MEASURE OF MAXIMAL ENTROPY FOR FINITE HORIZON SINAI BILLIARD FLOWS

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Abstract. Using recent work of Carrand [Ca] on equilibrium states for the billiard map, and bootstrapping via a "leapfrogging" method from [BD2], we construct the unique measure of maximal entropy (MME) for two-dimensional finite horizon Sinai (dispersive) billiard flows Φ^1 (and show it is Bernoulli), assuming the bound $h_{\mathrm{top}}(\Phi^1)\tau_{\min} > s_0 \log 2$, where $s_0 \in (0,1)$ quantifies the recurrence to singularities. This bound holds in many examples (it is expected to hold generically).

1. Introduction and Main Result

1.1. **Background.** Let Φ^t be a continuous flow on a compact manifold. The topological entropy of the flow, $h_{\mathrm{top}}(\Phi^1)$, is the supremum, over ergodic probability measures ν invariant under the (continuous) time-one map Φ^1 of the Kolmogorov entropy $h_\nu(\Phi^1)$. If a measure realising the supremum exists, it is called a measure of maximal entropy (MME) for the flow.

For geodesic flows, the study of the MME has a rich history. In the case of strictly negative curvature, the flow is Anosov, i.e. smooth and uniformly hyperbolic, and the pioneering works of Bowen [Bo2] and Margulis [Ma1, Ma2] half a century ago established existence, uniqueness, and mixing of the MME, leading to remarkable consequences, in particular on the structure (counting and equidistribution) of periodic orbits. For more general continuous flows, it became apparent [Bo0, Bo1, BW] that (flow) expansivity implies existence of the MME, and combined [Fr] with the (Bowen) specification property, also gives uniqueness.

Starting with the groundbreaking work of Knieper [Kn], most developments in the past 25 years have concerned smooth geodesic flows for which the hyperbolicity or compactness assumption are relaxed. In recent years, Climenhaga and Thompson [CT] have revisited the Bowen specification approach, which has allowed them to obtain several striking [CKW, B-T] results.

Sinai billiard flows, our object of study, are natural dynamical systems which are uniformly hyperbolic, but not differentiable (we refer to [CM] for a full-fledged introduction to mathematical billiards): A Sinai billiard table Q on the two-torus $\mathbb{T}^2$ is a set $Q = \mathbb{T}^2 \setminus \cup_i \mathcal{O}_i$, for finitely many pairwise disjoint convex closed domains $\mathcal{O}_i$ with C^3 boundaries having strictly positive curvature $\mathcal{K}$. The billiard flow Φ^t,

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$t \in \mathbb{R}$, is the motion of a point particle traveling in Q at unit speed and undergoing specular reflections[1] at the boundary of the scatterers $\mathcal{O}_i$. The associated billiard map $T : M \to M$, on the compact metric set $M = \partial Q \times [-\frac{\pi}{2}, \frac{\pi}{2}]$, is the first collision map on the boundary of Q. Grazing collisions cause discontinuities in the map T, but the flow is continuous (after identification of the incoming and outgoing angles). The map is expansive [BD1], but this property is not automatically[2] inherited by the flow, since neither the map nor the return time is continuous. In particular, it is not obvious that the flow satisfies a condition (such as asymptotic h-expansiveness [Mi]) sufficient for the upper-semi continuity of the Kolmogorov entropy (see [Ca, App. A–B]), and there does not appear to exist a quick way to prove the existence — let alone uniqueness — of a MME for the billiard flow.

The purpose of the present paper is to furnish mild conditions guaranteeing existence, uniqueness, and mixing (in fact, the Bernoulli property) of the MME for Sinai billiards. This can be viewed as a first step towards the much harder open problem of establishing equidistribution results for Sinai billiards.

Our proof is based on previous work of Carrand [Ca] (itself relying on [BD1]) and on [BD2]. These three papers use the[3] technique of transfer operators acting on anisotropic spaces, which was first introduced to billiards by Demers–Zhang [DZ1], and recently applied to construct the measure of maximal entropy of the billiard map [BD1].

1.2. **Results.** To state our main results, Theorem 1.4 and[4] Corollary 1.5, we introduce some basic notation. For $x \in M$, let $\tau(x)$ denote the flow time (return time) from x to $T(x)$, and set

$$\tau_{\min} = \inf \tau > 0 \, , \ \tau_{\max} = \sup \tau \, , \ \Lambda = 1 + 2\tau_{\min} \inf \mathcal{K} \, .$$

Throughout, we assume finite horizon, that is: there are no trajectories making only tangential collisions. Finite horizon implies $\tau_{\max} < \infty$.

Set

$$P(t) = \sup_{\mu: T\text{-invariant ergodic probability measure}} \{ h_\mu(T) - t \int \tau d\mu \} \, , \ t \geq 0 \, .$$

The real number $P(t)$ is called the pressure of the potential $-t\tau$ and a probability measure μ_t realising $P(t)$ is called an equilibrium measure for $-t\tau$.

Viewing Φ as the suspension of T under τ, Abramov's formula says that any ergodic probability measure ν invariant under the time-one map Φ^1 satisfies

$$\nu = \frac{\mu}{\int \tau d\mu} \otimes Leb \, , \tag{1.1}$$

where μ is an ergodic T-invariant probability measure, and, in addition,

$$h_\nu(\Phi^1) = \frac{h_\mu(T)}{\int \tau d\mu} \, . \tag{1.2}$$

[1]At a tangential collision, the reflection does not change the direction of the particle.

[2]See [BW] for a definition of expansiveness for the flow. See [Bo0, Ex. 1.6] for a weaker sufficient condition for existence.

[3]To our knowledge, the Climenhaga–Thompson specification approach has not been implemented yet for Sinai billiards.

[4]The condition (1.4) there is discussed in Lemma 1.3.

In the coordinates $x = (r, \varphi)$, where r is arclength along $\partial\mathcal{O}_i$ and φ is the post-collision angle with the normal to $\partial\mathcal{O}_i$, let $\mathcal{S}_0 = \{(r,\varphi) \in M : \varphi = \pm\frac{\pi}{2}\}$ denote the set of tangential collisions on M. Then for any $n \in \mathbb{Z}_*$, the set $\mathcal{S}_n = \cup_{i=0}^{-n} T^i \mathcal{S}_0$ is the singularity set of T^n. Following [BD1], define $\mathcal{M}_0^n$ to be the set of maximal connected components of $M \setminus \mathcal{S}_n$ for $n \geq 1$, and set

$$h_* = \lim_{n\to\infty} \frac{1}{n} \log \#\mathcal{M}_0^n$$

(existence of the limit is easy [BD1]). Then, for fixed $\varphi < \pi/2$ close to $\pi/2$ and large $n \in \mathbb{N}$, define $s_0(\varphi, n) \in (0, 1]$ to be the smallest number such that any orbit of length equal to n has at most $s_0 n$ collisions whose angles with the normal are larger than φ in absolute value. If

$$h_* > s_0 \log 2 \tag{1.3}$$

then [BD1] proves that $P(0) = h_*$, and there is a unique equilibrium measure $\mu_* = \mu_0$ for $t = 0$, which is the unique MME of T. There are many billiards [BD1, §2.4] satisfying (1.3), and in fact we do not know any billiard which violates it. (Note also that Demers and Korepanov showed [DK] that a conjecture of Bálint and Tóth, if true, implies that, generically, one can choose φ and n to make s_0 arbitrarily small.)

Using Abramov's formula, Carrand showed the following:

Proposition 1.1 ([Ca, Lemma 2.5, Cor. 2.6])**.** *The real number $t = h_{top}(\Phi^1) > 0$ is the unique t such that $P(t) = 0$. In addition, the set of equilibrium measures of T for $-h_{top}(\Phi^1)\tau$ is in bijection with the set of MMEs of the flow via* (1.1).

Denote $\Sigma_n\tau := \sum_{k=0}^{n-1} \tau \circ T^k$ (to avoid confusion with $\mathcal{S}_n$ and the notation S_n^δ below). We next state Carrand's main results (see also Proposition 3.1 below).

Theorem 1.2 ([Ca, Theorem 2.1, Theorem 1.2])**.** *(a) The following*[5] *limits exist:*

$$P_*(t) = \lim_{n\to\infty} \frac{1}{n} \log Q_n(t)\,, \text{ with } Q_n(t) = \sum_{A \in \mathcal{M}_0^n} |e^{-t\Sigma_n\tau}|_{C^0(A)}\,, \forall t \geq 0\,.$$

Moreover, $P_*(t) > P_*(s) \geq P(s)$ for all $0 \leq t < s$, and[6] *$t \mapsto P_*(t)$ is convex.*
(b) If $t \geq 0$ is such that

$$P_*(t) + t\tau_{\min} > s_0 \log 2\,, \tag{1.4}$$

and

$$\log\Lambda > t(\tau_{\max} - \tau_{\min})\,, \tag{1.5}$$

then there is a unique equilibrium measure μ_t for $-t\tau$. This measure charges all open sets, is Bernoulli, and $P_*(t) = P(t)$. Finally, μ_t is T-adapted,[7] *that is*

$$\int |\log d(x, \mathcal{S}_{\pm1})|\, d\mu_t < \infty\,. \tag{1.6}$$

[5] By [BD1] we always have $P_*(0) = h_* \geq P(0)$.

[6] The fact that $P_*(t)$ is strictly decreasing is immediate, see (3.5). Convexity follows from the Hölder inequality as in [BD2, Prop 2.6].

[7] To establish (1.6), Carrand shows that the μ_t measure of the ϵ-neighbourhood of $\mathcal{S}_{\pm1}$ is bounded by $C_t|\log\epsilon|^\gamma$ for $\gamma > 1$ and $C_t < \infty$.

In view of Proposition 1.1 and Theorem 1.2, to establish existence and uniqueness of the MME of the finite horizon flow Φ, it *suffices* to check (1.4) and (1.5) for $t = h_{\mathrm{top}}(\Phi^1) > 0$. We next discuss these conditions. The first one is very mild:

Lemma 1.3. *The bound* (1.4) *holds at* $t = h_{top}(\Phi^1)$ *as soon as*

$$h_{top}(\Phi^1)\tau_{\min} > s_0 \log 2\,. \tag{1.7}$$

The bound (1.7) *holds as soon as*

$$h_* \frac{\tau_{\min}}{\tau_{\max}} > s_0 \log 2\,. \tag{1.8}$$

If (1.4) *holds for some* $t' \geq 0$ *then it holds for all* $t \in [0, t']$*.*

It is not hard to find [Ca, Remark 5.6] billiards satisfying (1.7).

Proof. The first claim follows from Proposition 1.1 and the bound $P_*(t) \geq P(t)$ for all $t \geq 0$. The second claim holds because (1.2) implies $h_{\mathrm{top}}(\Phi^1) \geq \frac{h_*}{\int \tau d\mu_*} \geq \frac{h_*}{\tau_{\max}}$. Finally, the first claim of Lemma 3.3 below implies that $t \mapsto P_*(t) + t\tau_{\min}$ is nonincreasing. $\square$

The second condition (1.5) will require more efforts. Obviously, for any finite horizon billiard, there exists $\tilde{t} > 0$ such that (1.5) holds for all $t \in [0, \tilde{t}]$. However, we do[8] not know any billiard such that (1.5) holds for $t = h_{\mathrm{top}}(\Phi^1)$ (that is, $\log \Lambda > h_{\mathrm{top}}(\Phi^1)(\tau_{\max} - \tau_{\min})$). Fortunately, it turns out that (1.5) is not *necessary:* Assuming only finite horizon and (1.4) at $t = h_{\mathrm{top}}(\Phi^1)$, we will extend the conclusion of Theorem 1.2 to $t = h_{\mathrm{top}}(\Phi^1)$ by adapting the bootstrapping argument in [BD2, Lemma 3.10] (used there to cross the value $x = 1$ at which the pressure for $-x \log J^u T$ vanishes). This is our main result:

Theorem 1.4. *Let* T *be a finite horizon Sinai billiard map such that* (1.4) *holds at* $t = h_{top}(\Phi^1)$*. Then for all* $t \in [0, h_{top}(\Phi^1)]$*, we have* $P_*(t) = P(t)$*, and there exists a unique* T*-invariant probability measure* μ_t *realising* $P(t)$*. This measure charges all nonempty open sets, is Bernoulli and* T*-adapted.*

Our proof furnishes $t_\infty \geq h_{\mathrm{top}}(\Phi^1)$ such that the key Small Singular Pressure properties (3.1), (3.2), and (3.3) hold for all $t \in [0, t_\infty]$. If $t_\infty > h_{\mathrm{top}}(\Phi^1)$ and if (1.4) holds for some $t_2 \in (h_{\mathrm{top}}(\Phi^1), t_\infty]$, then the conclusion of Theorem 1.4 holds for all $t \in [0, t_2]$.

Theorem 1.2 and Proposition 1.1 of Carrand, combined with Theorem 1.4 and the proof of [Ca, Props. 7.1 and 7.2] for Bernoullicity of the flow, give:

Corollary 1.5. *Let* T *be a finite horizon Sinai billiard map such that* (1.4) *holds at* $t = h_{top}(\Phi^1)$*. Then*

$$\nu_* := \frac{\mu_{h_{top}(\Phi^1)}}{\int \tau \, d\mu_{h_{top}(\Phi^1)}} \otimes Leb$$

is the unique measure of maximal entropy of the billiard flow. This measure is Bernoulli, it charges all nonempty open sets, and it is flow adapted, that is[9]

$$\int_\Omega |\log d_\Omega(x, \mathcal{S}_0^\pm)| \, d\nu_* < \infty\,, \quad \Omega = Q \times \mathbb{S}^1\,, \tag{1.9}$$

[8]Note that (1.2) implies $h_{\mathrm{top}}(\Phi^1)(\tau_{\max} - \tau_{\min}) \leq h_*(\tau_{\max}/\tau_{\min} - 1)$.

[9]Note that (1.9) implies that $\log \|D\Phi_t\|$ is integrable for each $t \in [-\tau_{\min}, \tau_{\min}]$ so that, by subadditivity, it is integrable for each $t \in \mathbb{R}$.

where d_Ω is the Euclidean metric, $\mathcal{S}_0^- = \{\Phi_{-s}(z) : z \in \mathcal{S}_0\,,\, s \le \tau(T^{-1}z)\}$, and $\mathcal{S}_0^+ = \{\Phi_s(z) : z \in \mathcal{S}_0\,,\, s \le \tau(z)\}$.

Contrary to [BD2], homogeneity layers are not used for our potentials $-t\tau$. They are not needed because τ is piecewise Hölder and thus e^τ satisfies piecewise bounded distortion. The results of Carrand [Ca] that we build upon are based on bounds for transfer operators acting on Banach spaces of distributions defined with the logarithmic modulus of continuity of [BD1]. We could not find a Banach norm giving a spectral gap (there is no analogue of [BD2, Lemmas 3.3 and 3.4] for $\varsigma \ne 0$, see [Ca, Lemma 3.1] for $\gamma \ne 0$ where $(\log |W|/\log |W_i|)^\gamma$ replaces $(|W_i|/|W|)^\varsigma$). We thus do not have exponential mixing for $(T, \mu_{h_{\rm top}(\Phi^1)})$. (Even if we had, it would not immediately imply exponential mixing for (Φ^1, ν_*).)

The paper is organised as follows: Section 2 is devoted to recalling notation from [BD1] and to two basic lemmas on cone stable curves iterated by the billiard map. Section 3 is the core of the paper: In §3.1, after defining the Small Singular Pressure (SSP) conditions (3.1), (3.2), and (3.3) and stating Carrand's conditional Theorem 3.1, we reduce Theorem 1.4 to showing SSP for some $t \ge h_{\rm top}(\Phi^1)$ (Lemma 3.2). Then we set up the bootstrap mechanism, by introducing in (3.4) the supremum $t_\infty > 0$ of parameters satisfying SSP (this is the new idea). Lemma 3.3 embodies our version of the first ingredient of the bootstrap from [BD2, Definition 3.9] ("pressure gap"), constructing a "pivot" $t_* < t_\infty$ and its associated parameter $s_*(t_*) > t_\infty$. The key lemmas inspired by the second ingredient of bootstrapping [BD2, Lemmas 3.10–3.11] ("leapfrogging across t_* via the Hölder inequality"), are stated and proved in §3.2. Finally, Lemma 3.2 (and thus Theorem 1.4) is proved in §3.3: We assume for a contradiction that $t_\infty < h_{\rm top}(\Phi^1)$. Since $t_* < t_\infty$, this implies, by results from [Ca] recalled in Proposition 1.1 and Theorem 1.2(a), that the pressure of t_* is positive. Then, we exploit this positivity in order to pass over the pivot t_* via the key lemmas from §3.2, obtaining the desired contradiction.

Observe that using Carrand's [Ca] analysis of families more general than $g_t = -t\tau$, the results of the present paper extend to suitable one parameter-families g_t of piecewise Hölder potentials. We abstain from spelling out the details.

2. Notations. n-step Expansion. Growth Lemma

We recall here some facts about hyperbolicity and complexity of finite horizon Sinai billiards. There exist continuous families of stable and unstable cones, $\mathcal{C}^s$ and $\mathcal{C}^u$, which can be taken constant in M, and a constant $C_1 \in (0,1)$ such that,

$$\|DT^n(x)v\| \ge C_1\Lambda^n\|v\|\,,\ \forall v \in \mathcal{C}^u\,, \quad \|DT^{-n}(x)v\| \ge C_1\Lambda^n\|v\|\,,\ \forall v \in \mathcal{C}^s\,, \tag{2.1}$$

where, as before, $\Lambda = 1 + 2\tau_{\min}\mathcal{K}_{\min}$ is the minimum hyperbolicity constant.

A fundamental fact about this class of billiards is the linear bound on the growth in complexity due to Bunimovich [Ch, Lemma 5.2],

(2.2) There exists $K \ge 1$ such that for all $n \ge 0$, the number of curves in $\mathcal{S}_{\pm n}$ that intersect at a single point is at most Kn.

The parameter $\gamma > 1$ defining the Banach space norms in [Ca] is chosen so that $h_* > s_0\gamma \log 2$, which is possible due to (1.3). Next, choosing m so large that,

$$\tfrac{1}{m}\log(Km+1) < h_* - s_0\gamma\log 2\,,$$

we take $\delta_0 = \delta_0(m) \in (0, 1/C_1)$ so that any stable curve of length at most δ_0 can be cut by $\mathcal{S}_{-\ell}$ into at most $K\ell + 1$ connected components for all $0 \le \ell \le 2m$.

Let $\widehat{\mathcal{W}}^s$ be, as in [BD1, §5], the set of (cone-stable) curves whose tangent vectors lie in the stable cone for T, with length at most δ_0 and curvature bounded above by a constant $C_{\mathcal{K}}$ depending only on the table (homogeneity layers are not used). The constant $C_{\mathcal{K}}$ is chosen large enough that $T^{-1}\widehat{\mathcal{W}}^s \subset \widehat{\mathcal{W}}^s$, up to subdivision of curves. For $n \ge 1$, $\delta \in (0, \delta_0]$, and $W \in \widehat{\mathcal{W}}^s$, let $\mathcal{G}_n^\delta(W)$, $L_n^\delta(W)$, $S_n^\delta(W)$, and $\mathcal{I}_n^\delta(W)$ be as in [BD1, §5]: Set $\mathcal{G}_0^\delta(W) = W$ and define $\mathcal{G}_n^\delta(W)$ for $n \ge 1$ to be the set of smooth components of $T^{-1}W'$ for $W' \in \mathcal{G}_{n-1}^\delta(W)$, with elements longer than δ subdivided to have length between $\delta/2$ and δ. More precisely, if a smooth component U has length $\ell\delta + \rho$ with $\ell \ge 1$ and $0 \le \rho < \delta$, we decompose U into:

- either $\ell \ge 2$ pieces of length δ, if $\rho = 0$,
- or $\ell \ge 1$ piece(s) of length δ and one piece of length ρ, placed at one of the edges of U, if $\rho \ge \delta/2$,
- or $\ell - 1 \ge 0$ piece(s) of length δ, one piece of length $\delta/2$ (at one tip) and one piece of length $\rho + \delta/2$ (at the other tip), if $\rho \in (0, \delta/2)$.

Let $L_n^\delta(W)$ denote the set of curves in $\mathcal{G}_n^\delta(W)$ that have length at least $\delta/3$ and let $S_n^\delta(W) = \mathcal{G}_n^\delta(W) \setminus L_n^\delta(W)$. For $0 \le k < n$, we say that $U \in \mathcal{G}_k^\delta(W)$ is an ancestor of $V \in \mathcal{G}_n^\delta(W)$ if $T^{n-k}V \subseteq U$, and we define $\mathcal{I}_n^\delta(W)$ to be those curves in $\mathcal{G}_n^\delta(W)$ that have no ancestors of length at least $\delta/3$ (aside from perhaps W itself).

Finally, let $\delta_1 < \delta_0$ and $n_1 \ge m$ be chosen so that [BD1, eq. (5.6)] holds: For any stable curve W with $|W| \ge \delta_1/3$ and $n \ge n_1$,

$$\# L_n^{\delta_1}(W) \ge \tfrac{2}{3} \# \mathcal{G}_n^{\delta_1}(W) .$$

Up to replacing δ_1 by a smaller constant, we may and shall only consider values of δ of the form $\delta_0/2^N$ for $N \ge 0$. By induction on N, selecting the short tips in a compatible way when dividing δ by two, we require that[10] for all $W \in \widehat{\mathcal{W}}^s$,

$$\forall n \ge 1 \,, \text{ if } \delta'' < \delta' \text{ then } \forall\, U'' \in L_n^{\delta''}(W) \,,\, \exists!\, U' \in \mathcal{G}_n^{\delta'}(W) \text{ with } U'' \subset U' \,, \tag{2.3}$$

For $t \ge 0$, we introduce the following shorthand notation,

$$S_n^\delta(W,t) := \sum_{W_i \in S_n^\delta(W)} |e^{-t\Sigma_n \tau}|_{C^0(W_i)} \,,\ \mathcal{G}_n^\delta(W,t) := \sum_{W_i \in \mathcal{G}_n^\delta(W)} |e^{-t\Sigma_n \tau}|_{C^0(W_i)} \,,$$

and

$$L_n^\delta(W,t) := \mathcal{G}_n^\delta(W,t) - S_n^\delta(W,t) \,,\ \mathcal{I}_n^\delta(W,t) := \sum_{W_i \in \mathcal{I}_n^\delta(W)} |e^{-t\Sigma_n \tau}|_{C^0(W_i)} \,.$$

The lemma below replaces the usual one-step expansion (see [BD2, Lemma 3.1]):

Lemma 2.1 (n-Step Expansion)**.** *For any $t_0 > 0$ and $\theta_0 \in (e^{-\tau_{\min}}, e^{-\tau_{\min}/2})$ there exist a finite $n_0(t_0, \theta_0) \ge 2$ and $\bar\delta_0 = \frac{\delta_0}{2^N} > 0$ such that*

$$S_{n_0}^{\bar\delta_0}(W,t) \le \mathcal{G}_{n_0}^{\delta_0}(W,t) < \theta_0^{n_0 t} \,,\ \ \forall W \in \widehat{\mathcal{W}}^s \text{ with } |W| \le \bar\delta_0 \,,\ \forall t \ge t_0 \,. \tag{2.4}$$

[10]We use this in the proof of Lemma 3.7 below. An alternative way to guarantee (2.3) for a fixed length scale δ' is to define $\mathcal{G}_n^{\delta'}(W)$ as usual and treat it as the canonical partition of $T^{-n}W$. Then for any $\delta'' < \delta'/2$ one can define $\mathcal{G}_n^{\delta''}(W)$ as a refinement of $\mathcal{G}_n^{\delta'}(W)$, guaranteeing (2.3). This is done implicitly in the proof of [BD2, Lemma 3.11] and could be applied in our Lemma 3.7 below by taking $\delta' = \delta_{t_*}$ of that lemma. We do not adopt this approach since the canonical scale would not be chosen until nearly the end of our proof.

See also [Ca, Lemma 3.1(a)].

Proof. Clearly, $\sup -t\tau \le -t\tau_{\min} < 0$ if $t > 0$. For any $n_0 \ge 1$, there exists $\bar\delta_0(n_0) = \frac{\delta_0}{2^N}$ such that any $W \in \widehat{\mathcal{W}}^s$ with $|W| < \bar\delta_0$ is such that $T^{-n_0}(W)$ has at most (Kn_0+1) connected components [Ch, Lemma 5.2]. In addition using [CM, Ex. 4.50] as in [BD1, Proof of Lemma 5.1], we have $|T^{-j}W| \le C'|W|^{2^{-s_0 j}}$ for a uniform $C' > 0$ and all $j \ge 1$ (see also [Ca, Lemma 3.1]). Up to taking smaller $\bar\delta_0$, depending on δ_0 (and n_0), we can assume that $|T^{-j}W| \le \delta_0$ for all $0 \le j \le n_0$. Then, for $|W| \le \bar\delta_0$, there can be no additional subdivisions of $T^{-n_0}(W)$ due to pieces growing longer than δ_0, so that

$$\mathcal{G}^{\delta_0}_{n_0}(W,t) \le (Kn_0+1)e^{-tn_0\tau_{\min}}\,. \tag{2.5}$$

The same bound applies to $S^{\bar\delta_0}_{n_0}(W,t)$, since any element of $S^{\bar\delta_0}_{n_0}(W)$ must be created by a genuine cut by a singularity, not an additional subdivision due to pieces growing longer than $\bar\delta_0$. For any fixed $t_0 > 0$ and $\theta_0 \in (e^{-\tau_{\min}}, e^{-\tau_{\min}/2})$, we can find $n_0 = n_0(t_0,\theta_0) \ge 2$ such that $(Kn_0+1)^{1/n_0} \le \theta_0^{t_0} e^{\tau_{\min} t_0}$. Since $\theta_0^{t_0} e^{\tau_{\min} t_0} \le \theta_0^{t} e^{\tau_{\min} t}$ for all $t \ge t_0$, it follows that (2.4) holds for $\bar\delta_0 = \bar\delta_0(n_0,\delta_0)$. $\square$

Lemma 2.1 implies the following analogue[11] of [BD2, Lemmas 3.3–3.4, $\zeta = 0$]:

Lemma 2.2 (Growth Lemma). *Fix $\theta_0 \in (e^{-\tau_{\min}}, e^{-\tau_{\min}/2})$ and $t_0 > 0$. Suppose $\delta \le \delta_0$ and $m_1(\delta) \ge n_0(t_0,\theta_0)$ are such that any $W \in \widehat{\mathcal{W}}^s$ with $|W| \le \delta$ has the property that $W \setminus \mathcal{S}_{-j}$ comprises at most $Kj+1$ connected components for all $1 \le j \le 2m_1$. Then for any $t \ge t_0$ and each $W \in \widehat{\mathcal{W}}^s$ with $|W| \le \delta$, we have*

$$\mathcal{I}^{\delta}_n(W,t) \le \theta_0^{nt}\,,\ \forall n \ge m_1\,, \tag{2.6}$$

$$\mathcal{I}^{\delta}_n(W,t) \le Km_1\theta_0^{nt}\,,\ \forall n < m_1\,, \tag{2.7}$$

and

$$\mathcal{G}^{\delta}_n(W,t) \le \frac{4}{C_1\delta}\,Q_n(t)\,, \forall n \ge 1\,. \tag{2.8}$$

Proof. Let $n_0(t_0,\theta_0)$ and $\bar\delta_0(n_0,\delta_0)$ be given by Lemma 2.1. By choice of n_0, if $\varepsilon = \tau_{\min} + \log\theta_0 > 0$, then $(Kn_0+1)^{1/n_0} \le e^{\varepsilon t_0}$. Remark that $(Kn+1)^{1/n}$ decreases to 1 for $n \ge 2$ since $K \ge 1$. Thus $(Kn+1)^{1/n} \le e^{\varepsilon t_0}$ for all $n \ge n_0$. With this observation, for δ and m_1 as in the statement of the lemma, the bound (2.6) can be proved by induction on n (just like [BD2, Lemma 3.3] for $\zeta = 0$), writing $n = qm_1 + \ell$, with $q \ge 1$ and $0 \le \ell < m_1$, using $q-1$ times the bound (2.5) with m_1 iterates in place of n_0, and using it one last time with $m_1 + \ell$ iterates, since elements of $\mathcal{I}^{\delta}_n(W)$ have been short at each intermediate step.

For $n < m_1$, the bound (2.7) follows from the relation between δ and m_1.

Finally, to show (2.8), first note that, since each $W_i \in \mathcal{G}^{\delta}_n(W)$ is contained in a single element of $\mathcal{M}^n_0$, and since $|T^{-n}V| \ge C_1\Lambda^n|V|$ for any stable curve $|V|$ (due to (2.1)), there can be at most $2/(C_1\delta)+2$ elements of $\mathcal{G}^{\delta}_n(W)$ in one element of $\mathcal{M}^n_0$. Note also that $|e^{-t\Sigma_n\tau}|_{C^0(W_i)} \le |e^{-t\Sigma_n\tau}|_{C^0(A)}$ whenever $W_i \subset A \in \mathcal{M}^n_0$. This gives the required bound since $C_1\delta < 1$. $\square$

[11] See [Ca, Lemma 3.1(b)] for the replacement for [BD2, Lemmas 3.3–3.4, $\zeta \ne 0$], using a logarithmic weight with $\gamma > 0$ as in [BD1].

3. Bootstrapping

3.1. **Preparations: Small Singular Pressure. Two Bounds from** [Ca]**.** We say that Small Singular Pressure #1 (SSP.1) holds at $t \ge 0$ for $\varepsilon \in (0, 1/4]$ if

$$\text{there exist } \delta_t = \delta(\varepsilon) = \frac{\delta_0}{2^{N_t}} \in (0, \delta_1] \text{ and a finite } n_t = n_t(\varepsilon) \ge n_1 \tag{3.1}$$

$$\text{such that} \quad \frac{S_n^{\delta_t}(W,t)}{\mathcal{G}_n^{\delta_t}(W,t)} \le \varepsilon\,,\ \forall n \ge n_t\,,\ \forall W \in \widehat{\mathcal{W}}^s \text{ with } |W| \ge \delta_t/3\,,$$

and, in addition,

$$\sum_{n \ge n_t} \sup_{\substack{W \in \widehat{\mathcal{W}}^s \\ |W| \ge \delta_t/3}} \frac{e^{-nt\tau_{\min}}}{L_n^{\delta_t}(W,t)} < \infty \tag{3.2}$$

together with its "time-reversal," obtained by replacing T with its inverse T^{-1}, $\widehat{\mathcal{W}}^s$ by $\widehat{\mathcal{W}}^u$, and replacing τ with $\tau \circ T^{-1}$ (that is, replacing $\Sigma_n \tau$ with $\sum_{i=1}^n \tau \circ T^{-i} = (\Sigma_n \tau) \circ T^{-n}$), both hold.

Assume that (3.1) and (3.2) hold at $t \ge 0$ for $\varepsilon \le 1/4$, δ_t, and n_t. Then we say that Small Singular Pressure #2 (SSP.2) holds at t for ε if[12]

$$\text{for any } W \in \widehat{\mathcal{W}}^s \text{ there exists } n_t^*(|W|, \delta_t, \varepsilon) \in [n_t, \infty) \text{ such that} \tag{3.3}$$

$$\frac{S_n^{\delta_t}(W,t)}{\mathcal{G}_n^{\delta_t}(W,t)} \le 2\varepsilon\,,\ \forall n \ge n_t^*(|W|, \delta_t, \varepsilon)\,,$$

together with its time-reversal (in the sense defined above) both hold.

Note that the time-reversal of conditions (3.1), (3.2), and (3.3) involve stable curves for T^{-1}, that is, unstable curves for T. In view of the time reversibility of the billiard dynamics (see [CM, Sect. 2.14] for the precise involution ι), since $\tau \circ T^{-1} = \tau \circ \iota$, and $\tau \circ \iota$ is precisely the free flight time under T^{-1},the conditions for T and τ are equivalent[13] with those for $T^{-1} = \iota T \iota$ and $\tau \circ T^{-1} = \tau \circ \iota$.

To establish Theorem 1.2, Carrand proved[14] the following consequence of SSP:

Proposition 3.1 ([Ca, Theorem 1.2])**.** *Assume*[15] (1.4) *and that SSP.1 and SSP.2 hold*[16] *at $t > 0$ for $\varepsilon = 1/4$. Then there is a unique equilibrium measure μ_t for $-t\tau$, this measure is T-adapted, charges nonempty open sets, and is Bernoulli. In addition, $P_*(t) = P(t)$.*

Therefore, to show Theorem 1.4 it suffices to prove the following lemma:

Lemma 3.2. *There exists $t_2 \ge h_{top}(\Phi^1)$ such that* (3.1)*,* (3.2)*, and* (3.3) *hold at all $t \in [0, t_2]$ for $\varepsilon = 1/4$.*

---

[12] In the analogous condition of [BD1, Cor 5.3], there exists a uniform C_t such that $n_t^*(|W|, \delta_t, \varepsilon) = C_t n_t \frac{|\log(|W|/\delta_t)|}{|\log \varepsilon|}$.

[13] This equivalence does not always hold in [Ca] where $t\tau$ is replaced by a more general g.

[14] In particular, Carrand shows that (3.1) and (3.2) imply the analogues [Ca, Prop. 3.5 and 3.8] of [BD2, Prop. 3.14 and 3.15] for the Banach norm of [BD1]. He does not get a spectral gap.

[15] See also Lemma 1.3.

[16] SSP.1 suffices to construct the invariant measure μ_t and check it is T-adapted. SSP.2 is used to show ergodicity, which gives that μ_t is an equilibrium state for $-t\tau$, as well as the other claims.

Setting

$$t_C = \frac{\log \Lambda}{\tau_{\max} - \tau_{\min}} > 0\,,$$

[Ca, Lemmas 3.2 and 3.3 and Corollary 3.4] gives that, for any fixed $\varepsilon \in (0, 1/4]$, each $t \in [0, t_C]$ satisfies SSP (that is, (3.1), (3.2), and (3.3)) for $\delta_t(\varepsilon) > 0$, $n_t(\varepsilon) < \infty$, and $C_t < \infty$.

The starting point of our bootstrap argument is the following definition

$$t_\infty := \sup\{t' \geq 0 \text{ such that } (3.1), (3.2), \text{ and } (3.3) \text{ hold for all } 0 \leq t \leq t'\}\,. \tag{3.4}$$

We already know that $t_\infty \geq t_C > 0$. If $P(t_\infty) < 0$, then $t_\infty > h_{\text{top}}(\Phi^1)$, and we have shown Lemma 3.2. Otherwise, Lemma 3.7 below will establish that any $0 \leq t < s_*$ satisfies (3.1), (3.2), and (3.3) where $s_* > t_\infty$ is constructed in the next lemma (inspired by [BD2, Definition 3.9]).

Lemma 3.3 (Pressure gap: Constructing the "pivot" t_*)**.** *For all $t > 0$, the following limit exists and belongs to $[-\tau_{\max}, -\tau_{\min}]$:*

$$P'_-(t) := \lim_{s \uparrow t} \frac{P_*(t) - P_*(s)}{t - s}\,.$$

In addition, for any $\theta_0 \in (e^{-\tau_{\min}}, e^{-\tau_{\min}/2})$, defining

$$s_*(t) := \frac{t|P'_-(t)|}{|P'_-(t)| + (\log \theta_0)/2}\,,\ t \in (0, t_\infty)\,,$$

there exists $t_* \in (0, t_\infty)$ such that $s_* := s_*(t_*) > t_\infty$.

Remark 3.4. The parameter $s_*(t_*) > t_*$ is defined so that

$$\theta_0^{s_*/2} e^{|P'_-(t_*)|(s_* - t_*)} = 1\,.$$

The reason for this will become clear in the proof of Lemma 3.7.

Proof. Existence of the limit follows from the convexity of $P_*(t)$ which implies that left (and right) derivatives exist at every $t > 0$. Next, if $0 < s < t$, we have

$$\sum_{A \in \mathcal{M}_0^n} |e^{-t\Sigma_n \tau}|_{C^0(A)} \leq |e^{n(s-t)\tau_{\min}}| \sum_{A \in \mathcal{M}_0^n} |e^{-s\Sigma_n \tau}|_{C^0(A)}\,,\ \forall n \geq 1\,, \tag{3.5}$$

which implies $P'_-(t) \leq -\tau_{\min}$. A similar computation gives $P'_-(t) \geq -\tau_{\max}$.

Next, to construct t_*, we first check that

$$s_*(t) > t \cdot \big(1 + \frac{\tau_{\min}}{4\tau_{\max}}\big)\,,\ \forall t \in (0, t_\infty)\,. \tag{3.6}$$

Indeed, since

$$\frac{1}{1 - \frac{|\log \theta_0|}{2|P'_-(t)|}} \geq 1 + \frac{|\log \theta_0|}{2|P'_-(t)|}\,,$$

the bound (3.6) follows from the fact that $|P'_-(t)| \leq \tau_{\max}$ implies

$$\frac{|\log \theta_0|}{2|P'_-(t)|} \in \Big[\frac{\tau_{\min}}{4\tau_{\max}}, 1\big)\,.$$

Then, taking $t_* = t_\infty - \upsilon$ for $\upsilon \in (0, t_\infty)$, it suffices to pick $\upsilon > 0$ such that

$$\big(1 + \frac{\tau_{\min}}{4\tau_{\max}}\big)(t_\infty - \upsilon) > t_\infty\,.$$

Since $t_\infty \geq t_C = \log \Lambda/(\tau_{\max} - \tau_{\min})$, the above bound holds as soon as

$$\upsilon < \log \Lambda \cdot (\tau_{\max} - \tau_{\min})^{-1} \cdot \big(1 + 4\frac{\tau_{\max}}{\tau_{\min}}\big)^{-1} .$$

□

We record for further use two key bounds due to Carrand. Assume that (3.1) (3.2) hold for t, then by [Ca, Prop 3.5] there exists $c_{0,t} > 0$ such that

$$\mathcal{G}_n^{\delta_t}(W, t) \geq c_{0,t} e^{nP_*(t)} , \ \forall n \geq 1 , \forall W \in \widehat{\mathcal{W}}^s \text{ with } |W| \geq \delta_t/3 , \tag{3.7}$$

and by [Ca, Prop 3.8] there exists $c_{1,t} > 0$ such that

$$Q_n(t) \leq \frac{2}{c_{1,t}} e^{nP_*(t)} , \ \forall n \geq 1 , \tag{3.8}$$

Observe that (3.8) together with (2.8) give the upper bound

$$\mathcal{G}_n^{\delta}(W, t) \leq \frac{4}{C_1 \delta} Q_n(t) \leq \frac{8}{C_1 \delta c_{1,t}} e^{nP_*(t)} , \forall n \geq 1 , \forall \delta \leq \delta_0 . \tag{3.9}$$

Finally, (3.1) and (3.7) imply the following lower bound for any scale $\delta = \delta_0/2^N$.

Lemma 3.5. *For all $t \in (0, t_\infty)$ and $\delta = \delta_0/2^N$, there exists $c_{0,t}(\delta) > 0$ such that*

$$\mathcal{G}_n^{\delta}(W, t) \geq c_{0,t}(\delta) e^{nP_*(t)} , \ \forall n \geq 1 , \forall W \in \widehat{\mathcal{W}}^s \text{ with } |W| \geq \delta/3 . \tag{3.10}$$

The time reversal of the statement holds for T^{-1}.

Proof. First, assume $\delta < \delta_t$. Each element of $L_n^{\delta_t}(W)$ contains at least $\delta_t/(3\delta)$ elements of $\mathcal{G}_n^{\delta}(W)$. So if $|W| \geq \delta_t/3$, then (3.1) and bounded distortion for τ give

$$\mathcal{G}_n^{\delta}(W, t) \geq \frac{e^{-tC}\delta_t}{3\delta} L_n^{\delta_t}(W, t) \geq \frac{e^{-tC}\delta_t}{4\delta} \mathcal{G}_n^{\delta_t}(W, t) \geq \frac{e^{-tC}\delta_t c_{0,t}}{4\delta} e^{nP_*(t)} , \tag{3.11}$$

for all $n \geq n_t$, where we have used (3.7) in the last step.

Next, if $|W| \in [\delta/3, \delta_t/3)$, then there exists $n_W \leq C' \log(\delta_t/\delta)$ such that $T^{-n_W}(W)$ has a connected component V of length at least $\delta_t/3$. This is because while $T^{-n}W$ remains short, the number of components of $T^{-n}W$ is at most $Kn + 1$ by (2.2) while $|T^{-n}W| \geq C_1 \Lambda^n |W|$ according to (2.1). Thus setting $\bar{n} = \max\{n_W, n_t\}$, we apply (3.11) to V to estimate for $n \geq \bar{n}$.

$$\mathcal{G}_n^{\delta}(W, t) \geq \mathcal{G}_{n-\bar{n}}^{\delta}(V, t) e^{-\bar{n}\tau_{\max}} \geq e^{-\bar{n}(\tau_{\max} + P_*(t))} e^{-tC} \frac{\delta_t}{4\delta} c_{0,t} e^{nP_*(t)} ,$$

which proves (3.10) by definition of $\bar{n}$. If $n < \bar{n}$, then trivially

$$\mathcal{G}_n^{\delta}(W, t) \geq e^{-n\tau_{\max}} \geq e^{-n|\tau_{\max} + P_*(t)|} e^{nP_*(t)} \geq e^{-\bar{n}|\tau_{\max} + P_*(t)|} e^{nP_*(t)} .$$

Finally, if $\delta \geq \delta_t$, then since each element of $\mathcal{G}_n^{\delta}(W)$ contains at most $3\delta/\delta_t$ elements of $L_n^{\delta_t}(W)$ and $S_n^{\delta_t}(W) \subset S_n^{\delta}(W)$, we have

$$\mathcal{G}_n^{\delta_t}(W, t) = S_n^{\delta_t}(W, t) + L_n^{\delta_t}(W, t) \leq S_n^{\delta}(W, t) + \frac{3\delta}{\delta_t} \mathcal{G}_n^{\delta}(W, t) \leq \Big(1 + \frac{3\delta}{\delta_t}\Big) \mathcal{G}_n^{\delta}(W, t) ,$$

which gives the required lower bound on $\mathcal{G}_n^{\delta}(W, t)$, applying (3.7).

The time reversed statement of the lemma follows immediately using the reversibility of the billiard, as explained earlier. □

3.2. **Key Lemmas.** In view of Lemma 3.7 below, we adapt [BD2, Lemma 3.10]:

Lemma 3.6 (Leapfrogging via the Hölder Inequality)**.** *For all*[17] $t \geq t_*$ *and* $\kappa > 0$ *there exists* $\omega_\kappa = \omega_\kappa(t_*, t) > 0$ *such that for all* $W \in \widehat{\mathcal{W}}^s$ *with* $|W| \geq \delta_{t_*}/3$,

$$\mathcal{G}_n^\delta(W,t) \geq \frac{\omega_\kappa(t_*,t)}{\delta} \cdot e^{n(P_*(t_*)-(|P'_-(t_*)|+\kappa)(t-t_*))}\,, \qquad \forall \delta = \frac{\delta_0}{2^N} \leq \delta_{t_*}\,,\ \forall n \geq n_{t_*}\,. \tag{3.12}$$

In addition, for each $\delta = \frac{\delta_0}{2^N} < \delta_0$ *there exists* $\omega^*_\kappa = \omega^*_\kappa(t_*, t, \delta) > 0$ *such that for all* $W \in \widehat{\mathcal{W}}^s$ *with* $|W| \geq \delta/3$,

$$\mathcal{G}_n^\delta(W,t) \geq \omega^*_\kappa(t_*,t,\delta) \cdot e^{n(P_*(t_*)-(|P'_-(t_*)|+\kappa)(t-t_*))}\,,\ \forall n \geq 1\,. \tag{3.13}$$

Finally, the time reversals of (3.12) *and* (3.13) *also hold for the billiard map* T^{-1}.

The proof gives constants $\omega_\kappa(t_*, t)$ and $\omega^*_\kappa(t_*, t, \delta)$ which tend to zero as $t \to \infty$ (because the constant η in the proof tends to zero as $t \to \infty$).

Proof. We start with (3.12) (for $t \geq t_*$). Recall from the proof of (3.11) that for $u \in (0, t_\infty)$ and $\delta < \delta_u$, if $|W| \geq \delta_u/3$ and $n \geq n_u$, then

$$\mathcal{G}_n^\delta(W,u) \geq e^{-uC}\frac{\delta_u}{4\delta}c_{0,u}e^{nP_*(u)}\,,\ \forall \delta < \delta_u\,, \tag{3.14}$$

since each $V_i \in L_n^{\delta_u}(W)$ contains at least $\delta_u/3\delta$ elements of $\mathcal{G}_n^\delta(W)$.

Now, for $s \in (0, t_*)$, taking $\eta(s, t, t_*) \in (0, 1]$ such that $\eta t + (1 - \eta)s = t_*$, the Hölder inequality gives $\sum_i a_i^{t_*} \leq \left(\sum_i a_i^t\right)^\eta \left(\sum_i a_i^s\right)^{1-\eta}$ for any positive numbers a_i. It follows that for all $\delta \leq \delta_{t_*}$, each $W \in \widehat{\mathcal{W}}^s$ with $|W| \geq \delta_{t_*}/3$ and any $n \geq n_{t_*}$,

$$\begin{aligned}
\mathcal{G}_n^\delta(W,t) &\geq \frac{(\mathcal{G}_n^\delta(W,t_*))^{1/\eta}}{(\mathcal{G}_n^\delta(W,s))^{(1-\eta)/\eta}} \\
&\geq \left(e^{-t_*C}\frac{\delta_{t_*}}{4\delta}c_{0,t_*}e^{nP_*(t_*)}\right)^{1/\eta}\left(\frac{8}{C_1\delta c_{1,s}}e^{nP_*(s)}\right)^{1-1/\eta} \\
&= \frac{1}{\delta}\left(e^{-t_*C}\frac{\delta_{t_*}}{4}c_{0,t_*}\right)^{1/\eta}\left(\frac{8}{C_1 c_{1,s}}\right)^{1-1/\eta} e^{n(P_*(t_*)-P_*(s))\frac{1-\eta}{\eta}}e^{nP_*(t_*)}\,,
\end{aligned} \tag{3.15}$$

where we used (3.14) with $u = t_*$ for the lower bound in the numerator, and (3.9) for s for the upper bound in the denominator, recalling that $\{s, t_*\} \subset (0, t_\infty)$ and $\delta_{t_*} \leq \delta_1 < \delta_0$.

Since $\eta(s, t, t_*) = (t_* - s)/(t - s)$, we have

$$(P_*(t_*) - P_*(s))\frac{1-\eta}{\eta} = \frac{t-t_*}{t_*-s}(P_*(t_*) - P_*(s))\,.$$

Fix $\kappa > 0$ and choose $s = s(\kappa, t_*) \in (0, 1)$ close enough to t_* (i.e. small enough $\eta_\kappa = \eta(s(\kappa, t_*), t, t_*) > 0$) such that (since $0 < s < t_*$ and $P'_-(u) < 0$ for all $u > 0$)

$$(P_*(s) - P_*(t_*))/(t_* - s) \leq |P'_-(t_*)| + \kappa\,. \tag{3.16}$$

The bound (3.12) follows, setting, for $s = s(\kappa, t_*)$ (recall that η_κ depends on t),

$$\omega_\kappa(t_*,t) = \left(e^{-t_*C}\frac{\delta_{t_*}}{4}c_{0,t_*}\right)^{1/\eta_\kappa}\left(\frac{8}{C_1 c_{1,s}}\right)^{1-1/\eta_\kappa}\,.$$

[17]The same proof works replacing t_* by an arbitrary number in $(0, t_\infty)$, as long as $t \geq t_*$.

For (3.13), we use that (3.9) for s and Lemma 3.5 for t_* imply that for any $\delta \in (0, \delta_{t_*})$, for each $W \in \widehat{\mathcal{W}}^s$ with $|W| \geq \delta/3$, and all $n \geq 1$,

$$\mathcal{G}_n^\delta(W,t) \geq \frac{(\mathcal{G}_n^\delta(W,t_*))^{1/\eta}}{(\mathcal{G}_n^\delta(W,s))^{(1-\eta)/\eta}} \geq \big(c_{0,t_*}(\delta) \cdot e^{nP_*(t_*)}\big)^{1/\eta} \big(\frac{8}{C_1 \delta c_{1,s}} e^{nP_*(s)}\big)^{(\eta-1)/\eta}, \tag{3.17}$$

where we used (3.10) for t_*. We conclude by taking $s = s(\kappa, t_*) \in (0,1)$ close enough to t_* such that (3.16) holds, setting (again, η_κ depends on t)

$$\omega_\kappa^*(t_*, t, \delta) = c_{0,t_*}(\delta)^{1/\eta_\kappa} (8)^{1-1/\eta_\kappa} (C_1 \delta c_{1,s})^{1/\eta_\kappa - 1} .$$

□

Our second key lemma is inspired by [BD2, Lemma 3.11] (the proof below requires a more involved decomposition of orbits):

Lemma 3.7. *Let $t_* < t_\infty$ and $s_*(t_*) > t_\infty$ be as in Lemma 3.3. If $P(t_*) \geq 0$ then the SSP conditions* (3.1)*,* (3.2)*, and* (3.3) *hold at all $t \in [t_*, s_*)$ for $\varepsilon = 1/4$.*

Proof of Lemma 3.7. We first consider condition (3.1) of SSP.1.

By definition of s_* (recall that $\inf |P'_-(s)| > -\log \theta_0/2$)

$$\theta_0^{t'/2} e^{|P'_-(t_*)|(t'-t_*)} < 1\,, \quad \forall t_* \leq t' < s_*\,. \tag{3.18}$$

Thus for all $t' \in [t_*, s_*)$ there exists $\kappa_1 = \kappa(t_*, t') > 0$ such that

$$\bar\varepsilon := \sup_{t_* \leq t \leq t'} \big(\theta_0^{t/2} e^{(|P'_-(t_*)|+\kappa_1)(t-t_*)}\big) < 1\,. \tag{3.19}$$

For $m_1 \geq \max\{n_0(t_*, \theta_0), n_{t_*}\}$ to be chosen later depending on $\varepsilon = 1/4$, $\bar\varepsilon$, δ_{t_*}, and κ_1, pick $\delta_3(m_1) \in (0, \delta_{t_*}]$ (similarly to the choice of $\bar\delta_0$ in the proof of Lemma 2.1) so small that any stable curve of length at most δ_3 can be cut into at most $Kj+1$ connected components by $\mathcal{S}_{-j}$ for $0 \leq j \leq 2m_1$.

For $n \geq m_1$, write $n = \ell m_1 + r$, for some $0 \leq r < m_1$ and $\ell \geq 1$. Let $W \in \widehat{\mathcal{W}}^s$ with $|W| \geq \delta_3/3$. We group the curves $W_i \in S_n^{\delta_3}(W)$ with $|W_i| < \delta_3/3$, as in the proof of [BD2, Lemma 3.11], according to the largest $k \in \{0, \ldots, \ell-1\}$ such that $T^{(\ell-k)m_1+r} W_i \subset V_j \in L_{km_1}^{\delta_3}(W)$ (such a k must exist since $|W| \geq \delta_3/3$ while $|W_i| < \delta_3/3$). Denote[18] by $\bar{\mathcal{I}}_{(\ell-k)m_1+r}^{\delta_3}(V_j)$ the set of $W_i \in \mathcal{G}_n^{\delta_3}(W)$ thus associated with $V_j \in L_{km_1}^{\delta_3}(W)$ (such elements are known to be small only at iterates jm_1+r). For such W_i, $T^{(\ell-k')m_1+r}(W_i)$ is contained in an element of $\mathcal{G}_{m_1 k'}^{\delta_3}(W)$ shorter than $\delta_3/3$ for $k' < k$. So for $k > 0$, we may apply the inductive bound (2.6) since elements of $\bar{\mathcal{I}}_{(\ell-k)m_1+r}^{\delta_3}(V_j)$ can only be created by intersections with $\mathcal{S}_{-m_1}$ at the first $\ell-k-1$ iterates and with $\mathcal{S}_{-m_1-r}$ at the last step. For $k=0$, W itself may be longer than δ_3. Thus we first subdivide W into at most δ_0/δ_3 curves of length at most δ_3 and then apply (2.6) to each piece. This yields, for $t_* \leq t \leq t'$,

$$S_n^{\delta_3}(W,t) \leq \sum_{k=0}^{\ell-1} \sum_{V_j \in L_{km_1}^{\delta_3}(W)} |e^{-t\Sigma_{km_1}\tau}|_{C^0(V_j)} \sum_{W_i \in \bar{\mathcal{I}}_{(\ell-k)m_1+r}^{\delta_3}(V_j)} |e^{-t\Sigma_{(\ell-k)m_1+r}\tau}|_{C^0(W_i)}$$

---

[18]Note that $\bar{\mathcal{I}}_{(\ell-k)m_1+r}^{\delta}(V_j)$ was abusively denoted $\mathcal{I}_{(\ell-k)m_1+r}^{\delta}(V_j)$ in the proof of [BD1, Lemma 5.2], see footnote 23 there.

$$\leq \frac{\delta_0}{\delta_3}\theta_0^{tn} + \sum_{k=1}^{\ell-1} \sum_{V_j \in L_{km_1}^{\delta_3}(W)} |e^{-t\Sigma_{km_1}\tau}|_{C^0(V_j)} \theta_0^{t((\ell-k)m_1+r)} . \tag{3.20}$$

Next, recalling (2.3), for any $k \geq 1$, each $V_j \in L_{km_1}^{\delta_3}(W)$ is contained in an element $U_i \in \mathcal{G}_{km_1}^{\delta_{t_*}}(W)$. Since $|V_j| \geq \delta_3/3$, there are at most $3\delta_{t_*}/\delta_3$ different V_j corresponding to each fixed U_i. Then we group each $U_i \in \mathcal{G}_{km_1}^{\delta_{t_*}}(W)$ according to its most recent long ancestor $W_a \in L_j^{\delta_{t_*}}(W)$ for some $j \in [0, km_1]$. Note that $j = 0$ is possible if $|W| \geq \delta_{t_*}/3$. If $|W| < \delta_{t_*}/3$, and no such time j exists for U_i, then by convention we also associate the index $j = 0$ to such U_i. In either case, $U_i \in \mathcal{I}_{km_1}^{\delta_{t_*}}(W)$, and we may apply (2.6) after possibly subdividing W into at most δ_0/δ_{t_*} curves of length at most δ_{t_*}. Then, for $j \geq 1$, we apply (2.7) from Lemma 2.2 to each $\mathcal{I}_{km_1-j}^{\delta_{t_*}}(\cdot)$ (since $\delta_3 \leq \delta_{t_*}$, the constant $m_1(\delta_{t_*}) \leq m_1(\delta_3)$, so the bound holds with our chosen m_1, although it may not be optimal),

$$\begin{aligned} L_{km_1}^{\delta_3}(W,t) \leq \frac{3\delta_{t_*}}{\delta_3} \Bigg(& \sum_{U_i \in \mathcal{I}_{km_1}^{\delta_{t_*}}(W)} |e^{-t\Sigma_{km_1}\tau}|_{C^0(U_i)} \\ & + \sum_{j=1}^{km_1} \sum_{W_a \in L_j^{\delta_{t_*}}(W)} |e^{-t\Sigma_j\tau}|_{C^0(W_a)} \sum_{U_i \in \mathcal{I}_{km_1-j}^{\delta_{t_*}}(W_a)} |e^{-t\Sigma_{km_1-j}\tau}|_{C^0(U_i)} \Bigg) \\ \leq \frac{3\delta_{t_*}}{\delta_3} & \left(\frac{\delta_0}{\delta_{t_*}} \theta_0^{tkm_1} + \sum_{j=1}^{km_1} \sum_{W_a \in L_j^{\delta_{t_*}}(W)} |e^{-t\Sigma_j\tau}|_{C^0(W_a)} K m_1 \theta_0^{t(km_1-j)} \right) . \end{aligned}$$

Combining this estimate with (3.20) yields (summing over k for the $j = 0$ terms and adding the term corresponding to $k = 0$),

$$S_n^{\delta_3}(W,t) \leq \frac{3\delta_0}{\delta_3} \frac{n}{m_1} \theta_0^{tn} + \frac{3\delta_{t_*}}{\delta_3} \sum_{k=1}^{\ell-1} \sum_{j=1}^{km_1} K m_1 \theta_0^{t(n-j)} L_j^{\delta_{t_*}}(W,t) . \tag{3.21}$$

For fixed $k \in \{1, \ldots, \ell-1\}$, and for each $1 \leq j \leq km_1$ such that $L_j^{\delta_{t_*}}(W) \neq \emptyset$, the lower bound (3.12) in Lemma 3.6 and the distortion constant $e^{-tC} \geq e^{-t'C}$ imply (note that $n - j \geq \ell m_1 + r - km_1 \geq r + m_1 \geq n_{t_*}$),

$$\begin{aligned} \mathcal{G}_n^{\delta_3}(W,t) &\geq \sum_{W_a \in L_j^{\delta_{t_*}}(W)} e^{-tC} |e^{-t\Sigma_j\tau}|_{C^0(W_a)} \sum_{W_i \in \mathcal{G}_{n-j}^{\delta_3}(W_a)} |e^{-t\Sigma_{n-j}\tau}|_{C^0(W_i)} \\ &\geq \frac{\omega_{\kappa_1}(t_*,t)}{\delta_3 e^{t'C}} e^{(n-j)(P_*(t_*) - (|P_-'(t_*)| + \kappa_1)(t-t_*))} \sum_{W_a \in L_j^{\delta_{t_*}}(W)} |e^{-t\Sigma_j\tau}|_{C^0(W_a)} . \end{aligned} \tag{3.22}$$

Combining (3.21) with either (3.22) (for $j \geq 1$) or (3.13) from Lemma 3.6 (for $j = 0$) and setting $\Delta = 3e^{t'C}\delta_{t_*}Km_1$, yields (using that $P(t_*) \geq 0$),

$$\begin{aligned} \frac{S_n^{\delta_3}(W,t)}{\mathcal{G}_n^{\delta_3}(W,t)} \leq n & \frac{\frac{3\delta_0}{\delta_3 m_1}\theta_0^{tn}}{\omega_{\kappa_1}^*(t_*,t,\delta_3) e^{n(P_*(t_*) - (|P_-'(t_*)| + \kappa_1)(t-t_*))}} \\ & + \sum_{k=1}^{\ell-1} \sum_{j=1}^{km_1} \frac{\frac{3\delta_{t_*}}{\delta_3} K m_1 \theta_0^{t(n-j)} L_j^{\delta_{t_*}}(W,t)}{\frac{\omega_{\kappa_1}(t_*,t)}{\delta_3 e^{t'C}} e^{(n-j)(P_*(t_*) - (|P_-'(t_*)| + \kappa_1)(t-t_*))} L_j^{\delta_{t_*}}(W,t)} \end{aligned}$$

$$\leq \frac{3\delta_0}{\delta_3 \cdot \omega^*_{\kappa_1}(t_*, t, \delta_3) \cdot m_1} n (e^{-P_*(t_*)}\bar\varepsilon)^n + \frac{\Delta}{\omega_{\kappa_1}(t_*, t)} \sum_{k=1}^{\ell-1} \sum_{j=1}^{km_1} (e^{-(P_*(t_*)}\bar\varepsilon)^{n-j}$$

$$\leq \frac{3\delta_0}{\delta_3 \cdot \omega^*_{\kappa_1}(t_*, t, \delta_3) \cdot m_1} n \bar\varepsilon^n + \frac{\Delta}{\omega_{\kappa_1}(t_*, t)} \frac{1}{1-\bar\varepsilon} \sum_{k=1}^{\ell-1} \bar\varepsilon^{n-km_1}$$

$$\leq \frac{3\delta_0}{\delta_3 \cdot \omega^*_{\kappa_1}(t_*, t, \delta_3) \cdot m_1} n \bar\varepsilon^n + \frac{3e^{t'C}\delta_{t_*} K m_1}{\omega_{\kappa_1}(t_*, t) \cdot} \frac{\bar\varepsilon^{m_1}}{(1-\bar\varepsilon)(1-\bar\varepsilon^{m_1})} \,. \tag{3.23}$$

To establish (3.1), choose first $m_1 \geq n_{t_*}$ such that the second term is less than $\frac{\varepsilon}{2}$, setting $\delta_t := \delta_3(m_1)$, and then $n_t \geq m_1$ such that the first term is less than $\frac{\varepsilon}{2}$ for $n \geq n_t$.

We next show (3.2). For $n \geq n_t$, we deduce from (3.1) and (3.13) (for small $\kappa > 0$) that, for all $W \in \widehat{\mathcal{W}}^s$ with $|W| \geq \delta_t/3$,

$$L_n^{\delta_t}(W, t) \geq \frac{3}{4} \mathcal{G}_n^{\delta_t}(W, t) \geq \frac{3}{4} \omega^*_\kappa(t_*, t, \delta_t) e^{nP_*(t_*)} e^{-n(t-t_*)(|P'_-(t_*)|+\kappa)} \,.$$

Since $e^{-|P'_-(t_*)|(t-t_*)} > \theta_0^{t/2} \geq e^{-t\tau_{\min}/2}$ by (3.18), while $P_*(t_*) \geq 0$, it suffices to take κ such that $(t-t_*)\kappa + \frac{t}{2}\tau_{\min} < t\tau_{\min}$ to complete the proof of (3.2).

It remains to consider SSP.2. We may assume $|W| < \delta_{t_*}/3$ since otherwise (3.1) from SSP.1 implies (3.3) with $n_t^* = n_t$. As observed in the proof of [BD1, Cor. 5.3], there exists $\bar C_2$ (depending only on the billiard table) such that the first iterate ℓ_0 at which $\mathcal{G}_{\ell_0}^{\delta_{t_*}}(W)$ contains at least one element of length more than $\delta_{t_*}/3$ satisfies

$$\ell_0 \leq n_2 = n_2(\delta_{t_*}) := \bar C_2 |\log(|W|/\delta_{t_*})| \,.$$

Since $|W| < \delta_{t_*}/3$, it suffices to consider the term corresponding to $j = 0$ (and $k = 0$) in (3.23) (the other one is bounded by $\varepsilon/2$ for $n \geq m_1$ for m_1 chosen as above). For this purpose, for any $n = \ell m_1 + r \geq m_1$, the first term of (3.21) is replaced by

$$\frac{\delta_{t_*}}{3\delta_3} \theta_0^{tn} + \sum_{k=1}^{\ell-1} \frac{3\delta_{t_*}}{\delta_3} \theta_0^{tn} \leq \frac{3\delta_{t_*} n}{\delta_3 m_1} \theta_0^{tn} \,, \tag{3.24}$$

where we have applied (2.6) from Lemma 2.2. For any $n \geq \max\{n_2, m_1\}$, the bound (3.13) from Lemma 3.6 is replaced by

$$\mathcal{G}_n^{\delta_3}(W, t) \geq \omega^*_{\kappa_1}(t_*, t, \delta_3) \cdot e^{-tn_2\tau_{\max}} e^{(n-n_2)(P_*(t_*)-(|P'_-(t_*)|+\kappa_1)(t-t_*))} \,. \tag{3.25}$$

Dividing (3.24) by (3.25), the term corresponding to $j = 0$ in (3.23) is bounded by

$$\frac{3\delta_{t_*} \frac{n}{m_1} \theta_0^{tn}}{\delta_3 \cdot \omega^*_{\kappa_1}(t_*, t, \delta_3) \cdot e^{-tn_2\tau_{\max}} e^{(n-n_2)(P_*(t_*)-(|P'_-(t_*)|+\kappa_1)(t-t_*))}}$$
$$\leq \frac{3\delta_{t_*} e^{tn_2\tau_{\max}}}{m_1 \cdot \omega^*_{\kappa_1}(t_*, t, \delta_3) \cdot \delta_3} n \bar\varepsilon^{n-n_2} \,.$$

We conclude, since, if n_t^*/n_2 is large enough (depending on t, $\bar\varepsilon$, $\delta_3 = \delta_t$) then

$$n(\bar\varepsilon^{n/n_2} e^{t\tau_{\max}})^{n_2} < \frac{\varepsilon}{2} \cdot \frac{\bar\varepsilon^{n_2} \cdot m_1 \cdot \delta_3 \cdot \omega^*_{\kappa_1}(t_*, t, \delta_3)}{3\delta_{t_*}} \,,\ \forall n \geq n_t^* \,.$$

□

3.3. **Theorem 1.4: Proof of Lemma 3.2.** In view of the discussion above Lemma 3.2, it only remains to show Lemma 3.2 to establish Theorem 1.4:

Proof of Lemma 3.2. If $P(t_\infty) < 0$ we are done, as explained before Lemma 3.3. Assume for a contradiction that $P(t_\infty) \geq 0$. Let $t_* < t_\infty$ and $s_*(t_*) > t_\infty$ be as in Lemma 3.3, and fix $t_\infty < t_2 < s_*$. Then Lemma 3.7 applied to $\varepsilon = 1/4$ gives that the SSP conditions (3.1), (3.2), and (3.3) hold for all $t \in [0, t_2]$. Since $t_2 > t_\infty$, this is a contradiction. □

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