Thermodynamic Formalism for Dispersing Billiard Maps and Flows

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Overview of Mini-Course

Main Goal of Lectures: Introduce functional analytic framework to study transfer operators associated to hyperbolic systems, and use these tools to present recent progress regarding equilibrium states and topological pressure for dispersing billiard maps and flows.

Plan for Lectures

1. Introduction to Banach spaces for hyperbolic systems
   - Smooth expanding maps, contracting map, Baker's map
   - Geometry of dispersing billiards
2. Thermodynamic formalism for billiard map
   - Geometric potentials and topological pressure, $-t \log J^u T$, $t > 0$.
   - Measure of maximal entropy at $t = 0$, loss of spectral gap
3. Measure of maximal entropy for billiard flow
   - Family of potentials for map $-t\tau$, $t \geq 0$.
   - Equilibrium state for $t = h_{\mathrm{top}}(\Phi_1)$ yields MME for flow.

Transfer Operator or Ruelle-Perron-Frobenius Operator

Transformation $T : X \circlearrowleft$. **Transfer operator** $\mathcal{L}$ associated to T acts on a distribution μ by

$$\mathcal{L}\mu(\psi) = \mu(\psi \circ T), \qquad \psi \text{ a test function, say } C^\alpha.$$

If $d\mu = f dm$ is a measure abs. cont. w.r.t. m, then

$$\int \mathcal{L}f \, \psi \, dm = \int f \, \psi \circ T \, dm,$$

so that pointwise

$$\mathcal{L}f(x) = \sum_{y \in T^{-1}x} \frac{f(y)}{JT(y)},$$

where JT is the Jacobian of T with respect to m, represents the density of the measure $T_*\mu$, i.e. $d(T_*\mu) = \mathcal{L}f \, dm$.

$\mathcal{L} =$ Linear operator which governs evolution of measures, acting on some Banach space of functions, measures or distributions.

Weighted or Generalized Transfer Operator

Generalize the transfer operator by including a potential function g,

$$\mathcal{L}_g \mu(\psi) = \mu(e^g \, \psi \circ T) \, .$$

This allows the transfer operator to be used to study a variety of equilibrium states associated with some classes of potentials (often Hölder continuous). For example, the measure of maximal entropy.

In this case, one constructs an invariant measure μ using the left and right maximal eigenvectors of $\mathcal{L}_g$:

$\mathcal{L}_g \nu = \lambda \nu$ and $\mathcal{L}_g^* \tilde{\nu} = \lambda \tilde{\nu}$, where $\mathcal{L}_g^*$ is the dual to $\mathcal{L}_g$ on a suitable Banach space. Then

$$\mu(\psi) = \nu(\psi \tilde{\nu}),$$

is an invariant measure for T (Parry construction).

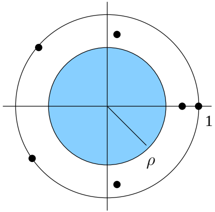
Today: Discuss the case $g = 0$.

Spectral Decomposition of $\mathcal{L}$ via Quasi-Compactness

Goal: Use spectral properties of $\mathcal{L}$ acting on an appropriate Banach space to gain dynamical information about T.

Method: Prove $\mathcal{L}$ is **quasi-compact** on some Banach space $\mathcal{B}$:
$\exists \rho < 1$ s.t. the spectrum of $\mathcal{L}$ outside disk of radius ρ is finite-dimensional.

- Eigenspace corresponding to $1 =$ invariant measures
- Periodic behavior of $\mathcal{L}$ corresponds to eigenvalues other than 1 on the unit circle



- If 1 is a simple eigenvalue and we can eliminate periodicity, we can conclude that $\mathcal{L}$ has a spectral gap

Consequences of the Spectral Gap

$$\int f\,\psi \circ T^n\,dm = \mu_f(\psi \circ T^n) = \mathcal{L}^n \mu_f(\psi)\,, \quad \text{where } d\mu_f = f dm.$$

The presence of a spectral gap allows us to establish exponential decay of correlations and convergence to equilibrium, along with many limit theorems:

- Central Limit Theorem
- Large deviation estimates
- Almost-sure invariance principles

The functional analytic framework gives a unified (and often simplified) approach for handling perturbations as well, either through classical perturbation theory, or the weakened form due to [Keller, Liverani '99].

How can we apply this approach to specific systems?

Quasi-Compactness via Dynamical Inequalities

Dynamical method to estimate the essential spectral radius [Hennion '93] following [Doeblin, Fortet '37], [Ionescu-Tulcea, Marinescu '50], [Lasota, Yorke '73].

Essential ingredients:

- Two Banach spaces $(\mathcal{B}, \|\cdot\|)$ and $(\mathcal{B}_w, |\cdot|_w)$, with an embedding $\mathcal{B} \hookrightarrow \mathcal{B}_w$ such that $|f|_w \le \|f\|$ for $f \in \mathcal{B}$
- The unit ball of $\mathcal{B}$ is compactly embedded in $\mathcal{B}_w$
- (Lasota-Yorke/Doeblin-Fortet inequalities) $\exists\ C > 0$ and $\rho < 1$ such that for all $f \in \mathcal{B}$, $n \ge 0$,

$$\|\mathcal{L}^n f\| \le C\rho^n \|f\| + C|f|_w$$
$$|\mathcal{L}^n f|_w \le C|f|_w$$

Then $\mathcal{L} : \mathcal{B} \circlearrowleft$ has essential spectral radius $\le \rho$.

(Note: The above inequalities imply that the spectral radius is ≤ 1, but for reasonable choices of $\mathcal{B}$, it is actually 1.)

Ex 1: Expanding maps of the interval

[Lasota, Yorke '73]

$T : [0,1] \circlearrowleft$, piecewise C^2, $\exists \lambda < 1$ s.t. $|T'| \geq \lambda^{-1} > 1$
m denotes Lebesgue measure

- Weak space, $\mathcal{B}_w = L^1(m)$
- Strong space, $\mathcal{B} = BV$ with norm

$$\|f\|_{BV} = \sup_{\psi \in C^1, |\psi|_\infty \leq 1} \int f\, \psi'\, dm$$

- $\mathcal{L}f(x) = \sum_{y \in T^{-1}x} \frac{f(y)}{|T'(y)|}$ for $f \in L^1(m)$

One Lasota-Yorke inequality is immediate: $|\mathcal{L}^n f|_1 \leq |f|_1$ since

$$\int |\mathcal{L}f|\, dm \leq \int \mathcal{L}|f|\, dm = \int |f|\, dm\,.$$

Ex 1: Expanding maps of the interval

Estimate in the smooth case:

$$\int \mathcal{L} f \, \psi' \, dm = \int f \, \psi' \circ T \, dm = \int f \left(\frac{\psi \circ T}{T'} \right)' dm + \int f \frac{\psi \circ T}{(T')^2} T'' \, dm$$
$$\leq \|f\|_{BV} |\psi|_\infty \lambda + |f|_1 |\psi|_\infty C_{\mathsf{dist}}$$

Taking appropriate suprema,

$$\|\mathcal{L} f\|_{BV} \leq \lambda \|f\|_{BV} + C|f|_1$$

The case with discontinuities is handled similarly by splitting the integral over intervals of differentiability for T.

So $\mathcal{L}$ acting on BV is quasi-compact. If T is mixing, then $\mathcal{L}$ has a spectral gap.

Note: essential spectral radius bounded by $\lambda = \sup_{x \in I} \frac{1}{|T'(x)|} < 1$.

Ex 2: A contracting map of the interval

[Liverani '04]

$T : [0,1] \circlearrowleft$, $T \in C^1$, $\exists \lambda < 1$ s.t. $|T'| \le \lambda$, $\exists c \in I, T(c) = c$

- Expect convergence of measures to δ_c so usual function spaces will not work
- Consider spaces of distributions: Dual spaces to Hölder continuous test functions

$$|\psi|_{C^\alpha} = |\psi|_{C^0} + H^\alpha(\psi)\,, \qquad H^\alpha(\psi) = \sup_{x \neq y} \frac{|\psi(x) - \psi(y)|}{|x-y|^\alpha}$$

Let $f \in C^1(I)$ and let $d\mu = f dm$. Choose $\alpha < 1$ and define

$$|\mu|_w = \sup_{|\psi|_{C^1} \le 1} |\mu(\psi)| \quad \text{and} \quad \|\mu\| = \sup_{|\psi|_{C^\alpha} \le 1} |\mu(\psi)|$$

- $\mathcal{B}$ is the completion of C^1 in the $\| \cdot \|$-norm
- $\mathcal{B}_w$ is the completion of C^1 in the $| \cdot |_w$-norm
- Unit ball of $\mathcal{B}$ compact in $\mathcal{B}_w$ since C^1 compact in C^α

Ex 2: A contracting map of the interval

For $\psi \in C^\alpha$, let $\overline{\psi} = \int \psi \circ T \, dm$. Then

$$\mathcal{L}\mu(\psi) = \mu(\psi \circ T - \overline{\psi}) + \mu(\overline{\psi}) \le \|\mu\| |\psi \circ T - \overline{\psi}|_{C^\alpha} + |\mu|_w |\overline{\psi}|_{C^1}$$

Estimate $|\psi \circ T - \overline{\psi}|_{C^\alpha}$ by

$$|\psi \circ T(x) - \overline{\psi}(x)| = |\psi \circ T(x) - \psi \circ T(z)| \le |\psi|_{C^\alpha} \lambda^\alpha$$
$$|\psi \circ T(x) - \overline{\psi}(x) - \psi \circ T(y) + \overline{\psi}(y)| \le |\psi|_{C^\alpha} \lambda^\alpha |x - y|^\alpha$$

Also, $|\overline{\psi}|_{C^1} \le |\psi|_\infty = 1$, so that

$$\|\mathcal{L}\mu\| \le \lambda^\alpha \|\mu\| + |\mu|_w$$

A similar estimate shows that $|\mathcal{L}\mu|_w \le |\mu|_w$

• **Note: We cannot choose $\alpha = 0$ so $\mathcal{B}$ must be larger than the space of measures**

Norms for Hyperbolic systems

Conclusions we can draw from these simple examples:

- When T is expanding, $\mathcal{L}$ improves regularity of functions
- When T is contracting, $\mathcal{L}$ improves regularity in certain spaces of distributions

Moral: Hyperbolic systems have both contracting and expanding directions so by choosing spaces of distributions that are regular in the unstable direction and by averaging (integrating) along stable curves, we are able to define norms in which $\mathcal{L}$ improves regularity.

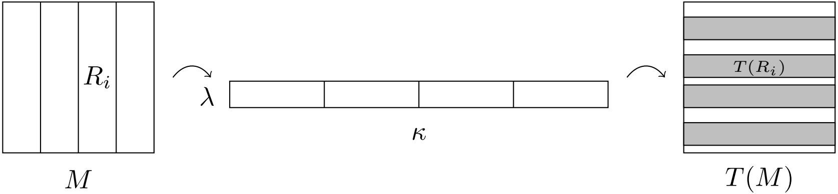
By integrating against Hölder continuous functions on stable curves, we are in spirit defining a notion that is dual to that of **standard pairs**, developed by Dolgopyat and Chernov, which considers the evolution of Hölder densities on unstable curves.

Ex 3: Generalized Baker's Map

$M = [0,1]^2$. Fix $\kappa \in \mathbb{N}$, $\kappa \geq 2$, and $\lambda \in \mathbb{R}$ such that $0 < \lambda \leq 1/\kappa$.

Define a generalized (κ, λ) Baker's transformation $T_{\kappa,\lambda}$:

- Subdivide M into κ vertical rectangles R_i of width $1/\kappa$.
- $T_{\kappa,\lambda}$ affine on each R_i: expands by factor κ horizontally, contracts by factor λ vertically
- $\{T_{\kappa,\lambda}(R_i)\}_i$ have disjoint interiors.



The map $T = T_{\kappa,\lambda}$ with $\kappa = 4$ and $\lambda < 1/4$.

If $\lambda = 1/\kappa$, then T is area preserving; otherwise, dissipative.

Transfer Operator

$\mathcal{W}^{s/u} := \{$vertical/horizontal line segments of length 1 in $M\}$
$\mathcal{W}^{s/u} =$ local stable/unstable manifolds for $T = T_{\kappa,\lambda}$

For $\alpha \in [0,1]$, define $|\psi|_{C^\alpha(\mathcal{W}^s)} = \sup_{W \in \mathcal{W}^s} |\psi|_{C^\alpha(W)}$.
If $\psi \in C^\alpha(\mathcal{W}^s)$, then $\psi \circ T \in C^\alpha(\mathcal{W}^s)$.

Now define $\mathcal{L}$ acting on $(C^\alpha(\mathcal{W}^s))^*$ by

$$\mathcal{L}f(\psi) = f(\psi \circ T), \quad \forall \psi \in C^\alpha(\mathcal{W}^s), f \in (C^\alpha(\mathcal{W}^s))^*$$

If $f \in C^1(M)$, then associate f with the measure $f dm$, $m =$ Lebesgue measure. Then pointwise,

$$\mathcal{L}f(x) = \frac{f \circ T^{-1}(x)}{\kappa\lambda}$$

Note $\mathcal{L}f = 0$ on $M \setminus T(M)$ and m is conformal, i.e. $\mathcal{L}^* m = m$.

Definition of Norms

Let $f \in C^1(\mathcal{W}^u)$.

Define the **weak norm** of f by

$$|f|_w = \sup_{W \in \mathcal{W}^s} \sup_{\substack{\psi \in C^1(W) \\ |\psi|_{C^1(W)} \le 1}} \int_W f \, \psi \, dm_W$$

$m_w =$ arclength measure on W.

Let $\alpha \in (0,1)$ and define the **strong stable norm** of f by

$$\|f\|_s = \sup_{W \in \mathcal{W}^s} \sup_{\substack{\psi \in C^\alpha(W) \\ |\psi|_{C^\alpha(W)} \le 1}} \int_W f \, \psi \, dm_W$$

On each $W \in \mathcal{W}^s$, these are simply the norms for the contracting map.

Definition of Norms

The strong norm should provide regularity in the unstable direction.

Write $W \in \mathcal{W}^s$ in coordinates:

$$W = \{(s,t) \in M : s = s_W, t \in [0,1]\}$$

Then define $d(W_1, W_2) = |s_{W_1} - s_{W_2}|$,
and for test functions $\psi_i \in C^1(W_i)$, define

$$d_0(\psi_1, \psi_2) = \sup_{t \in [0,1]} |\psi_1(s_{W_1}, t) - \psi_2(s_{W_2}, t)|$$

Choose $\beta \in (0,1)$ with $\beta \le 1 - \alpha$.

Define the **strong unstable norm** of f by

$$\|f\|_u = \sup_{W_1, W_2 \in \mathcal{W}^s} \sup_{\substack{\psi_i \in C^1(W_i) \\ |\psi_i|_{C^1(W_i)} \le 1 \\ d_0(\psi_1, \psi_2) = 0}} d(W_1, W_2)^{-\beta} \left| \int_{W_1} f\, \psi_1 - \int_{W_2} f\, \psi_2 \right|$$

Banach spaces $\mathcal{B}$ and $\mathcal{B}_w$

The **strong norm** of f is $\|f\|_{\mathcal{B}} = \|f\|_s + \|f\|_u$

Define the weak space $\mathcal{B}_w$ to be the completion of $C^1(\mathcal{W}^u)$ in the $|\cdot|_w$ norm.

Define the strong space $\mathcal{B}$ to be the completion of $C^1(\mathcal{W}^u)$ in the $\|\cdot\|_{\mathcal{B}}$ norm.

Lemma (Embedding Lemma)

We have the following sequence of continuous embeddings,

$$C^1(M) \hookrightarrow \mathcal{B} \hookrightarrow \mathcal{B}_w \hookrightarrow (C^1(\mathcal{W}^s))^*.$$

Moreover, the embedding $\mathcal{B} \hookrightarrow \mathcal{B}_w$ is relatively compact.

Proof of Relative Compactness

Fix $\varepsilon > 0$. Let $C^1_1(W)$ denote the unit ball of $C^1(W)$.

- Choose $\{\psi_i\}_{i=1}^{N_\varepsilon} \subset C^1([0,1])$ which forms an ε-cover of $C^1_1(W)$ in the $C^\alpha(W)$ norm for all $W \in \mathcal{W}^s$.
- Choose $\{W_j\}_{j=1}^{J_\varepsilon} \subset \mathcal{W}^s$ which forms an ε-cover of $\mathcal{W}^s$ in the metric $d(\cdot,\cdot)$.

Take $f \in C^1(\mathcal{W}^u)$, $W \in \mathcal{W}^s$, $\psi \in C^1_1(W)$. Choose ψ_i s.t. $|\psi - \psi_i|_{C^\alpha(W)} \le \varepsilon$ and W_j s.t. $d(W, W_j) \le \varepsilon$. Then,

$$\left| \int_W f\psi - \int_{W_j} f\psi_i \right| \le \left| \int_W f(\psi - \psi_i) \right| + \left| \int_W f\,\psi_i - \int_{W_j} f\,\psi_i \right|$$
$$\le \|f\|_s |\psi - \psi_i|_{C^\alpha} + d(W, W_j)^\beta \|f\|_u \le \varepsilon^\beta \|f\|_{\mathcal{B}}$$

Taking the supremum over W and ψ implies that

$$\min_{i,j} \big| |f|_w - \ell_{i,j}(f) \big| \le \varepsilon^\beta \|f\|_{\mathcal{B}}\,, \quad \text{where } \ell_{i,j}(f) = \int_{W_j} f\,\psi_i\,. \qquad \square$$

Dynamical Inequalities

Proposition

For any $n \geq 0$ and $f \in \mathcal{B}$,

$$\begin{aligned} \|\mathcal{L}^n f\|_s &\leq \lambda^{\alpha n}\|f\|_s + |f|_w, & (1) \\ \|\mathcal{L}^n f\|_u &\leq \kappa^{-\beta n}\|f\|_u, & (2) \\ |\mathcal{L}^n f|_w &\leq |f|_w. & (3) \end{aligned}$$

Proof: By density of $C^1(\mathcal{W}^u)$ in both $\mathcal{B}$ and $\mathcal{B}_w$, it suffices to prove the bounds for $f \in C^1(\mathcal{W}^u)$.

Proofs of (1) and (3) are similar to those for the contracting map. We will prove (2).

Proof of Strong Unstable Norm Contraction

(2) W^1, $W^2 \in \mathcal{W}^s$, $|\psi_j|_{C^1(W^j)} \le 1$ s.t. $d_0(\psi_1, \psi_2) = 0$.

There is a 1-1 correspondence between elements of $T^{-n}W^1 = \cup_i W_i^1$ and $T^{-n}W^2 = \cup_i W_i^2$: For each i, W_i^1, W_i^2 lie in a vertical rectangle on which T^n is smooth.

Also, since T preserves horizontal lines, $d_0(\psi_1 \circ T^n, \psi_2 \circ T^n) = 0$ on each pair W_i^1, W_i^2.

$$\int_{W^1} \mathcal{L}^n f\, \psi_1 - \int_{W^2} \mathcal{L}^n f\, \psi_2 = \kappa^{-n} \sum_i \int_{W_i^1} f\, \psi_1 \circ T^n - \int_{W_i^2} f\, \psi_2 \circ T^n$$
$$\le \kappa^{-n} \sum_i d(W_i^1, W_i^2)^\beta \|f\|_u \le \kappa^{-\beta n} d(W^1, W^2)^\beta \|f\|_u$$

Dividing through by $d(W^1, W^2)^\beta$ and taking the appropriate suprema proves (2): $\|\mathcal{L}^n f\|_u \le \kappa^{-\beta n} \|f\|_u$. □

Quasi-Compactness and a Spectral Gap

Theorem

$\mathcal{L}$ is quasi-compact as an operator of $\mathcal{B}$ with spectral radius 1 and essential spectral radius at most $\max\{\lambda^\alpha, \kappa^{-\beta}\} < 1$.
Moreover, $\mathcal{L}$ has a spectral gap on $\mathcal{B}$.

The upper bounds on the essential spectral radius and the spectral radius follow from the dynamical inequality:

$$\|\mathcal{L}^n f\|_{\mathcal{B}} = \|\mathcal{L}^n f\|_s + \|\mathcal{L}^n f\|_u \le \max\{\lambda^{\alpha n}, \kappa^{-\beta n}\}\|f\|_{\mathcal{B}} + |f|_w \,.$$

The fact that $\mathcal{L}^* m = m$ implies that the spectral radius is 1, (since 1 is in the spectrum of $\mathcal{L}^*$ and so also of $\mathcal{L}$) so that $\mathcal{L}$ is quasi-compact. Also, the peripheral spectrum contains no Jordan blocks since $\|\mathcal{L}^n\|_{\mathcal{B}}$ is uniformly bounded.

Peripheral Spectrum: Sketch of Proof

From quasi-compactness and the absence of Jordan blocks,

$$\mathcal{L} = \sum_{j=0}^{N} e^{2\pi i \theta_j} \Pi_j + R, \quad \|R\|_{\mathcal{B}} < 1, \ \Pi_j \Pi_k = R \Pi_j = \Pi_j R = 0$$

Since there are no Jordan blocks, $\Pi_j = \lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} e^{-2\pi i \theta_j k} \mathcal{L}^k$.

Set $\theta_0 = 0$ and $\mu_0 = \Pi_0 1$. Let $\mathbb{V}_j = \Pi_j(C^1)$.

a) **Elements of $\mathbb{V} = \oplus_j \mathbb{V}_j$ are measures abs. cont. wrt μ_0**

- Since $\Pi_j(C^1) = \mathbb{V}_j$, for each $\mu \in \mathbb{V}_j$, $\exists f \in C^1(M)$ s.t.

$$|\mu(\psi)| = |\Pi_j f(\psi)| \leq \lim_n \sum_{k=0}^{n-1} |f(\psi \circ T^k)| \leq |f|_\infty |\psi|_\infty ,$$

and also $\mu(\psi) \leq |f|_\infty \mu_0(\psi)$ if $\psi \geq 0$.

Peripheral Spectrum: Sketch of Proof

b) $\exists$ **finite** $\# q_k \in \mathbb{N}$ **st** $\cup_{j=0}^N \{\theta_j\} = \cup_k \{\frac{p}{q_k} : 0 \le p < q_k, p \in \mathbb{N}\}$

- Let $\mu \in \mathbb{V}_j$. By (a) $\exists f_\mu \in L^\infty(\mu_0)$ s.t. $d\mu = f_\mu d\mu_0$. Then,

$$e^{2\pi i\theta_j}\mu(\psi) = \mu(\psi \circ T) = \int \psi \circ T\, f_\mu d\mu_0 = \int \psi\, f_\mu \circ T^{-1} d\mu_0$$

so $f_\mu \circ T^{-1} = e^{2\pi i\theta_j} f_\mu$. For $k > 1$, $\mu_k = (f_\mu)^k \mu_0$ satisfies $\mathcal{L}\mu_k = e^{2\pi i\theta_j k}\mu_k$, i.e. $k\theta_j$ is in the peripheral spectrum of $\mathcal{L}$.

c) M **has a single ergodic component of pos.** μ_0 **measure.**

- Use the fact that $\mathcal{W}^s$ and $\mathcal{W}^u$ fully cross M and the definition of $\mu_0 = \Pi_0 1$ as a limit.

(c) implies 1 is a simple eigenvalue of $\mathcal{L}$.

If $\mu \in \mathbb{V}_j$, then $\theta_j = p/q$ by (b) so that $\mathcal{L}^q\mu = \mu$. But if $T = T_{\kappa,\lambda}$, then $T^q = T_{\kappa^q,\lambda^q}$ is another generalized Baker's map, so that 1 is a simple eigenvalue of $\mathcal{L}^q$ as well. Thus $\mu = \mu_0$ and $\theta_j = 0$, i.e. 1 is the only eigenvalue of modulus 1 and it is simple. □

Applications to Hyperbolic Maps

- Real-analytic hyperbolic diffeomorphisms [Rugh '94], [Fried '95]
- Anosov and Axiom A diffeomorphisms [Blank, Keller, Liverani '01], [Baladi '05], [Gouëzel, Liverani '06, '08], [Baladi, Tsujii '07], [Faure, Roy, Sjöstrand '08]
- Piecewise hyperbolic maps [D., Liverani '08], [Baladi, Gouëzel '09, '10]
- Planar billiard maps
  - Dispersing billiards and perturbations [D., Zhang '11,'13, '14]
  - Measure of maximal entropy for map [Baladi, D., '20]
  - Geometric potentials [Baladi, D. '22]
  - More general Hölder potentials [Carrand, preprint '22]
  - Measure of maximal entropy for flow [Baladi, Carrand, D. preprint '22]

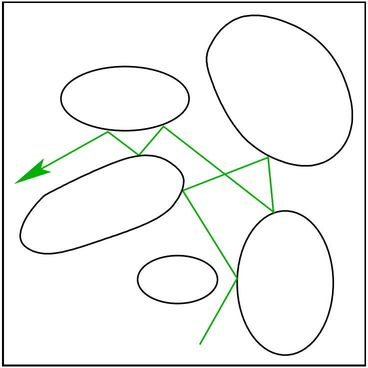
Part Two: Geometry of Dispersing Billiards

Goal for this section: Recall some geometric facts about dispersing billiards that we will use in subsequent lectures: hyperbolicity, distortion, complexity, growth lemma.

Reference: N. Chernov and R. Markarian, *Chaotic Billiards*, Mathematical Surveys and Monographs **127** (2006), 330 pp.

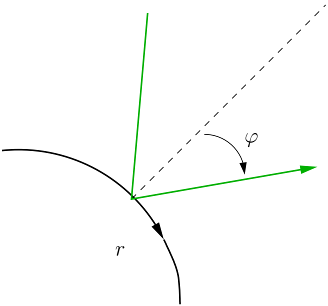
Periodic Lorentz Gas (Sinai Billiard) [Sinai '68]

- Billiard table $\mathcal{Q} = \mathbb{T}^2 \setminus \cup_i B_i$; scatterers B_i.
- Boundaries of scatterers are $\mathcal{C}^3$ and have strictly positive curvature.
- Billiard flow is given by a point particle moving at unit speed with elastic collisions at the boundary

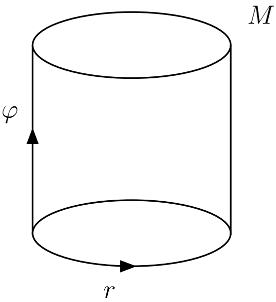


Finite horizon condition: there is an upper bound on the free flight time between collisions. Otherwise **Infinite horizon**.

The Associated Billiard Map



- $r =$ position coordinate oriented clockwise on boundary of scatterer ∂B_i
- $\varphi =$ angle outgoing trajectory makes with normal to scatterer



$M = \big(\cup_i \, \partial B_i \big) \times [-\frac{\pi}{2}, \frac{\pi}{2}]$, the natural "collision" cross-section for the billiard flow.

$T : (r, \varphi) \to (r', \varphi')$ is the first return map: the **billiard map**.

- a hyperbolic map with singularities

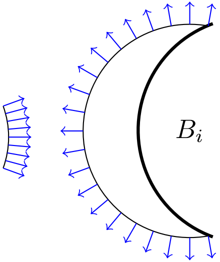
Statistical Properties with respect to SRB Measure

T preserves a smooth invariant measure on M, $\mu_{\text{SRB}} = \cos\varphi\, dr\, d\varphi$

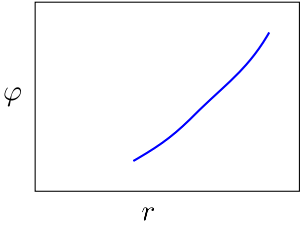
With respect to this measure, many statistical properties have been proved using a variety of techniques.

- μ_{SRB} is ergodic [Sinai '70] and Bernoulli [Gallavotti, Ornstein '74]
- Countable Markov partitions and Markov "sieves" [Bunimovich, Sinai '80, '81], [Bunimovich, Chernov, Sinai '90, '91]
  - Central Limit Theorem
- Young Towers
  - exponential decay of correlations, [Young '98]
  - almost sure invariance principle [Melbourne, Nicol '05]
  - local moderate and large deviations, [Melbourne, Nicol '08], [Young, Rey-Bellet '08]
- Coupling arguments via standard pairs [Chernov '06], [Chernov, Dolgopyat '09]
- Transfer operator techniques [D., Zhang '11, '13, '14]

Hyperbolicity away from Singularities



A dispersing wavefront before and after collision.



The wavefront projects to a curve with positive slope on B_i.

Positive slope in $M \implies$ unstable curve

Negative slope in $M \implies$ stable curve

Hyperbolicity: Stable and Unstable Cones

For both finite and infinite horizon, two global families of cones:

$$\mathcal{C}^u = \left\{ (dr, d\varphi) : \mathcal{K}_{\min} \le \frac{d\varphi}{dr} \le \mathcal{K}_{\max} + \frac{1}{\tau_{\min}} \right\}$$

$$\mathcal{C}^s = \left\{ (dr, d\varphi) : -\mathcal{K}_{\min} \ge \frac{d\varphi}{dr} \ge -\mathcal{K}_{\max} - \frac{1}{\tau_{\min}} \right\}$$

$\tau_{\min} > 0$ is minimum time between consecutive collisions

$\mathcal{K}_{\min\,/\,\max} =$ min/max curvature of scatterers

Strict invariance:

$$DT(x)\mathcal{C}^u \subsetneq \mathcal{C}^u \quad \text{and} \quad DT(x)^{-1}\mathcal{C}^s \subsetneq \mathcal{C}^s$$

Minimum expansion: $\Lambda := 1 + 2\mathcal{K}_{\min}\tau_{\min}$.

$$\exists C_0 > 0 \quad \text{s.t.} \quad \|DT^n(x)v\| \ge C_0\Lambda^n\|v\| \quad \forall v \in \mathcal{C}^u$$

and similarly for stable cone under $DT^{-n}(x)$.

Invariant Families of Stable/Unstable Curves

- Call a smooth curve $W \subset M$ **stable** (or cone-stable) if the tangent vector to W at each point belongs to $\mathcal{C}^s$.
- Define

$$\widehat{\mathcal{W}}^s = \{\text{stable curves with curvature bounded by } D_0 > 0 \\ \text{and length at most } \delta_0 > 0\ \}$$

  Since T is piecewise C^2 and uniformly hyperbolic away from its singularities, we can choose $D_0 > 0$ such that $\widehat{\mathcal{W}}^s$ is invariant under T^{-1}, up to subdivision of long curves.
- Define $\mathcal{W}^s \subset \widehat{\mathcal{W}}^s$ real local stable manifolds
- Similarly, define a T-invariant set $\widehat{\mathcal{W}}^u$ of (cone-) unstable curves, and local unstable manifolds $\mathcal{W}^u$.

Singularities

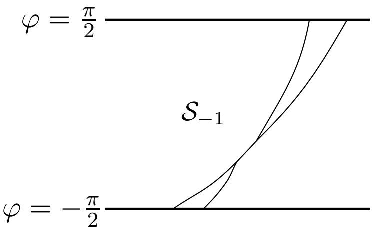
Tangential collisions create singularity curves for T.

- Let $\mathcal{S}_0 = \{\varphi = \pm\frac{\pi}{2}\}$.
- $\mathcal{S}_n = \bigcup_{i=0}^{n} T^{-i}\mathcal{S}_0$ is the singularity set for T^n, $n \geq 1$.
- $\mathcal{S}_{-n} = \bigcup_{i=0}^{n} T^{i}\mathcal{S}_0$ is the singularity set for T^{-n}, $n \geq 1$.

T^n is discontinuous at the set of decreasing curves $\mathcal{S}_n$ and T^{-n} is discontinuous at the set of increasing curves $\mathcal{S}_{-n}$.

Important fact: $\mathcal{S}_n$ is uniformly transverse to $\mathcal{C}^u$ and $\mathcal{S}_{-n}$ is uniformly transverse to $\mathcal{C}^s$.

- **Continuation of Singularities**
  Every curve in $\mathcal{S}_n \setminus \mathcal{S}_0$ is part of a monotonic piecewise smooth curve belonging to $\mathcal{S}_n$ which terminates on $\mathcal{S}_0$.



Linear Bound on Complexity

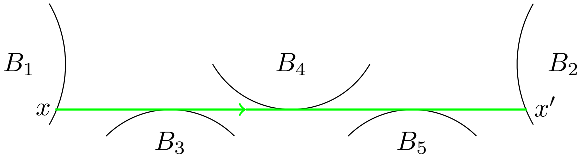
Want expansion due to hyperbolicity to beat cutting due to singularities. In the finite horizon case, there is a linear bound due to Bunimovich.

For $x \in M$, let $N(\mathcal{S}_n, x)$ denote the number of singularity curves in $\mathcal{S}_n$ that meet at x. Define $N(\mathcal{S}_n) = \sup_{x \in M} N(\mathcal{S}_n, x)$.

Lemma (Bunimovich, Chernov, Sinai '90)

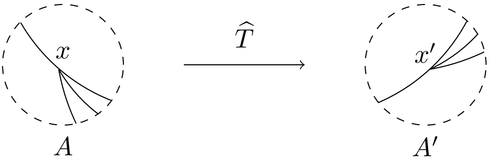
Assume finite horizon. There exists $K > 0$ depending only on the configuration of scatterers such that $N(\mathcal{S}_n) \leq Kn$ for all $n \geq 1$.

Idea of Proof. Let $x, x' \in M$ lie on a straight billiard trajectory with one or more tangential collisions between.



Linear Bound on Complexity

Let A, A' be neighborhoods of x, x' in M, partitioned into sectors $A_1, \dots A_k \subset A$ and $A'_1, \dots A'_k \subset A$ such that $T^{n_j} A_j = A'_j$.
Note $k \le \tau_{\max}/\tau_{\min}$. Set $\widehat{T}|_{A_j} := T^{n_j}$



- Assume $N(\mathcal{S}_{n-1}) \le K(n-1)$.
- Let $N(\mathcal{S}_i | A'_j, x')$ denote the number of curves in $\mathcal{S}_i$ passing through x' and lying in A'_j.
- $N(\mathcal{S}_n, x) \le k + \sum_j N(\mathcal{S}_{n-n_j} | A'_j, x') \le k + \sum_j N(\mathcal{S}_{n-1} | A'_j, x')$
- So $N(\mathcal{S}_n, x) \le k + K(n-1) \le Kn$ if $k \le K$. $\square$

The proof uses that the flow is continuous.

Distortion Control and Extended Singularity Set

When $T(x)$ is near $\mathcal{S}_0$, $DT(x)$ becomes large:

$$\|DT(x)|_{E^u}\| \sim \frac{1}{\cos\varphi(Tx)} \sim d(x, \mathcal{S}_1)^{-1/2}, \quad \|DT(x)|_{E^s}\| \sim \cos\varphi(x)$$

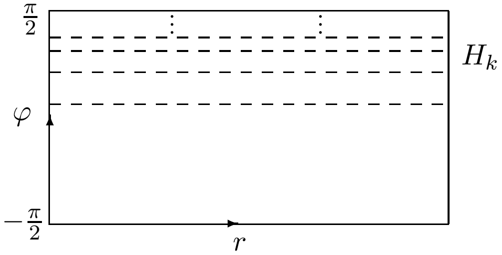
Indeed, $\det DT(x) = \frac{\cos\varphi(x)}{\cos\varphi(Tx)}$, which can be viewed as the product of stable and unstable Jacobians.

To control distortion, partition M into **homogeneity strips** $H_{\pm k}$,

$$H_k = \left\{ \frac{\pi}{2} - \frac{1}{k^q} < \varphi < \frac{\pi}{2} - \frac{1}{(k+1)^q} \right\}$$

and similarly for H_{-k}, $|k| \geq k_0$. Standard choice: $q = 2$

Distortion Control and Extended Singularity Set



- Define $\widehat{\mathcal{W}}^s_{\mathbb{H}}$ = homogeneous elements of $\widehat{\mathcal{W}}^s$
- Distortion depends on the exponent q: If $T^i W \subset \widehat{\mathcal{W}}^s_{\mathbb{H}}$, $i = 0, \ldots, n$, then

$$\log \frac{J_W T^n(x)}{J_W T^n(y)} \le C_d d(x,y)^{1/(q+1)} \qquad \forall x, y \in W$$

- But the singularity set becomes countable: $\mathcal{S}_0^{\mathbb{H}} := \mathcal{S}_0 \cup (\cup_{|k| \ge k_0} \partial H_k)$, and $\mathcal{S}^{\mathbb{H}}_{\pm n} = \bigcup_{i=0}^n T^{\mp i} \mathcal{S}_0^{\mathbb{H}}$.
- Need a new complexity bound.

One-step Expansion [Chernov '06]

Define an adapted metric in the tangent space $dx = (dr, d\varphi)$ by,

$$\|dx\|_* = \frac{\mathcal{K}(x) + |\mathcal{V}|}{\sqrt{1+\mathcal{V}^2}} \|dx\| \,,$$

where $\mathcal{V} = d\varphi/dr$ and $\mathcal{K}(x)$ is the curvature of the scatterer at x.

- $\|DT(x)^{-1} dx\|_* \geq \Lambda \|dx\|_*$ for all stable vectors dx, where $\Lambda = 1 + 2\mathcal{K}_{\min}\tau_{\min}$.

For $V \in \widehat{\mathcal{W}}^s_{\mathbb{H}}$, let $\{V_i\}_i =$ homogeneous connected comp. of $T^{-1}V$.

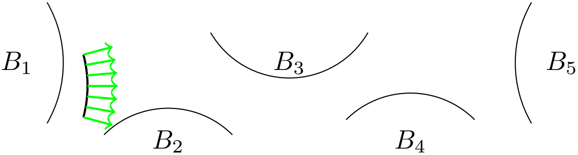
Lemma (One-step expansion)

There exists $\theta < 1$ such that for all $V \in \widehat{\mathcal{W}}^s_{\mathbb{H}}$,

$$\limsup_{\delta \downarrow 0} \sup_{|V| \leq \delta} \sum_{V_i} |J_{V_i} T|_* < \theta \,,$$

where $|\cdot|_*$ denotes the sup norm in the adapted metric.

Proof of One-step Expansion for Finite Horizon



- A short stable curve can be cut by at most $\tau_{\max}/\tau_{\min}$ tangential collisions under T^{-1}.
- All but one of these collisions is nearly grazing.
- Near the grazing collisions, $V_k \subset H_k$ and,

$$\sum_{k \geq k_0} |J_{V_k} T|_* \leq C \sum_{k \geq k_0} k^{-q} \leq C' k_0^{1-q} \quad \text{if } q > 1$$

- Fix $\varepsilon > 0$ and choose k_0 large enough that $k_0^{q-1} \frac{\tau_{\max}}{\tau_{\min}} \leq \varepsilon$.
- Choose $\delta_0 > 0$ so that a stable curve of length δ_0 must map into homogeneity strips of index $|k| \geq k_0$ at the nearly tangential collisions.
- Then $\theta = \Lambda^{-1} + \varepsilon$ satisfies the lemma. □

Growth Lemma

Consequence of one-step expansion is the following growth lemma.

- For $W \in \widehat{\mathcal{W}}^s$, partition $T^{-1}W$ into maximal connected homogeneous components. Subdivide any curve longer than δ_0 into curves of length between $\delta_0/2$ and δ_0. Call this collection $\mathcal{G}_1(W)$.
- Define inductively, $\mathcal{G}_n(W) = \{\mathcal{G}_1(W_i) : W_i \in \mathcal{G}_{n-1}(W)\}$.
- Let $L_n(W)$ denote those $W_i \in \mathcal{G}_n(W)$ such that $|W_i| \geq \delta_0/3$.
- Let $\mathcal{I}_n(W)$ denote those $W_i \in \mathcal{G}_n(W)$ such that $T^j W_i \subset V_j \in \mathcal{G}_{n-j}(W)$ with $|V_j| < \delta_0/3$ for all $j = 0, \ldots, n-1$. W is the **most recent long ancestor** of W_i.

Lemma

There exists $C_1 > 0$ such that for all $W \in \widehat{\mathcal{W}}^s$ and all $n \geq 1$,

$$\sum_{W_i \in \mathcal{G}_n(W)} |J_{W_i} T^n|_{C^0(W_i)} \leq C_1 \, .$$

Proof of Growth Lemma

Organize $W_i \in \mathcal{G}_n(W)$ by most recent long ancestor.

- If $W_i \in L_n(W)$, then W_i is its own most recent long ancestor.
- Otherwise, $W_i \in \mathcal{I}_j(V_k)$ for some $V_k \in L_{n-j}(W)$, $j \geq 1$.
- Or $W \in \mathcal{I}_n(W)$, whether W is long or short.

$$\begin{aligned}
\sum_{W_i \in \mathcal{G}_n(W)} |J_{W_i} T^n|_{C^0} &\leq \sum_{j=1}^{n} \sum_{V_k \in \mathcal{G}_{n-j}(W)} |J_{V_k} T^{n-j}|_{C^0} \sum_{W_i \in \mathcal{I}_j(V_k)} |J_{W_i} T^j|_{C^0} \\
&\leq C_* \theta^n + \sum_{j=1}^{n-1} \sum_{V_k \in \mathcal{G}_{n-j}(W)} |J_{V_k} T^{n-j}|_{C^0} C_* \theta^j \\
&\leq C_* \theta^n + \sum_{j=1}^{n-1} \sum_{V_k \in \mathcal{G}_{n-j}(W)} e^{C_d \delta_0^{1/q}} \frac{|T^{n-j} V_k|}{|V_k|} C_* \theta^j \\
&\leq C_* \theta^n + \sum_{j=1}^{n-1} C' \delta_0^{-1} |W| \theta^j \leq C_* \theta^n + C'' \delta_0^{-1} |W|
\end{aligned}$$