + 2\mathcal{K}_{\min}\tau_{\min}$.

A Definition of Topological Pressure

- Define $\mathcal{S}_n = \cup_{i=0}^n T^{-i}\mathcal{S}_0$,
  $\mathcal{S}_n^{\mathbb{H}} = \cup_{i=0}^n T^{-i}\mathcal{S}_0^{\mathbb{H}}$
- Let $\mathcal{M}_0^n$ = connected components of $M \setminus \mathcal{S}_n$,
- $\mathcal{M}_0^{n,\mathbb{H}}$ = connected components of $M \setminus (\mathcal{S}_{n-1}^{\mathbb{H}} \cup T^{-n}\mathcal{S}_0)$



$M \setminus \mathcal{S}_n$

Define for $t > 0$,

- $$Q_n(t) := \sum_{A \in \mathcal{M}_0^{n,\mathbb{H}}} \sup_{x \in A \cap M'} |J^s T^n(x)|^t, \qquad M' = M \setminus (\cup_{n\in\mathbb{Z}}\mathcal{S}_n)$$
- $$P_*(t) := \lim_{n\to\infty} \frac{1}{n} \log Q_n(t)$$
- The limit exists since the sequence $\log Q_n(t)$ is subadditive: $Q_{n+k}(t) \le Q_n(t)Q_k(t)$. It follows, $Q_n(t) \ge e^{nP_*(t)}$.

Properties of $P_*(t)$ and Variational Inequality

Theorem

For a finite horizon Sinai billiard:

a) $P_*(t)$ is a convex, continuous, decreasing function for $t > 0$;

b) $P_*(t)$ satisfies a variational inequality,

$$P_*(t) \geq P(t) = \sup \left\{ h_\mu(T) - t \int \log J^u T \, d\mu : \mu \ T\text{-inv. prob.} \right\}$$

Proof. (a) follows from $Q_n(\alpha t + (1-\alpha)s) \leq Q_n(t)^\alpha Q_n(s)^{1-\alpha}$.
(b) relies on the continuation of singularities property. This implies that setting $\mathcal{P} = \mathcal{M}_0^1$, then the elements of $\mathcal{P}_{-n}^n = \bigvee_{i=-n}^n T^{-i}\mathcal{P}$ are simply connected. This plus the uniform hyperbolicity of T implies $\mathcal{P}$ is a generating partition. Then using that $\int_M \log J^s T \, d\mu = -\int_M \log J^u T \, d\mu$ for an invariant measure μ, a standard estimate (e.g. [Walters '82]) implies $h_\mu(T) - t \int_M \log J^u T \, d\mu \leq P_*(t)$. □

Definition of $t_* > 1$

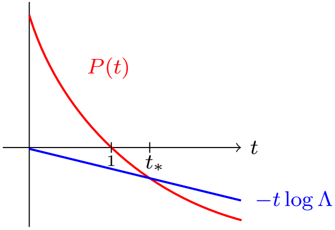
Want to prove that $P_*(t) = P(t)$ for $t \in (0, t_*)$ for some $t_* > 1$.

To do this, need to prove exact exponential growth of $Q_n(t)$:

$$\exists C_2 > 0 \;\; s.t. \;\; e^{nP_*(t)} \le Q_n(t) \le C_2 e^{nP_*(t)} \,,$$

and uniform growth along stable curves,

$$\exists c_0 > 0 \, s.t. \, \forall W \in \widehat{\mathcal{W}}^s, |W| \ge \delta_1, \sum_{W_i \in \mathcal{G}_n(W)} |J_{W_i} T^n|^t_{C^0(W_i)} \ge c_0 Q_n(t) \,.$$



- $t_* := \sup\{t > 0 : -t \log \Lambda < P(t)\}$
  Pressure Gap: $\Lambda^{-t} < e^{P(t)}$ for $t < t_*$
- Fix $t_0 > 0$ and $t_1 < t_*$.
- Choose $\theta < 1$ s.t. the intersection of $t \log \theta$ and $P(t)$ is to the right of t_1.
- Choose q, k_0 and δ_0 so that the one-step expansion holds for θ uniformly for all $t \in [t_0, t_1]$.

Growth lemmas and prevalence of 'long' partition elements

For $\delta_1 < \delta_0$, let $\mathcal{G}_n^{\delta_1}(W)$ denote the analogous collection as $\mathcal{G}_n(W)$, but with respect to the length scale δ_1 rather than δ_0.

Lemma ('Long' elements of $\mathcal{G}_n(W)$ carry most weight)

$\forall \varepsilon > 0\ \exists \delta_1, n_1 > 0$ s.t $\forall W \in \widehat{\mathcal{W}}^s$ with $|W| \geq \delta_1/3$ and all $n \geq n_1$,

$$\sum_{\substack{W_i \in \mathcal{G}_n^{\delta_1}(W_i) \\ |W_i| < \delta_1/3}} |J_{W_i} T^n|^t_{C^0(W_i)} \leq \varepsilon \sum_{W_i \in \mathcal{G}_n^{\delta_1}(W)} |J_{W_i} T^n|^t_{C^0(W_i)}$$

Define $\mathcal{A}_n(\delta) = \{A \in \mathcal{M}_0^{n,\mathbb{H}} : \mathrm{diam}^u(T^n A) \geq \delta/3\}$.

Lemma ('Long' elements of $\mathcal{M}_{-n}^{0,\mathbb{H}}$ carry most weight)

There exist $\delta_2 > 0$ and $c_0 > 0$ such that

$$\sum_{A \in \mathcal{A}_n(\delta_2)} \sup_{x \in A} |J^s T^n(x)|^t \geq c_0\, Q_n(t)\,,\ \forall n \in \mathbb{N}\,,\ \forall t \in [t_0, t_1]\,.$$

Uniform Growth for $W \in \widehat{\mathcal{W}}^s$ and Supermultiplicativity

Proposition

a) There exists $c_1 > 0$ s.t. for any $W \in \widehat{\mathcal{W}}^s$ with $|W| \geq \delta_1/3$,

$$\sum_{W_i \in \mathcal{G}_n(W)} |J_{W_i} T^n|^t_{C^0(W_i)} \geq c_1 Q_n(t)\,,\ \forall n \geq 1\,,\ \forall t \in [t_0, t_1]\,.$$

b) There exists $c_2 > 0$ s.t. for all $k, n \geq 1$,

$$Q_{n+k}(t) \geq c_2 Q_n(t) Q_k(t)\,.$$

(b) follows from (a) and first growth lemma, since

$$\sum_{W_i \in \mathcal{G}^{\delta_1}_{n+k}(W)} |J_{W_i} T^{n+k}|^t_{C^0} \geq C \sum_{V_j \in L^{\delta_1}_n(W)} |J_{V_j} T^n|^t_{C^0} \sum_{W_i \in \mathcal{G}^{\delta_1}_k(V_j)} |J_{W_i} T^k|^t_{C^0}$$

Immediate corollary of (b) is **exact exponential growth of $Q_n(t)$**:

$$e^{nP_*(t)} \leq Q_n(t) \leq 2c_2^{-1} e^{nP_*(t)} \quad \forall n \geq 1\,,\ \forall t \in [t_0, t_1]\,.$$

Definition of Norms: Weak Norm

Fix $0 < \alpha \le 1/(q+1)$.

For $f \in C^1(M)$, define the **weak norm** of f by

$$|f|_w = \sup_{W \in \mathcal{W}^s_H} \sup_{\substack{\psi \in \mathcal{C}^\alpha(W) \\ |\psi|_{\mathcal{C}^\alpha(W)} \le 1}} \int_W f \, \psi \, dm_W \, .$$

Define $\mathcal{B}_w$ to be the completion of $C^1(M)$ in the $|\cdot|_w$ norm.

Remark: Norms defined on stable manifolds $\mathcal{W}^s$ rather than cone-stable curves $\widehat{\mathcal{W}}^s$. We make this choice because J^sT varies Hölder continuously along $W \in \mathcal{W}^s$, but only measurably transverse to stable direction. These norms are not well suited to study perturbations of the dynamics.

For $t = 1$, J^sT disappears and one can use $\widehat{\mathcal{W}}^s$ instead. Such norms are robust under perturbations [D., Zhang '13].

Definition of Norms: Strong Norm

Choose $p > q + 1$, $\beta \in (1/p, \alpha)$ and $\gamma < \min\{1/p, \alpha - \beta\}$.

Define the **strong stable norm** of f by

$$\|f\|_s = \sup_{W \in \mathcal{W}^s_H} \sup_{\substack{\psi \in \mathcal{C}^\beta(W) \\ |\psi|_{\mathcal{C}^\beta(W)} \le |W|^{-1/p}}} \int_W f\, \psi \, dm_W$$

Define the **strong unstable norm** of f by

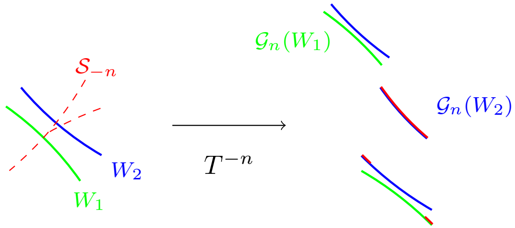
$$\|f\|_u = \sup_{\varepsilon \le \varepsilon_0} \sup_{\substack{W_1, W_2 \in \mathcal{W}^s_H \\ d(W_1, W_2) \le \varepsilon}} \sup_{\substack{|\psi_i|_{\mathcal{C}^\alpha(W_i)} \le 1 \\ d_0(\psi_1, \psi_2) = 0}} \varepsilon^{-\gamma} \left| \int_{W_1} f \psi_1 - \int_{W_2} f \psi_2 \right|$$

The **strong norm** of f is defined to be $\|f\|_{\mathcal{B}} = \|f\|_s + c_u \|f\|_u$,

Define $\mathcal{B}$ to be the completion of $C^1(M)$ in the $\| \cdot \|_{\mathcal{B}}$ norm.

Lasota-Yorke: Unmatched Pieces

For strong unstable norm, estimate $\left| \int_{W_1} \mathcal{L}_t^n f \, \psi_1 - \int_{W_2} \mathcal{L}_t^n f \, \psi_2 \right|$



- Unmatched pieces have length at most $\Lambda^{-j}\varepsilon$ is they are cut by a singularity curve at time $-j$.
- Use the strong stable norm to estimate,

$$\int_{W_i} \mathcal{L}_t^n f \, \psi = \int_{V_j} \mathcal{L}_t^{n-j} f \, \psi \circ T^j |J_{V_j} T^j|^t \leq \Lambda^{-j/p} \varepsilon^{1/p} \|\mathcal{L}_t^{n-j} f\|_s |J_{V_j} T^j|^t_{C^0}$$

- $\| \cdot \|_s$ acts as 'weak norm' for $\| \cdot \|_u$ to control unmatched pieces.

Banach Spaces

Theorem ([Baladi, D. '22])

- We have a sequence of continuous inclusions,

$$\mathcal{C}^1(M) \subset \mathcal{B} \subset \mathcal{B}_w \subset (\mathcal{C}^\alpha(M))^*.$$

- The embedding of the unit ball of $\mathcal{B}$ in $\mathcal{B}_w$ is compact.
- There exist $C, C_n > 0$ such that for all $f \in \mathcal{B}$, $n \geq 0$,

$$|\mathcal{L}_t^n f|_w \leq C Q_n(t) |f|_w \, ,$$
$$\|\mathcal{L}_t^n f\|_s \leq C \big(\Lambda^{-(\beta - 1/p)n} Q_n(t) + \theta^{(t-1/p)n} \big) \|f\|_s + C_n |f|_w)$$
$$\|\mathcal{L}_t^n f\|_u \leq C Q_n(t) \big(n^\gamma \Lambda^{-\gamma n} \|f\|_u + C_n \|f\|_s \big) \, .$$

Implies the spectral radius of $\mathcal{L}_t$ on $\mathcal{B}$ is at most $e^{P_*(t)}$ and its essential spectral radius $< e^{P_*(t)}$ **if** $\theta^t < e^{P_*(t)}$ (**pressure gap**).

To prove $\mathcal{L}_t$ is quasi-compact, we need a **lower bound** on the spectral radius.

Lower Bound on Spectral Radius

The lower bound follows from our uniform growth result:

There exists $c_1 > 0$ s.t. for any $W \in \widehat{\mathcal{W}}^s$ with $|W| \geq \delta_1/3$,

$$\sum_{W_i \in \mathcal{G}_n(W)} |J_{W_i} T^n|^t_{C^0(W_i)} \geq c_1 Q_n(t)\,,\ \forall n \geq 1\,,\ \forall t \in [t_0, 1]\,.$$

Let $W \in \mathcal{W}^s_H$ with $|W| \geq \delta_1/3$, choose $\psi \equiv 1$. For any $n \geq 1$,

$$\int_W \mathcal{L}^n_t 1 = \sum_{W_i \in \mathcal{G}_n(W)} \int_{W_i} |J_{W_i} T^n|^t \geq e^{-C_d} \sum_{W_i \in \mathcal{G}_n(W)} |J_{W_i} T^n|^t_{C^0(W_i)}$$
$$\geq e^{-C_d} c_1 Q_n(t) \geq e^{-C_d} c_1 e^{nP_*(t)}$$

Thus $\|\mathcal{L}^n 1\|_s \geq C e^{nP_*(t)}$ and so the spectral radius of $\mathcal{L}$ is $e^{P_*(t)}$.

Spectral Decomposition of $\mathcal{L}_t$

Our exact exponential growth implies:

$$\|\mathcal{L}_t^n\|_{\mathcal{B}} \le C Q_n(t) \le C' e^{nP_*(t)} ,$$

so that the peripheral spectrum of $\mathcal{L}_t$ has no Jordan blocks.

There exist a finite set $\{\theta_j\}_{j=0}^N$, $\theta_0 = 0$, linear operators $\Pi_j, R : \mathcal{B} \circlearrowleft$ satisfying $\Pi_i \Pi_j = \Pi_j R = R \Pi_j = 0$ with spectral radius of $R < 1$, such that

$$e^{-P_*(t)} \mathcal{L}_t = \sum_{j=1}^{N} e^{2\pi i \theta_j} \Pi_j + R$$

Proof of spectral gap follows similar lines as for Baker's map: Define $\nu_t = \Pi_0 1$. Show all eigenvectors corresponding to the peripheral spectrum are measures absolutely continuous wrt ν_t, and θ_j must be rational. Use mixing to show 1 is simple for $\mathcal{L}_t^k$ for $k \ge 1$. (Lack of smoothness complicates argument.)

A Spectral Gap for $\mathcal{L}_t$

Theorem ([Baladi, D. '22])

For each $t_0 > 0$ and $t_1 < t_*$, there exists a Banach space $\mathcal{B} = \mathcal{B}(t_0, t_1)$ such that $\mathcal{L}_t$ has a spectral gap:

- $e^{P_*(t)}$ is the eigenvalue of maximum modulus, it is simple, and the remainder of the spectrum of $\mathcal{L}_t$ is contained in a disk of radius $\bar{\sigma} e^{P_*(t)}$, where $\bar{\sigma} < 1$ is uniform for $t \in [t_0, t_1]$.

Letting ν_t and $\tilde{\nu}_t$ denote the maximal right and and left eigenvectors for $\mathcal{L}_t$, define

$$\mu_t(\psi) = \frac{\langle \nu_t, \psi \tilde{\nu}_t \rangle}{\langle \nu_t, \tilde{\nu}_t \rangle}, \quad \psi \in C^\alpha(M).$$

Then μ_t is an invariant probability measure for T, and enjoys exponential decay of correlations against Hölder observables.

μ_t has no atoms, gives 0 weight to any C^1 curve and is positive on open sets. Moreover, $\int |\log d(x, \mathcal{S}_{\pm 1})| \, d\mu_t < \infty$.

Entropy of μ_t and a Variational Principle

Define $B(x, n, \varepsilon) = \{y \in M : d(T^{-i}x, T^{-i}y) \le \varepsilon, \forall i \in [0, n]\}$.

Proposition (Measure of Bowen Balls)

There exists $C > 0$ s.t. for all $x \in M$, $n \ge 1$, and $y \in B(x, n, \varepsilon)$,

$$\mu_t(B(x, n, \varepsilon)) \le Ce^{-nP_*(t) + t \log J^s T^n (T^{-n}y)}.$$

- [Brin, Katok '81] $\implies$ for μ_t-a.e. $x \in M$,

$$\lim_{\varepsilon \to 0} \limsup_{n \to \infty} -\frac{1}{n} \log \mu_t(B(x, n, \varepsilon)) = h_{\mu_t}(T).$$

- This plus the Proposition implies

$$h_{\mu_t}(T) \ge P_*(t) - t \int \log J^s T \, d\mu_t = P_*(t) + t \int \log J^u T \, d\mu_t$$

- But $P_*(t) \ge h_{\mu_t}(T) - t \int \log J^u T \, d\mu_t$ since $P_*(t) \ge P(t)$.
- Conclude: $P_*(t) = h_{\mu_t}(T) - t \int \log J^u T \, d\mu_t = P(t)$.

Part Two: Measure of Maximal Entropy

Goal for this section: Discuss the case $t = 0$. We must modify the Banach spaces and we lose the spectral gap. Yet we are able to maintain enough control of the transfer operator to construct a unique measure of maximal entropy.

Reference: V. Baladi and M. Demers, *On the measure of maximal entropy for finite horizon Sinai billiard maps*, Journal Amer. Math. Soc. (2020).

- Transfer operator for geometric potential with $t = 0$,

$$\mathcal{L}_0 f = \frac{f \circ T^{-1}}{J^s T \circ T^{-1}}$$

  - $J^s T \approx \cos \varphi$ so the potential is unbounded
  - $J^s T$ is not continuous on any open set

Weight Function for Topological Entropy

To control the evolution of $\mathcal{L}_0^n f$, must control integrals of the type,

$$\int_W \mathcal{L}_0^n f \, \psi \, dm_W = \int_{T^{-n}W} f \, \psi \circ T^n \, dm_{T^{-n}W} \, .$$

- $W \in \mathcal{W}^s$, the set of local stable manifolds with uniformly bounded curvature
- m_W is arclength measure on W
- $\psi \in C^\alpha(W)$ is a Hölder continuous test function
- f is an element of our Banach space (closure of $C^1(M)$ in some norm)

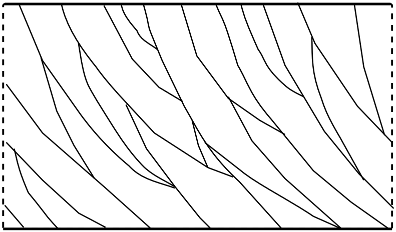
$T^{-n}W = \cup_i W_i$, smooth, connected components.

We need to estimate precisely how $\sum_{W_i} 1$ grows as a function of n and W. **Without a Jacobian, the growth lemmas will look different; we cannot use homogeneity strips.**

Step 1: A Definition of Topological Entropy

- Let $\mathcal{M}^n_{-k}$ = connected components of $M \setminus (\mathcal{S}_{-k} \cup \mathcal{S}_n)$.
- Define

$$h_* = \lim_{n \to \infty} \frac{1}{n} \log \# \mathcal{M}_0^n$$



$M \setminus \mathcal{S}_n$

- The limit exists since the sequence $\log \# \mathcal{M}_0^n$ is subadditive: $\# \mathcal{M}_0^{n+m} \le \# \mathcal{M}_0^n \cdot \# \mathcal{M}_0^m$.
- h_* is the exponential rate of growth of the number of pieces created by the discontinuities of T. It does not depend on a choice of metric.
- h_* satisfies a variational inequality,

$$h_* \ge \sup\{h_\mu(T) : \mu \text{ is a } T\text{-invariant Borel prob. measure}\}$$

$\mathcal{G}_n(W)$ and Linear Complexity Bound

To obtain a precise estimate on the spectral radius of $\mathcal{L}_0$, we will need precise estimates on the growth rates of $\#\mathcal{M}_0^n$ and $\#\mathcal{G}_n(W)$.

- Define $\mathcal{G}_n(W)$ without homogeneity strips
- For $W \in \widehat{\mathcal{W}}^s$, define $\mathcal{G}_1(W)$ to be the maximal, connected components of $T^{-1}W$ subdivided to length at most δ_0 (t.b.d.)
- Define $\mathcal{G}_n(W) = \{\mathcal{G}_1(W_i) : W_i \in \mathcal{G}_{n-1}(W)\}$.

Recall the linear complexity bound.

For $x \in M$, let $N(\mathcal{S}_n, x)$ denote the number of singularity curves in $\mathcal{S}_n$ that meet at x. Define $N(\mathcal{S}_n) = \sup_{x \in M} N(\mathcal{S}_n, x)$.

Lemma (Bunimovich, Chernov, Sinai '90)

Assume finite horizon. There exists $K > 0$ depending only on the configuration of scatterers such that $N(\mathcal{S}_n) \le Kn$ for all $n \ge 1$.

Choose n_0 s.t. $n_0^{-1} \log(Kn_0 + 1) < h_*$. Choose δ_0 s.t. any stable curve of length $\le \delta_0$ is cut into at most $Kn_0 + 1$ pieces by $\mathcal{S}_{-n_0}$.

Fragmentation Lemma (Growth Lemma)

Let $L_n^\delta(W) = \{W_i \in \mathcal{G}_n^\delta(W) : |W| \geq \delta/3\}$.
$Sh_n^\delta(W) = \mathcal{G}_n^\delta(W) \setminus L_n^\delta(W)$.

Lemma ([Baladi, D. '20])

For all $\varepsilon > 0$ there exists $n_1, \delta > 0$ s.t. for all $n \geq n_1$,

$$\#Sh_n^\delta(W) \leq \varepsilon \#\mathcal{G}_n^\delta(W) \quad \textit{for all } W \in \widehat{\mathcal{W}}^s \textit{ with } |W| \geq \delta/3.$$

Idea of Proof: Choose $\varepsilon > 0$ and n_1 s.t. $3C_0^{-1}(Kn_1 + 1)\Lambda^{-n_1} < \varepsilon$.
Choose $\delta > 0$ s.t. if $|W| < \delta$ then $T^{-n_1}W$ comprises at most $Kn_1 + 1$ connected components of length at most δ_0.

Then $Sh_{n_1}^\delta(W)$ contains at most $Kn_1 + 1$ elements while $|T^{-n_1}W| \geq C_0\Lambda^{n_1}\delta/3$, where $\Lambda = 1 + 2\mathcal{K}_{\min}\tau_{\min}$.

Thus $\#\mathcal{G}_{n_1}^\delta(W) \geq C_0\Lambda^{n_1}/3$ and so $\frac{\#Sh_{n_1}^\delta(W)}{\#\mathcal{G}_{n_1}^\delta(W)} \leq \varepsilon$ by choice of n_1.

Argument can be iterated, grouping by most recent long ancestor.

Fragmentation of $\mathcal{M}_0^n$

This lemma can also be formulated for elements of $\mathcal{M}_0^n$ and $\mathcal{M}_{-n}^0$.

Let δ_1, $n_1 \geq n_0$ correspond to $\varepsilon = 1/4$ in fragmentation lemma:
For all $W \in \widehat{\mathcal{W}}^s$ with $|W| \geq \delta_1/3$,

$$\#L_n^{\delta_1}(W) \geq \tfrac{3}{4}\#\mathcal{G}_n^{\delta_1}(W)\,, \quad \forall n \geq n_1\,.$$

$$\text{Define } \; L_s(\mathcal{M}_0^n) := \{A \in \mathcal{M}_0^n : \text{diam}^s(A) \geq \delta_1/3\}$$
$$L_u(\mathcal{M}_{-n}^0) := \{B \in \mathcal{M}_{-n}^0 : \text{diam}^u(B) \geq \delta_1/3\}$$

Lemma

There exists $c_0 > 0$ s.t. for all $n \geq 1$,

$$\#L_s(\mathcal{M}_0^n) \geq c_0\delta_1\#\mathcal{M}_0^n \quad \textit{and} \quad \#L_u(\mathcal{M}_{-n}^0) \geq c_0\delta_1\#\mathcal{M}_{-n}^0\,.$$

Fragmentation Lemmas $\implies$ Uniform Bounds on Growth

Proposition

a) $\exists\, c_1 > 0$ such that for any $W \in \widehat{\mathcal{W}}^s$ with $|W| \geq \delta_1/3$,

$$\#\mathcal{G}_n(W) \geq c_1 \# \mathcal{M}_0^n \qquad \forall n \geq 1 .$$

b) There exists $c_2 > 0$ such that for all $k, n \geq 1$,

$$\#\mathcal{M}_0^{n+k} \geq c_2 \#\mathcal{M}_0^n \cdot \#\mathcal{M}_0^k .$$

(b) implies **exact exponential growth** of $\#\mathcal{M}_0^n$,

$$e^{nh_*} \leq \#\mathcal{M}_0^n \leq 2c_2^{-1} e^{nh_*} \quad \text{for all } n \geq 1.$$

(a) + fragmentation lemma $\implies$ (b) since

$$\#\mathcal{G}_{n+k}(W) \geq \sum_{V_j \in L_n^{\delta_1}(W)} \#\mathcal{G}_k(V_j) \geq \#L_n^{\delta_1}(W) c_1 \#\mathcal{M}_0^k$$

$$\geq \tfrac{3c_1}{4} \#\mathcal{G}_n^{\delta_1}(W) \#\mathcal{M}_0^k \geq \frac{3c_1^2}{4} \#\mathcal{M}_0^n \#\mathcal{M}_0^k$$

New Assumption: 'Sparse Recurrence' to Singularities

In order to leverage these growth and fragmentation lemmas to control $\mathcal{L}_0$, we need the following additional assumption on T.

- Fix $n_0 \in \mathbb{N}$ and an angle φ_0 close to $\pi/2$.
- Let $s_0 \in (0,1)$ be the smallest number such that any orbit of length n_0 has at most $s_0 n_0$ collisions with $|\varphi| \geq \varphi_0$.

Finite horizon guarantees that we can always choose n_0 and φ_0 so that $s_0 < 1$. (Indeed, no triple tangencies implies that $s_0 \leq \frac{2}{3}$.)

Assumption: $h_* > s_0 \log 2$

Fact: If W is a local stable manifold, then $|T^{-1}W| \leq C|W|^{1/2}$.

Our assumption ensures that the growth due to tangential collisions does not exceed the exponential rate of growth given by h_*.

Toy Calculation in Previous Norms

Recall that the strong stable norm for $t > 0$ was

$$\|f\|_s = \sup_{W \in \mathcal{W}^s} \sup_{|\psi|_{C^\alpha(W)} \le |W|^{-1/p}} \int_W f\psi \, dm_W \, ,$$

and the weight $|W|^{-1/p}$ was needed to control the contribution from unmatched pieces in the strong unstable norm estimate.

But now we have no Jacobian or homogeneity strips. So suppose $W \in \mathcal{W}^s$ s.t. $T^{-1}W$ has a single component with $|T^{-1}W| \approx |W|^{1/2}$. Then if $\psi = |W|^{-1/p}$,

$$\int_W \mathcal{L}_0 f \, \psi = |W|^{-1/p} \int_{T^{-1}W} f \le \|f\|_s \frac{|T^{-1}W|^{1/p}}{|W|^{1/p}} \approx \|f\|_s |W|^{-1/2p}$$

and taking sup over $W \in \mathcal{W}^s$ yields ∞. The spectral radius of $\mathcal{L}_0 = \infty$ on such a space, for any $p > 0$.

To avoid this, we use a logarithmic weight instead.

Definition of Norms: Weak Norm

Choose $\alpha, \beta, \varsigma > 0$ and $\gamma > 1$ such that

$$\beta < \alpha \leq 1/3, \quad 2^{s_0 \gamma} < e^{h_*}, \quad \varsigma < \gamma \, .$$

Choose n_0 so that

$$\frac{1}{n_0} \log(K n_0 + 1) < h_* - \gamma s_0 \log 2 \, ,$$

where K is from the linear bound on complexity.

Fix the length scale $\delta_0 > 0$ so that any $W \in \mathcal{W}^s$ (with $|W| \leq \delta_0$) is cut into at most $K n_0 + 1$ pieces by $\mathcal{S}_{-n_0}$.

For $f \in C^1(M)$, define the **weak norm** of f by

$$|f|_w = \sup_{W \in \mathcal{W}^s} \sup_{\substack{\psi \in \mathcal{C}^\alpha(W) \\ |\psi|_{\mathcal{C}^\alpha(W)} \leq 1}} \int_W f \, \psi \, dm_W \, .$$

Define $\mathcal{B}_w$ to be the completion of $C^1(M)$ in the $| \cdot |_w$ norm.

Definition of Norms: Strong Norm

Define the **strong stable norm** of f by

$$\|f\|_s = \sup_{W \in \mathcal{W}^s} \sup_{\substack{\psi \in \mathcal{C}^\beta(W) \\ |\psi|_{\mathcal{C}^\beta(W)} \le |\log|W||^\gamma}} \int_W f\,\psi\,dm_W$$

Define the **strong unstable norm** of f by

$$\|f\|_u = \sup_{\varepsilon \le \varepsilon_0} \sup_{\substack{W_1, W_2 \in \mathcal{W}^s \\ d(W_1, W_2) \le \varepsilon}} \sup_{\substack{|\psi_i|_{\mathcal{C}^\alpha(W_i)} \le 1 \\ d_0(\psi_1, \psi_2) = 0}} |\log \varepsilon|^\varsigma \left| \int_{W_1} f \psi_1 - \int_{W_2} f \psi_2 \right|$$

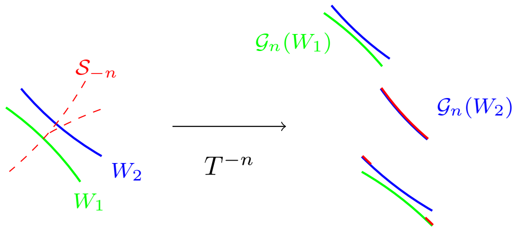
The **strong norm** of f is defined to be $\|f\|_{\mathcal{B}} = \|f\|_s + \|f\|_u$,

Define $\mathcal{B}$ to be the completion of $C^1(M)$ in the $\|\cdot\|_{\mathcal{B}}$ norm.

No contraction of $\|\cdot\|_u$

The logarithmic modulus of continuity in the strong unstable norm prevents contraction of $\|\cdot\|_u$.

For strong unstable norm, estimate $\left|\int_{W_1} \mathcal{L}_0^n f\, \psi_1 - \int_{W_2} \mathcal{L}_0^n f\, \psi_2\right|$



- If $d(W^1, W^2) \le \varepsilon$, and if $W_i^1 \in \mathcal{G}_n(W^1)$, $W_i^2 \in \mathcal{G}_n(W_i^2)$ are matched, then $d(W_i^1, W_i^2) \le C\Lambda^{-n}\varepsilon$.
- But the contraction is $\frac{|\log C\Lambda^{-n}\varepsilon|^\varsigma}{|\log \varepsilon|^\varsigma}$, and taking the supremum over $\varepsilon > 0$ yields 1.

Banach Spaces and Inequalities

Theorem ([Baldi, D. '20])

- We have a sequence of inclusions,

$$\mathcal{C}^1(M) \subset \mathcal{B} \subset \mathcal{B}_w \subset (\mathcal{C}^\alpha(M))^*.$$

- The embedding of the unit ball of $\mathcal{B}$ in $\mathcal{B}_w$ is compact.
- Assume $h_* > s_0 \log 2$. There exists $C > 0$ such that for all $f \in \mathcal{B}$, $n \geq 0$,

$$\begin{aligned} |\mathcal{L}^n f|_w &\leq C|f|_w \# \mathcal{M}_0^n \\ \|\mathcal{L}^n f\|_s &\leq C(\sigma^n \|f\|_s + |f|_w) \# \mathcal{M}_0^n \,, \quad \text{for some } \sigma < 1 \\ \|\mathcal{L}^n f\|_u &\leq C(\|f\|_u + \|f\|_s) \# \mathcal{M}_0^n \end{aligned}$$

The inequalities above are not true Lasota-Yorke inequalities due to lack of contraction in the strong unstable norm.

Bounds on the Spectral Radius of $\mathcal{L}_0$

Although we do not prove quasi-compactness of $\mathcal{L}_0$ on $\mathcal{B}$, we do have good control of $\|\mathcal{L}_0^n\|_{\mathcal{B}}$.

- Our upper bound $\#\mathcal{M}_0^n \le C_2 e^{nh_*}$ plus our 'Lasota-Yorke' inequalities imply that $\|\mathcal{L}_0^n\|_{\mathcal{B}} \le C e^{nh_*}$, for all $n \ge 1$.
- Our lower bound on $\#\mathcal{G}_n(W)$ implies that

$$\|\mathcal{L}_0^n 1\|_s \ge |\mathcal{L}_0^n 1|_w \ge \int_W \mathcal{L}_0^n 1 = \sum_{W_i \in \mathcal{G}_n^{\delta_1}(W)} |W_i|$$

$$\ge \frac{\delta_1}{3} \frac{3}{4} \#\mathcal{G}_n^{\delta_1}(W) \ge C e^{nh_*} .$$

This implies that the sequence $e^{-nh_*}\mathcal{L}_0^n 1$ is uniformly bounded away from 0 and ∞ in the strong norm. We use this fact to construct an eigenmeasure for $\mathcal{L}_0$ with eigenvalue e^{h_*}.

Construction of μ_*

- The sequence

$$\nu_n = \frac{1}{n}\sum_{k=0}^{n-1} e^{-kh_*}\mathcal{L}_0^k 1, \text{ is uniformly bounded in } \mathcal{B}.$$

  By compactness, a subsequence converges in $\mathcal{B}_w$.
  Let $\nu \in \mathcal{B}_w$ be a limit point of ν_n. ν is a measure.

- Similarly, let $\tilde{\nu} \in (B_w)^*$ be a limit point of the sequence

$$\frac{1}{n}\sum_{k=0}^{n-1} e^{-kh_*}(\mathcal{L}_0^*)^k(d\mu_{\mathsf{SRB}}).$$

- Define $$\mu_*(\psi) = \frac{\tilde{\nu}(\psi\nu)}{\tilde{\nu}(\nu)}, \quad \text{for } \psi \in C^1(M).$$

Since $\mathcal{L}_0\nu = e^{h_*}\nu$ and $\mathcal{L}_0^*\tilde{\nu} = e^{h_*}\tilde{\nu}$, we have $\mu_*(\psi \circ T) = \mu_*(\psi)$, i.e. μ_* is an invariant measure for T.

Hyperbolicity of μ_*

Key Fact: Although $\nu \in \mathcal{B}_w$, it follows from the convergence of ν_n to ν in the $|\cdot|_w$ norm that $\|\nu\|_{\mathcal{B}} < \infty$.

This implies estimates of the form:

- For any $k \in \mathbb{Z}$, $\exists C_k > 0$ s.t.

  $$\nu(\mathcal{N}_\varepsilon(\mathcal{S}_k)) \le C_k(-\log \varepsilon)^{-\gamma}, \qquad \mu_*(\mathcal{N}_\varepsilon(\mathcal{S}_k)) \le C_k(-\log \varepsilon)^{-\gamma}.$$

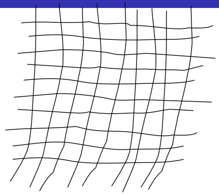
  $\mathcal{N}_\varepsilon(\mathcal{S}_k) = \varepsilon$-neighborhood of $\mathcal{S}_k$ in M, $\gamma > 1$.

- $\displaystyle\int_M -\log d(x, \mathcal{S}_{\pm 1})\, d\mu_*(x) < \infty$ (μ_* is T-adapted).

- μ_*-a.e. $x \in M$ has a stable and unstable manifold of positive length. The same is true with respect to ν.

Ergodicity of μ_*

Since μ_* is hyperbolic, we cover a full measure set of M with Cantor rectangles, and study the properties of μ_* on each rectangle.



A Cantor Rectangle R

Lemma (Absolute continuity of holonomy)

On each Cantor rectangle R, the holonomy map sliding along unstable manifolds in R is absolutely continuous with respect to the conditional measures of μ_* on stable manifolds.

That $\|\nu\|_{\mathcal{B}} < \infty$ is crucial to the proof of the lemma.

Consequences:

- Each Cantor rectangle R belongs to one ergodic component.
- Since T is topologically mixing, we can force images of rectangles to overlap $\implies$ (T^n, μ_*) is ergodic for all n.

Mixing and Bernoulli Property of μ_*

- The local product structure of the Cantor rectangles, together with a global argument showing that a full measure set of points on each component of M can be connected by a network of stable/unstable manifolds, enables us to prove that (T, μ_*) is K-mixing, following techniques of [Pesin '77, '92].

- K-mixing + hyperbolicity + absolute continuity of μ_* + bounds on $\mu_*(\mathcal{N}_\varepsilon(\mathcal{S}_{\pm 1}))$
  $\Longrightarrow$ the partition $\mathcal{M}_{-1}^1$ is **very weakly Bernoulli**, following the technique of [Chernov, Haskell '96].

  Since $\bigvee_{n=-\infty}^{\infty} T^{-n}(\mathcal{M}_{-1}^1)$ generates the full σ-algebra for T, this implies by [Ornstein, Weiss '73] that (T, μ_*) is Bernoulli.

Entropy of μ_*

Define $B(x, n, \varepsilon) = \{y \in M : d(T^{-i}x, T^{-i}y) \le \varepsilon, \forall i \in [0, n]\}$.

Proposition (Measure of Bowen Balls)

There exists $C > 0$ s.t. for all $x \in M$ and $n \ge 1$,

$$\mu_*(B(x, n, \varepsilon)) \le C e^{-nh_*}.$$

- [Brin, Katok '81] $\implies$ for μ_*-a.e. $x \in M$,

$$\lim_{\varepsilon \to 0} \limsup_{n \to \infty} -\frac{1}{n} \log \mu_*(B(x, n, \varepsilon)) = h_{\mu_*}(T^{-1}) = h_{\mu_*}(T).$$

- This plus the Proposition implies $h_{\mu_*}(T) \ge h_*$
- But $h_* \ge h_{\mu_*}(T)$ by Theorem 1.
- Conclude: $h_* = h_{\mu_*}(T)$.

Uniqueness of μ_*

The usual Bowen argument for uniqueness uses

$$\forall \varepsilon > 0, \exists C > 0 \text{ s.t. for } \mu_*\text{-a.e. } x \in M, \ \ \mu_*(B(x,n,\varepsilon)) \geq Ce^{-nh_*}.$$

This **fails for billiards** due to rate of approach to singularity set.
Rather: $\forall \eta > 0$ and μ_*-a.e. $x \in M$,

$$\exists C = C(\eta, x) > 0 \text{ s.t. } \mu_*(B(x,n,\varepsilon)) \geq Ce^{-n(h_*+\eta)}.$$

This is not sufficient for the Bowen argument.

However: fix $\delta > 0$ small. For each $n \geq 1$ 'most' $x \in M$ belong to an element of $\mathcal{M}_0^j$ at time $n/2 \leq j \leq n$ that satisfies,

$$\text{diam}^s(A) \geq \delta \quad \text{and} \quad \text{diam}^u(T^j A) \geq \delta$$
$$\implies \mu_*(A) \geq C_\delta e^{-jh_*}, \qquad \text{for some } C_\delta > 0.$$

Together with a time shift to group elements of $\mathcal{M}_0^n$ according to $\mathcal{M}_0^j$, this is sufficient to adapt the Bowen argument.

Variational Principle and Measure of Maximal Entropy

Theorem ([Baladi, D. '20])

Let T be the billiard map corresponding to a finite horizon periodic Lorentz gas. Assume $h_* > s_0 \log 2$. Then,

$$h_* = \lim_{n\to\infty} \frac{1}{n} \log \# \mathcal{M}_0^n = \sup_\mu h_\mu(T).$$

Moreover, there exists a unique T-invariant measure μ_* such that

- $h_{\mu_*}(T) = h_*$
- $h_* = P(0) = \lim_{t\downarrow 0} P(t) = \lim_{t\downarrow 0} P_*(t)$
- $h_* = h_{\mathsf{top}}(T, M')$
- (T, μ_*) is Bernoulli and positive on open sets
- $\int -\log d(x, \mathcal{S}_{\pm 1})\, d\mu_*(x) < \infty$

Last item implies that μ_* is T-adapted. By [Lima, Matheus '18], Buzzi '20], $\exists C > 0$ such that $P_n(T) \geq C e^{n h_*}$, for n large.