

THERMODYNAMIC FORMALISM FOR DISPERSING BILLIARDS

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ABSTRACT. For any finite horizon Sinai billiard map T on the two-torus, we find $t_* > 1$ such that for each $t \in (0, t_*)$ there exists a unique equilibrium state μ_t for $-t \log J^u T$, and μ_t is T -adapted. (In particular, the SRB measure is the unique equilibrium state for $-\log J^u T$.) We show that μ_t is exponentially mixing for Hölder observables, and the pressure function $P(t) = \sup_{\mu} \{h_{\mu} - \int t \log J^u T d\mu\}$ is analytic on $(0, t_*)$. In addition, $P(t)$ is strictly convex if and only if $\log J^u T$ is not μ_t -a.e. cohomologous to a constant, while, if there exist $t_a \neq t_b$ with $\mu_{t_a} = \mu_{t_b}$, then $P(t)$ is affine on $(0, t_*)$. An additional sparse recurrence condition gives $\lim_{t \downarrow 0} P(t) = P(0)$.

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1. INTRODUCTION AND STATEMENT OF MAIN RESULTS

1.1. Set-up. A Sinai or dispersive billiard table Q on the two-torus \mathbb{T}^2 is a set $Q = \mathbb{T}^2 \setminus \cup_{i=1}^{\Omega} \mathcal{O}_i$, for some finite number $\Omega \geq 1$ of pairwise disjoint closed domains \mathcal{O}_i (the obstacles, or scatterers) with C^3 boundaries having strictly positive curvature \mathcal{K} . (In particular, the domains are strictly convex.) The billiard flow, also called a periodic Lorentz gas, is the motion of a point particle traveling in Q at unit speed and undergoing specular reflections at the boundary of the scatterers. (At a tangential — also called grazing — collision, the reflection does not change the direction of the particle.)

We study here the associated billiard map $T : M \rightarrow M$ on the compact set $M = \partial Q \times [-\frac{\pi}{2}, \frac{\pi}{2}]$, defined to be the first collision map on the boundary of Q . We use the standard coordinates $x = (r, \varphi)$, where r is arclength along $\partial \mathcal{O}_i$ and φ is the angle the post-collision trajectory makes with the normal to $\partial \mathcal{O}_i$. Grazing collisions cause discontinuities in the map T . We remark, however, that since the flow is continuous, the map T is well-defined and bijective on M . There is no need to reduce the domain to a smaller set.

For $x \in M$, let $\tau(x)$ denote the distance from x to $T(x)$ (the free flight time). Set $\mathcal{K}_{\max} = \sup \mathcal{K} < \infty$, $\mathcal{K}_{\min} = \inf \mathcal{K} > 0$, and $\tau_{\min} = \inf \tau > 0$. Then [CM] the cones in \mathbb{R}^2 defined by

$$C^u = \{(dr, d\varphi) : \mathcal{K}_{\min} \leq \frac{d\varphi}{dr} \leq \mathcal{K}_{\max} + \frac{1}{\tau_{\min}}\}, \quad C^s = \{(dr, d\varphi) : -\mathcal{K}_{\min} \geq \frac{d\varphi}{dr} \geq -\mathcal{K}_{\max} - \frac{1}{\tau_{\min}}\}$$

are strictly invariant under DT and DT^{-1} , respectively, whenever these derivatives exist.

The map T is uniformly hyperbolic, in the following sense: Let

$$(1.1) \quad \Lambda := 1 + 2\tau_{\min}\mathcal{K}_{\min} > 1.$$

Then there exists $C_1 > 0$ such that, for all x for which $DT^n(x)$, respectively $DT^{-n}(x)$, is defined,

$$(1.2) \quad \|DT^n(x)v\| \geq C_1\Lambda^n\|v\|, \quad \forall v \in C^u, \quad \|DT^{-n}(x)v\| \geq C_1\Lambda^n\|v\|, \quad \forall v \in C^s, \quad \forall n \geq 0.$$

Let $\mathcal{S}_0 = \{(r, \varphi) \in M : \varphi = \pm \frac{\pi}{2}\}$ denote the set of tangential collisions on M . Then

$$(1.3) \quad \mathcal{S}_n = \cup_{i=0}^{-n} T^i \mathcal{S}_0, \quad n \in \mathbb{Z},$$

is the singularity set for T^n . In other words, there exists $n \in \mathbb{Z}$ such that $DT^n(x)$ is not defined if and only if x belongs to the (invariant and dense, [CM, Lemma 4.55]) set of curves $\cup_{m \in \mathbb{Z}} \mathcal{S}_m$. Let

$$(1.4) \quad M' = M \setminus \cup_{m \in \mathbb{Z}} \mathcal{S}_m.$$

The spaces $E^u(x)$ and $E^s(x)$ are defined at any $x \in M'$. Indeed, for each $n \geq 0$, let $x_n = T^n x$, and consider $v_n = DT^{-n}(x_n)v / \|DT^{-n}(x_n)v\|$ for some $v \in C^s$. Since $x \in M'$, we have that $DT^{-n}(x_n)$ is well-defined for each $n \geq 0$. By uniform hyperbolicity, the sequence v_n converges to a vector v_∞ . The direction of v_∞ is $E^s(x)$. Similarly, for $y \in M \setminus \cup_{m \leq 0} \mathcal{S}_m$, consider $y_n = T^{-n}y$ and $u_n = DT^n(y_n)u / \|DT^n(y_n)u\|$, for $n \geq 0$ and $u \in C^u$. The limit of u_n is $E^u(y)$.

We have [CM, Theorem 4.66, Theorem 4.75] that Lebesgue($M' \setminus M$) = $\mu_{\text{SRB}}(M' \setminus M) = 0$, where $\mu_{\text{SRB}} = (2|\partial Q|)^{-1} \cos \varphi \, dr d\varphi$ is the unique absolutely continuous invariant measure. Also, at each $x \in M'$, the unstable and stable Jacobians $J^u T(x)$ and $J^s T(x)$, with respect to arclength along unstable, respectively stable, manifolds, are well-defined and nonzero. Note also that, if $J_{\text{Leb}} T$ denotes the Jacobian of T with respect to Lebesgue, then, setting $E(x) = \sin(\angle(E^s(x), E^u(x)))$,

$$(1.5) \quad J_{\text{Leb}} T(x) = \frac{\cos(\varphi(x))}{\cos(\varphi(T(x)))} = J^u T(x) \cdot J^s T(x) \cdot \frac{E \circ T(x)}{E(x)}, \quad \forall x \in M'.$$

We assume that the billiard table Q has *finite horizon*, i.e., the billiard flow on Q does not have any trajectories making only tangential collisions. This implies (but is not¹ equivalent to) $\tau_{\max} := \sup \tau < \infty$, see [BD, Remark 1.1].

¹We need the stronger condition e.g. in the proof of Proposition 5.2.

1.2. Potentials and Pressure. Theorems 1.1–1.2. Corollaries 1.3–1.6. The Operator \mathcal{L}_t . Since T admits a finite generating partition (see the beginning of Section 2.3), it follows that for any T -invariant probability measure² μ , the Kolmogorov entropy $h_\mu(T)$ is finite ([W, Theorem 4.10, Theorem 4.17]).

Fix $t \geq 0$. Let μ be a T -invariant probability measure μ . If $\mu(M \setminus M') = 0$, define the *pressure of μ* for the (so-called *geometric potential*) $-t \log J^u T$ by

$$P_\mu(-t \log J^u T) = h_\mu(T) - t \int_M \log J^u T d\mu.$$

If $\mu(M \setminus M') \neq 0$, we set $\int_M \log J^u T d\mu = \infty$, so that $P_\mu(-t \log J^u T) = -\infty$ if $t > 0$. Due to the invariance of μ , the bound (1.2) implies that $\int_M \log J^u T d\mu = \lim_{n \rightarrow \infty} \frac{1}{n} \int_M \log J^u T^n d\mu \geq \log \Lambda$, thus the integral is either well-defined and nonnegative or infinite. (See Remark 2.3 for more comments.) It is known that

$$(1.6) \quad \chi^u := \int_M \log J^u T d\mu_{\text{SRB}} = h_{\mu_{\text{SRB}}}(T) \in (\log \Lambda, \infty),$$

so that $P_{\mu_{\text{SRB}}}(-\log J^u T) = 0$ (this is the Pesin entropy formula, see e.g. [CM, Theorem 3.42]).

For a bounded function $g : M \rightarrow \mathbb{R}$, we set $P_\mu(-t \log J^u T + g) = P_\mu(-t \log J^u T) + \int g d\mu$, and we define the *pressure $P(t, g)$ of the potential $-t \log J^u T + g$* by

$$(1.7) \quad P(t, g) := \sup\{P_\mu(-t \log J^u T + g) : \mu \text{ a } T\text{-invariant probability measure}\}, \quad P(t) := P(t, 0).$$

We call μ an *equilibrium state* for the potential $-t \log J^u T + g$ if $P_\mu(-t \log J^u T + g) = P(t, g)$.

The case $t = 0, g = 0$, corresponds to the measure of maximal entropy. Under an additional condition of “sparse recurrence to the singularity set” (see Definition 5.4), a measure μ_0 with $P(0) = P_{\mu_0}(0)$ was recently constructed in [BD] (μ_0 was called μ_* there), shown to be mixing (in fact, Bernoulli), to be the unique measure μ satisfying $P_\mu(0) = P(0)$, and to satisfy the T -adapted condition (1.9) below. (The speed of mixing of μ_0 is not known.)

For $t = 1$, we mentioned above that $P_{\mu_{\text{SRB}}}(-\log J^u T) = 0$. In addition, μ_{SRB} is T -adapted and, for any T -invariant probability measure μ giving small enough weight to neighbourhoods of singularity sets [KS, Part IV, Theorem 1.1], the Ruelle inequality $P_\mu(-\log J^u T) \leq 0$ holds. The measure μ_{SRB} is mixing, in fact, correlations for Hölder observables decay exponentially [Y].

For t in a small interval³ around 1, [CWZ] established the existence of equilibrium states for the potential $-t \log J^u T$ using a Young tower construction with exponential tails, proving that these measures are exponentially mixing on Hölder observables and are unique in the class of measures that lift to the Young tower.

We establish a thermodynamic formalism for Sinai billiards for $t \in (0, t_*)$, with $t_* > 1$ defined by

$$(1.8) \quad t_* := \sup\{t > 0 : \Lambda^{-t} < e^{P(t)}\} = \sup\left\{t > 0 : t > -\frac{P(t)}{\log \Lambda}\right\}.$$

(That $t_* > 1$ follows since $\Lambda > 1$ from (1.1), while $P(t) \geq 0$ for $t \leq 1$.) The definition of t_* can be viewed as a pressure gap condition, controlling by $P(t)$ the contribution from pieces that constantly get cut by the singularities. In particular, for any $t < t_*$, we may⁴ choose $\theta \in (\Lambda^{-1}, 1)$ in the one-step expansion Lemma 3.1 so that $\theta^t < e^{P(t)}$. This complexity bound permits us to prove the required growth lemmas essential to our analysis. Our first main result is the following theorem:

Theorem 1.1 (Thermodynamic Formalism for Sinai Billiards). *For each $t \in (0, t_*)$, the potential $-t \log J^u T$ admits a unique equilibrium state μ_t . The measure μ_t is mixing, gives positive mass to*

²All probability measures in the present work are Borel measures.

³The interval depends on the exponential rate of return (itself close to 1) to the Young tower coupling magnet.

⁴It is in fact enough to require there that $\theta^t < e^{P_*(t)}$.

any nonempty open set, and does not have atoms. Moreover, μ_t is T -adapted, that is⁵

$$(1.9) \quad \int |\log d(x, \mathcal{S}_{\pm 1})| d\mu_t < \infty.$$

In addition, μ_t has exponential decay of correlations for Hölder observables. Finally, if T satisfies the sparse recurrence condition then $\lim_{t \downarrow 0} P(t) = P(0)$.

We prove Theorem 1.1 for $t \in (0, t_*)$ in three steps:

- First, we introduce in Section 2.2 an equivalent (topological) expression $P_*(t)$ for $P(t)$, generalising what was done in [BD] for $t = 0$, and we show that $P_*(t)$ is convex and strictly decreasing (Proposition 2.6), and that $P(t) \leq P_*(t)$ (Proposition 2.4), for all $t > 0$.
- Next, for $t \in (0, t_*)$, we prove the following properties for the transfer operator

$$(1.10) \quad \mathcal{L}_t f = \frac{f \circ T^{-1}}{|J^s T|^{1-t} \circ T^{-1}}$$

acting on an anisotropic Banach⁶ space \mathcal{B} (Theorem 4.1): The operator \mathcal{L}_t has spectral radius $e^{P_*(t)}$, essential spectral radius strictly smaller than $e^{P_*(t)}$, and the maximal eigenvectors of \mathcal{L}_t and its dual give rise to a T -invariant probability measure μ_t . In addition, \mathcal{L}_t has a spectral gap on \mathcal{B} , so that μ_t is exponentially mixing on Hölder observables.

- Finally, in Section 5, still for $t \in (0, t_*)$, we show that $P_{\mu_t}(-t \log J^u T) = P_*(t)$, so that $P(t) = P_*(t)$ (Corollary 5.3), as well as the remaining claims about μ_t : in particular that μ_t is the unique equilibrium state among all T -invariant Borel probability measures realising the variational principle $P(t) = P_*(t)$ (Theorem 2.5), and that sparse recurrence implies that $P(t)$ tends to $P(0)$ as $t \downarrow 0$ (Proposition 5.5). Our proof of uniqueness also gives a more general variational principle, $P(t, g) = P_*(t, g)$, Theorem 5.8.

We use the Banach spaces \mathcal{B} introduced in [DZ2], except that we work with (exact) stable manifolds \mathcal{W}^s (as in [BD]) instead of cone stable curves $\widehat{\mathcal{W}}^s$ (see Section 2.1). More importantly, we must tune the parameters used to define $\mathcal{B} = \mathcal{B}(t_0, t_1)$ in Section 4.1 to an interval $[t_0, t_1] \subset (0, t_*)$ containing t . In particular, the decay rate k^{-q} defining the homogeneity strips (2.1) in [DZ2] was $q = 2$, while we need to assume $qt > 1$ here (due to (3.2)). Also, we need to let the parameter p used in the definition (4.3) of the strong stable norm tend to infinity when $t \rightarrow t_*$ (see Lemma 4.7). It follows that our bound for the essential spectral radius of \mathcal{L}_t on $\mathcal{B}(t_0, t_1)$ deteriorates as $t_0 \rightarrow 0$ or $t_1 \rightarrow t_*$, and we lose the spectral gap in both limits.

The keys to the proof of the spectral Theorem 4.1 are the delicate growth lemmas given in Sect. 3. To prove these growth lemmas, subtle modifications of the fundamental ideas of Chernov [CM] and of the original techniques introduced in [DZ1, BD] were necessary. In particular, the analysis for $t > 1$ required a new bootstrap argument (see the beginning of Sect. 3 and Sect. 3.4 and 3.6).

In Section 6, a more careful study of the operator \mathcal{L}_t yields our second main result:

Theorem 1.2 (Strict Convexity). *The function $t \mapsto P(t)$ is analytic on $(0, t_*)$, with*

$$(1.11) \quad P'(t) = \int \log J^s T d\mu_t = - \int \log J^u T d\mu_t < 0,$$

and

$$(1.12) \quad P''(t) = \sum_{k \geq 0} \left[\int (\log J^s T \circ T^k) \log J^s T d\mu_t - (P'(t))^2 \right] \geq 0.$$

Moreover, $P''(t) = 0$ if and only if $\log J^s T = f - f \circ T + \int \log J^s T d\mu_t$ (μ_t a.e.) for some $f \in L^2(\mu_t)$. Finally, both $t \mapsto \int \log J^u T d\mu_t$ and $t \mapsto h_{\mu_t}$ are decreasing functions of t .

⁵The T -adapted property appears in particular in the work of Lima–Matheus [LM].

⁶We attract the reader's attention to Lemma 4.3 showing $\mathcal{L}_t(C^1) \subset \mathcal{B}$, which furnishes the proof of [BD, Lemma 4.9], which had been omitted there, see Remark 4.4.

The formula for $P'(t)$ in (1.11) implies that, if there exist $t_a \neq t_b$ in $[0, t_*)$ such that $\mu_{t_a} = \mu_{t_b}$, then $P(t)$ is not strictly convex: indeed, $P'(t)$ is constant on $[t_a, t_b]$. By analyticity, we then deduce that $P(t)$ must be affine on $(0, t_*)$. Therefore, we get an immediate corollary of Theorem 1.2:

Corollary 1.3. *If there exist $t_a \neq t_b$ in $(0, t_*)$ such that $\mu_{t_a} = \mu_{t_b}$ then $P(t)$ is affine on $(0, t_*)$, and $\log J^s T$ is μ_t a.e. cohomologous to its average $\int \log J^s T d\mu_t$ for all $t \in (0, t_*)$.*

We expect that there does not exist any Sinai billiard table such that $\log J^s T$ is μ_t a.e. cohomologous to a constant on M' for some $t \in [0, t_*)$. If we only want to verify that $\mu_0 \neq \mu_1 = \mu_{\text{SRB}}$, it is enough to show that $P''(1) \neq 0$. Note that in [BD], assuming sparse recurrence (see Definition 5.4), we showed that $\mu_0 = \mu_{\text{SRB}}$ (i.e., $\mu_0 = \mu_1$) only if $\frac{1}{p} \log |\det(DT^{-p}|_{E^s(x)})| = P(0)$ for every nongrazing periodic orbit $T^p(x) = x$.

The proof of analyticity of $P(t)$ via analyticity of \mathcal{L}_t in Theorem 1.2 gives the following two corollaries.

Corollary 1.4 (Continuity of Equilibrium States). *For each $\psi \in C^1(M)$, $\mu_t(\psi)$ is analytic for $t \in (0, t_*)$. Moreover, the measures μ_t vary continuously in the weak topology.*

Corollary 1.5 (Uniform Rates of Mixing). *The exponential rate of mixing of μ_t for C^1 observables is uniformly bounded away from 1 in any compact subinterval of $(0, t_*)$.*

In addition, the proof of the claim on $P''(t) = 0$ in Theorem 1.2 gives:⁷

Corollary 1.6 (Central Limit Theorem). *For any $t \in (0, t_*)$ such that $P''(t) \neq 0$, setting $\chi_t := P'(t)$ and $\sigma_t := P''(t)$, we have $\lim_{k \rightarrow \infty} \mu_t(\frac{1}{\sqrt{k}} \sum_{j=0}^{k-1} (\log J^s T - \chi_t) \circ T^j \leq z) = \frac{1}{\sqrt{2\pi\sigma_t}} \int_{-\infty}^z e^{-v^2/(2\sigma_t^2)} dv$, for any $z \in \mathbb{R}$.*

We next motivate *heuristically* the choice of the weight $1/|J^s T|^{1-t}$ in (1.10), by analogy with the theory for smooth hyperbolic T . For a transitive Anosov diffeomorphism T , the transfer operator whose maximal left and right eigenvectors on an anisotropic Banach space give rise to μ_t is $\tilde{\mathcal{L}}_t(f) = (f/(|J^u T|^t J^s T)) \circ T^{-1}$ (see [GL] or [Ba, Chapter 7]). A coboundary argument, reflecting the fact that C^1 functions are interpreted as distributions via integration with respect to the SRB measure $\mu_{\text{SRB}} = (2|\partial Q|)^{-1} \cos \varphi drd\varphi$ here (see below Proposition 4.2), but with respect to Lebesgue in [GL, Ba], will replace $1/(|J^u T|^t J^s T)$ by $1/|J^s T|^{1-t}$: Indeed, (1.5) gives (on M')

$$\begin{aligned} -\log(|J^u T|^t J^s T) &= -\log|J^s T J^u T|^t - \log|J^s T|^{1-t} \\ (1.13) \qquad &= -t \log \left(\frac{E \cos \varphi}{(E \cos \varphi) \circ T} \right) - (1-t) \log J^s T. \end{aligned}$$

The first term of (1.13) is a coboundary. Thus we can expect that the operators $\tilde{\mathcal{L}}_t$ and \mathcal{L}_t from (1.10) have isomorphic spectral data, which motivates intuitively our study of \mathcal{L}_t .

The rest of the paper is organized as follows. In Section 2, after defining our notion of topological pressure $P_*(t, g)$ that we will connect to the measure-theoretic pressure $P(t, g)$, we state our strong variational principle and prove the preliminary result that $P_*(t, g) \geq P(t, g)$. In Section 3, we carry out the main growth lemmas and estimates needed to prove the exact exponential growth of the topological complexity $Q_n(t, g)$ (defined in Sect. 2.2). These estimates are uniform for $t \in [t_0, t_1] \subset (0, t_*)$. In Section 4, we define the Banach spaces on which our operators \mathcal{L}_t act and prove inequalities which furnish a spectral gap, again uniform for $t \in [t_0, t_1]$. Section 5 establishes the main properties of the measure μ_t constructed from the maximal eigenvectors of \mathcal{L}_t and $(\mathcal{L}_t)^*$; in particular, μ_t is the unique equilibrium state with pressure equal to $P_*(t)$. Its existence provides the strong version of the variational principle $P_*(t) = P(t)$ and completes the proof of Theorem 1.1 (see also the sketch provided after the statement of Theorem 1.1 above). Finally, Section 6 proves the analyticity of the pressure function and conditions for its strict convexity as stated in Theorem 1.2.

⁷Our approach gives other limit theorems (large deviation estimates, invariance principles, see [DZ1, Sect. 6]).

2. TOPOLOGICAL FORMULATION $P_*(t, g)$ FOR $P(t, g)$. VARIATIONAL PRINCIPLE (THEOREM 2.5)

2.1. Hyperbolicity and Distortion. $\mathcal{W}^s, \widehat{\mathcal{W}}^s, \mathcal{W}_{\mathbb{H}}^s, \widehat{\mathcal{W}}_{\mathbb{H}}^s$. **Families** $\mathcal{M}_{-k}^n, \mathcal{M}_{-k}^{n, \mathbb{H}}$. For $n > 0$, following [BD], define \mathcal{M}_0^n to be the set of maximal connected components of $M \setminus \mathcal{S}_n$, and \mathcal{M}_{-n}^0 to be the maximal connected components of $M \setminus \mathcal{S}_{-n}$. Set $\mathcal{M}_{-k}^n = \mathcal{M}_{-k}^0 \vee \mathcal{M}_0^n$. Note that if $A \in \mathcal{M}_0^n$, then $T^k A \in \mathcal{M}_{-k}^{n-k}$ for each $0 \leq k \leq n$, and $T^k A$ is a union of elements of \mathcal{M}_{-k}^0 for each $k > n$.

To control distortion, we introduce homogeneity strips whose spacing depends on $t_0 \in (0, 1)$ if $t \geq t_0$. Choose⁸ $q = q(t_0) > 1$ such that $qt_0 \geq 2$. For fixed $k_0 \in \mathbb{N}$ define

$$(2.1) \quad \mathbb{H}_k = \{(r, \varphi) \in M : (k+1)^{-q} \leq \frac{\pi}{2} - \varphi < k^{-q}\}, \quad \text{for } k \geq k_0,$$

and similarly \mathbb{H}_{-k} is defined approaching $\varphi = -\pi/2$. A connected component of \mathbb{H}_k , for some $|k| \geq k_0$, or of the set $\mathbb{H}_0 = \{(r, \varphi) : k_0^{-q} \leq \min\{\frac{\pi}{2} - \varphi, \frac{\pi}{2} + \varphi\}\}$ is called a *homogeneity strip*. We let \mathcal{H} denote the partition of M into homogeneity strips. Let $\mathcal{S}_0^{\mathbb{H}} = \mathcal{S}_0 \cup (\cup_{|k| \geq k_0} \partial \mathbb{H}_k)$ and, for $n \in \mathbb{Z}$, let $\mathcal{S}_n^{\mathbb{H}} = \cup_{i=0}^{-n} T^i \mathcal{S}_0^{\mathbb{H}}$ denote the *extended singularity set* for T^n .

Fix⁹ $\delta_0 \in (0, 1)$. Let \mathcal{W}^s denote the set of all nontrivial connected subsets W of local stable manifolds of T of length at most δ_0 . Such curves have curvature bounded above by a fixed constant [CM, Prop 4.29], and $T^{-n} \mathcal{W}^s = \mathcal{W}^s$ for all $n \geq 1$, up to subdivision of curves according to the length scale δ_0 . Let $\mathcal{W}_{\mathbb{H}}^s \subset \mathcal{W}^s$ denote the set of nontrivial connected subsets W of elements of \mathcal{W}^s with the property that $T^n W$ belongs to a single homogeneity strip for each $n \geq 0$. Such curves are called¹⁰ *homogeneous stable manifolds*.

We call a C^2 curve $W \subset M$ (cone) stable if at each point x in W , the tangent vector $\mathcal{T}_x W$ to W lies in \mathcal{C}^s . We denote by $\widehat{\mathcal{W}}^s$ the set of (cone) stable curves with second derivative bounded by a constant chosen sufficiently large ([CM, Prop 4.29]) so that $T^{-n} \widehat{\mathcal{W}}^s \subset \widehat{\mathcal{W}}^s$ for all $n \geq 1$, up to subdivision of curves according to δ_0 . Finally, $\widehat{\mathcal{W}}_{\mathbb{H}}^s \subset \widehat{\mathcal{W}}^s$ is the set of elements of $\widehat{\mathcal{W}}^s$ contained in a single homogeneity strip, while \mathcal{W}_H^s is the set of elements of \mathcal{W}^s that are contained in a single homogeneity strip. Such curves are called *weakly homogeneous* (cone) stable curves and stable manifolds, respectively. Obviously, $\mathcal{W}_{\mathbb{H}}^s \subset \mathcal{W}_H^s \subset \mathcal{W}^s \subset \widehat{\mathcal{W}}^s$ and $\mathcal{W}_H^s \subset \widehat{\mathcal{W}}_H^s$.

For every $W \in \widehat{\mathcal{W}}^s$, let $C^1(W)$ denote the space of C^1 functions on W , and for every $\eta \in (0, 1)$ let $C^\eta(W)$ denote the closure¹¹ of $C^1(W)$ for the η -Hölder norm defined by

$$(2.2) \quad |\psi|_{C^\eta(W)} = \sup_W |\psi| + H_W^\eta(\psi), \quad H_W^\eta(\psi) = \sup_{\substack{x, y \in W \\ x \neq y}} \frac{|\psi(x) - \psi(y)|}{d(x, y)^\eta}.$$

The following lemma extends standard distortion bounds for homogeneous curves to all exponents $t > 0$. (See Lemma 6.2 for a further generalisation.)

Lemma 2.1. *There exists $\bar{\delta}_0 > 0$ and $C_d > 0$, depending on k_0 and q , such that for all $\delta_0 < \bar{\delta}_0$, all $n \geq 0$, and any $W \in T^{-n} \widehat{\mathcal{W}}^s$ such that $T^i W \in \widehat{\mathcal{W}}_H^s$ for each $i = 0, \dots, n-1$, we have*

$$\left| 1 - \frac{|J_W T^n(x)|^t}{|J_W T^n(y)|^t} \right| \leq 2^t C_d d(x, y)^{1/(q+1)}, \quad \forall x, y \in W, \quad \forall t > 0,$$

where $J_W T^n(x) = |\det(DT_x^n | \mathcal{T}_x W)|$ denotes the Jacobian of T^n along W , and $d(\cdot, \cdot)$ denotes distance in M .

⁸The standard choice for $t = 1$ is $q = 2$.

⁹The index $k_0 = k_0(t_0, t_1)$ and the length scale $\delta_0 = \delta_0(t_0, t_1) < 1$ will be chosen in Definition 3.2.

¹⁰In [CM], these curves are called H -manifolds. This strong notion of homogeneity is needed to prove Hölder continuity of the conditional densities of the SRB measure decomposed along stable manifolds – needed to get valid test functions for our spaces — using the asymptotic limit of the ratio of stable Jacobians, forward iterates must be contained in a single homogeneity strip (so that the ratio remains bounded).

¹¹Using the closure of C^1 will give injectivity of the inclusion of the strong space in the weak one in Proposition 4.2.

Proof. There exists $C_d < \infty$, independent of δ_0 , but depending on k_0 and q such that

$$(2.3) \quad \left| 1 - \frac{J_W T^n(x)}{J_W T^n(y)} \right| \leq C_d d(x, y)^{1/(q+1)}, \quad \forall x, y \in W, \quad \forall W \text{ as in the lemma.}$$

(For $q = 2$, see e.g. [CM, Lemma 5.27] or [DZ1, App. A]. The proofs there give (2.3) for all $q > 1$.)

For $t \leq 1$, the estimate is an immediate consequence of (2.3), since for all $A > 0$, we have $|1 - A^t| \leq |1 - A|$. Now choose $\bar{\delta}_0$ such that $C_d \bar{\delta}_0^{1/(q+1)} \leq 3/4$. Then, for $t > 1$, we set $A = \frac{J_W T^n(x)}{J_W T^n(y)}$. By (2.3), this implies that $1/4 \leq A \leq 2$ if $\delta_0 < \bar{\delta}_0$. For A in this range, we have, again using (2.3), that $|1 - A^t| \leq 2^t |1 - A| \leq 2^t C_d d(x, y)^{1/(q+1)}$. \square

Next, recalling that $\mathcal{S}_k^{\mathbb{H}} = \cup_{i=0}^{-k} T^i \mathcal{S}_0^{\mathbb{H}}$, define for $n \geq 1$,

$$(2.4) \quad \begin{aligned} \mathcal{M}_0^{n, \mathbb{H}} &= \text{maximal connected components of } M \setminus \left(T^{-n} \mathcal{S}_0 \cup \mathcal{S}_{n-1}^{\mathbb{H}} \right), \\ \mathcal{M}_{-n}^{0, \mathbb{H}} &= \text{maximal connected components of } M \setminus \left(\mathcal{S}_0 \cup T(\mathcal{S}_{-(n-1)}^{\mathbb{H}}) \right), \\ \mathcal{M}_{-k}^{n, \mathbb{H}} &= \mathcal{M}_{-k}^{0, \mathbb{H}} \vee \mathcal{M}_0^{n, \mathbb{H}}, \quad k \geq 1. \end{aligned}$$

We comment on the use of $\mathcal{S}_0^{\mathbb{H}}$ in (2.4). First notice (just like for the sets \mathcal{M}_{-k}^n defined in the beginning of this subsection) that if $A \in \mathcal{M}_0^{n, \mathbb{H}}$, then $T^k A \in \mathcal{M}_{-k}^{n-k, \mathbb{H}}$ for each $0 \leq k \leq n$, and $T^k A$ is a union of elements of $\mathcal{M}_{-k}^{0, \mathbb{H}}$ for each $k > n$. Next, if $W \in \widehat{\mathcal{W}}_H^s$ is such that $V = T^{-1}W$ is a single curve, then $J_W T^{-1}(x) \approx 1/\cos \varphi(T^{-1}x)$ while $J_V T(y) \approx \cos \varphi(y)$. Thus by (2.3), the definitions in (2.4) guarantee that for any $W \in \widehat{\mathcal{W}}_H^s$ such that $W \subset A \in \mathcal{M}_{-n}^{0, \mathbb{H}}$, the Jacobian $J_W T^{-n}$ has bounded distortion on W , while $J_{T^{-n}W} T^n$ has bounded distortion on $T^{-n}W$ (which is contained in a single element of $\mathcal{M}_0^{n, \mathbb{H}}$).

We shall also need the following distortion bound.

Lemma 2.2 (Distortion Relative to $\mathcal{M}_{-n}^{0, \mathbb{H}}$). *There exists $C > 0$ such that for all $n \geq 1$, for all $U, V \in \widehat{\mathcal{W}}_H^s$ such that $U, V \subset A \in \mathcal{M}_{-n}^{0, \mathbb{H}}$, and all¹² $u \in \bar{U} \setminus \mathcal{S}_{-n}$, $v \in \bar{V} \setminus \mathcal{S}_{-n}$,*

$$\left| \log \frac{J_U T^{-n}(u)}{J_V T^{-n}(v)} \right| \leq C.$$

The bound above is more general (and weaker) than the usual distortion bound along stable curves given by (2.3) or between stable curves given by [CM, Theorem 5.42] (or more generally [DZ1, Appendix A]) since we do not assume that the points u, v in \bar{A} , with $A \in \mathcal{M}_{-n}^{0, \mathbb{H}}$, lie on the same stable or unstable curve.

Proof. Let $n \geq 1$, $u \in \bar{U}$, $v \in \bar{V}$, be as in the statement of the lemma. Define $u_i = T^{-i}u$, $v_i = T^{-i}v$ for $i = 0, \dots, n$, and notice that u_i, v_i belong to the closure of the same element of $\mathcal{M}_{-n+i}^{i, \mathbb{H}}$. By the uniform hyperbolicity of T , for $i = 0, \dots, n$, if $A \in \mathcal{M}_{-n+i}^{i, \mathbb{H}}$, then $\text{diam}^u(\bar{A}) \leq C\Lambda^{-i}$ and $\text{diam}^s(\bar{A}) \leq C\Lambda^{-n+i}$, where $\text{diam}^u(B)$ is the maximum length of an unstable curve in B , and $\text{diam}^s(B)$ is the maximum length of a stable curve in B . Thus, due to the uniform transversality of \mathcal{C}^s and \mathcal{C}^u , we have

$$(2.5) \quad d(u_i, v_i) \leq \bar{C} \max\{\Lambda^{-i}, \Lambda^{-n+i}\}.$$

By the time reversal of [CM, eq. (5.24)], we have that

$$(2.6) \quad \log J_{U_i} T^{-1}(u_i) = \log \frac{\cos \varphi(u_i) + \tau(u_{i+1})(\mathcal{K}(u_i) - \mathcal{V}(u_i))}{\cos \varphi(u_{i+1})} + \log \frac{\sqrt{1 + \mathcal{V}(u_{i+1})^2}}{\sqrt{1 + \mathcal{V}(u_i)^2}},$$

¹² \bar{U} denotes the closure of U in M . The distortion bounds on U and V extend trivially to the boundaries of homogeneity strips, but not to real singularity lines, hence $\bar{U} \setminus \mathcal{S}_{-n}$.

where $\mathcal{V}(u_i) = \frac{d\varphi}{dr}(u_i) < 0$ is the slope of the tangent line to U_i at u_i . Summing over i , the last term above telescopes and the sum is uniformly bounded away from 0 and ∞ , giving,

$$(2.7) \quad \left| \log \frac{J_U T^{-n}(u)}{J_V T^{-n}(v)} \right| \leq C + \sum_{i=0}^{n-1} \left| \log \frac{\cos \varphi(v_{i+1})}{\cos \varphi(u_{i+1})} \right| + \left| \log \frac{\cos \varphi(u_i) + \tau(u_{i+1})(\mathcal{K}(u_i) - \mathcal{V}(u_i))}{\cos \varphi(v_i) + \tau(v_{i+1})(\mathcal{K}(v_i) - \mathcal{V}(v_i))} \right|$$

Since u_{i+1}, v_{i+1} lie in the same homogeneity strip for each i , using (2.1) we have

$$(2.8) \quad \left| \log \frac{\cos \varphi(v_{i+1})}{\cos \varphi(u_{i+1})} \right| \leq C \frac{|\varphi(u_{i+1}) - \varphi(v_{i+1})|}{\cos \varphi(u_{i+1})} \leq C d(u_{i+1}, v_{i+1})^{1/(q+1)}.$$

Next, the terms in the second set on the right-hand side of (2.7) are bounded and the denominator in the expression is at least $\tau_{\min} \mathcal{K}_{\min} > 0$. Moreover, \mathcal{K} is differentiable while τ is 1/2-Hölder continuous.¹³ Thus following [CM, eq. (5.26)], we have

$$\sum_{i=0}^{n-1} \left| \log \frac{\cos \varphi(u_i) + \tau(u_{i+1})(\mathcal{K}(u_i) - \mathcal{V}(u_i))}{\cos \varphi(v_i) + \tau(v_{i+1})(\mathcal{K}(v_i) - \mathcal{V}(v_i))} \right| \leq C \sum_{i=0}^{n-1} d(u_{i+1}, v_{i+1})^{1/2} + d(u_i, v_i) + |\Delta \mathcal{V}_i|,$$

where $\Delta \mathcal{V}_i = \mathcal{V}(u_i) - \mathcal{V}(v_i)$. By (2.5), the sums over all terms in (2.7) involving $d(u_i, v_i)$ are uniformly bounded in n . It remains to estimate $\sum_{i=0}^{n-1} |\Delta \mathcal{V}_i|$. By [CM, eq. (5.29)] and (2.5), we bound $|\Delta \mathcal{V}_i|$ by

$$\begin{aligned} C \left(|\Delta \mathcal{V}_0| \Lambda^{-i} + \sum_{j=0}^i \Lambda^{-j} d(u_{i-j}, v_{i-j})^{1/2} \right) &\leq C \left(|\Delta \mathcal{V}_0| \Lambda^{-i} + \sum_{j=0}^i \Lambda^{-j} (\Lambda^{(-i+j)/2} + \Lambda^{(-n+i-j)/2}) \right) \\ &\leq C' \left(|\Delta \mathcal{V}_0| \Lambda^{-i} + \Lambda^{-i/2} + \Lambda^{(-n+i)/2} \right). \end{aligned}$$

Summing over i completes the proof of the lemma. \square

Remark 2.3 (About $\int_M \log J^u T d\mu$). *We note for further use a consequence of the proof above: It is clear from (2.6) that for any T -invariant probability measure μ with $\mu(M \setminus M') = 0$,*

$$\int_M \log J^s T d\mu = -\infty \iff \int_M \log \cos \varphi d\mu = -\infty,$$

since all other terms in (2.6) are bounded away from zero and infinity. Similarly, the time reversal of (2.6) (see [CM, eq. (5.24)]) implies

$$\int_M \log J^u T d\mu = \infty \iff \int_M \log \cos \varphi \circ T d\mu = -\infty.$$

Thus, applying (1.5), for any T -invariant probability measure μ on M ,

$$(2.9) \quad \text{if } \mu(M \setminus M') = 0 \text{ then } \int_M \log J^u T d\mu = - \int_M \log J^s T d\mu,$$

where equality holds also when both sides are infinite.

2.2. Topological Formulation $P_*(t, g)$ of the Pressure $P(t, g)$. Theorem 2.5. In view of our proof of uniqueness in §5.5 (which uses differentiability of the pressure), for a C^1 function $g : M \rightarrow \mathbb{R}$ and $n \geq 1$, we set $S_n g = \sum_{i=0}^{n-1} g \circ T^i$. The hyperbolicity of T implies the following distortion bounds: There exists $C_* < \infty$ such that for all $n \geq 1$ and all $W \in \widehat{\mathcal{W}}^s$ such that $T^i W \in \widehat{\mathcal{W}}^s, \forall 0 \leq i \leq n$,

$$(2.10) \quad |e^{S_n g(x) - S_n g(y)} - 1| \leq C_* |\nabla g|_{C^0} d(x, y), \forall x, y \in W.$$

Recalling (1.4), we define (aside from §5.5 we only need $g \equiv 0$),

$$(2.11) \quad Q_n(t, g) = \sum_{A \in \mathcal{M}_0^{n, \mathbb{H}}} \sup_{x \in A \cap M'} |J^s T^n(x)|^t e^{S_n g(x)}, \quad Q_n(t) = Q_n(t, 0), \quad n \geq 1,$$

¹³We cannot take advantage of the smoother bounds on τ given by [CM, eq. (5.28)] since our points u_i and v_i may lie on different stable or unstable manifolds.

and

$$(2.12) \quad P_*(t, g) = \limsup_{n \rightarrow \infty} \frac{1}{n} \log Q_n(t, g), \quad P_*(t) := P_*(t, 0).$$

We will show the following result in Section 2.3:

Proposition 2.4 (Topological Pressure). *For all $t > 0$ and $g \in C^1$, we have¹⁴ $P(t, g) \leq P_*(t, g)$.*

For $t_* > 1$ given by (1.8), the analysis carried out in Sections 3–5 will yield:

Theorem 2.5 ((Strong¹⁵) Variational Principle). *If $t \in (0, t_*)$, then $P_*(t) = P(t)$ and the supremum is attained at the unique invariant measure μ_t from Theorem 1.1.*

Proof. This follows from Proposition 2.4, Theorem 4.1, Corollary 5.3, and Proposition 5.7. \square

Theorem 5.8 will give the generalisation of the above strong form of the variational principle to $P(t, g) = P_*(t, g)$ for suitable g .

We first establish basic properties of $P_*(t, g)$:

Proposition 2.6. *For each $t > 0$ and $g \in C^1$ the limsup (2.12) defining $P_*(t, g)$ is a limit in $(-\infty, \infty)$. The function $t \mapsto P_*(t, g)$ is convex and strictly decreasing on $(0, \infty)$.*

Remark 2.7. *It is not hard to show, using Lemma 2.2, that for each $t > 0$, there exists $C_D > 0$ such that $Q_n(t) \leq C_D^t \sum_{A \in \mathcal{M}_0^{n, \mathbb{H}}} \inf_{x \in A \cap M'} |J^s T^n(x)|^t$ for all $n \geq 1$, so that replacing the supremum by an infimum in the definition of $Q_n(t)$ does not change the value of $P_*(t)$.*

Proof of Proposition 2.6. We first set $g = 0$. The partition \mathcal{M}_0^1 is finite, and each element of \mathcal{M}_0^1 is subdivided by curves in $\mathcal{S}_0^{\mathbb{H}}$ to comprise a union of elements of $\mathcal{M}_0^{1, \mathbb{H}}$, according to (2.4). Thus,

$$Q_1(t) = \sum_{A \in \mathcal{M}_0^{1, \mathbb{H}}} \sup_{x \in A \cap M'} |J^s T(x)|^t \leq C \sum_{E \in \mathcal{M}_0^1} \sum_k \sup_{x \in \mathbb{H}_k} |\cos \varphi(x)|^t \leq C \# \mathcal{M}_0^1 \sum_k k^{-qt},$$

where the sum over k includes $k = 0$ and $|k| \geq k_0$; the sum converges since $qt \geq 2 > 1$.

We next show that $Q_n(t)$ is submultiplicative:

$$(2.13) \quad \begin{aligned} Q_{n+k}(t) &= \sum_{A \in \mathcal{M}_0^{n, \mathbb{H}}} \sum_{\substack{B \in \mathcal{M}_0^{n+k, \mathbb{H}} \\ B \subset A}} \sup_{x \in B \cap M'} |J^s T^{n+k}(x)|^t \\ &\leq \sum_{A \in \mathcal{M}_0^{n, \mathbb{H}}} \sup_{y \in A \cap M'} |J^s T^n(y)|^t \sum_{\substack{B \in \mathcal{M}_0^{n+k, \mathbb{H}} \\ B \subset A}} \sup_{x \in B \cap M'} |J^s T^k(T^n x)|^t, \quad \forall k, n \geq 1. \end{aligned}$$

Notice that if $B, B' \subset A \in \mathcal{M}_0^{n, \mathbb{H}}$ are distinct elements of $\mathcal{M}_0^{n+k, \mathbb{H}}$, then $T^n B, T^n B' \in \mathcal{M}_{-n}^{k, \mathbb{H}} = \mathcal{M}_{-n}^{0, \mathbb{H}} \vee \mathcal{M}_0^{k, \mathbb{H}}$ are both contained in $T^n A \in \mathcal{M}_{-n}^{0, \mathbb{H}}$ and so must lie in distinct elements of $\mathcal{M}_0^{k, \mathbb{H}}$. Thus the inner sum in (2.13) is bounded by $Q_k(t)$ for each A , and the outer sum is bounded by $Q_n(t)$, proving submultiplicativity. If $g \neq 0$, it is easy to see that we also have $Q_{n+k}(t, g) \leq Q_n(t, g)Q_k(t, g)$. Therefore, since $Q_1(t, g) < \infty$, the sequence in (2.12) converges to a limit in $[-\infty, \infty)$.

To see that $P_*(t, g) > -\infty$, let x_p be a periodic point of period p with no tangential collisions,¹⁶ and let χ_p^- denote the negative Lyapunov exponent of x_p . Then, $Q_{np}(t, g) \geq |J^s T^{np}(x_p)|^t e^{S_{np}g(x_p)} = e^{npt\chi_p^-} e^{nS_{np}g(x_p)}$, and so $P_*(t, g) \geq t\chi_p^- + \frac{1}{p}S_{pg}(x_p) > -\infty$.

¹⁴Recall our convention that $\int_M \log J^u T d\mu = \infty$ if $\mu(M \setminus M') > 0$.

¹⁵By "strong" we mean that the supremum is a maximum, and it is attained at a unique measure.

¹⁶Such a periodic point always exists. For example, since two adjacent scatterers are in convex opposition, there is a period 2 orbit whose trajectory is normal to both scatterers.

To prove convexity, pick $t, t' > 0$ and $\alpha \in [0, 1]$. Then using the Hölder inequality,

$$\begin{aligned} Q_n(\alpha t + (1 - \alpha)t', g) &= \sum_{A \in \mathcal{M}_0^{n, \mathbb{H}}} \sup_{x \in A \cap M'} |J^s T^n|^{\alpha t + (1 - \alpha)t'} e^{\alpha S_n g(x)} e^{(1 - \alpha)S_n g(x)} \\ &\leq \left(\sum_{A \in \mathcal{M}_0^{n, \mathbb{H}}} \sup_{x \in A \cap M'} |J^s T^n|^t e^{S_n g(x)} \right)^\alpha \left(\sum_{A \in \mathcal{M}_0^{n, \mathbb{H}}} \sup_{x \in A \cap M'} |J^s T^n|^{t'} e^{S_n g(x)} \right)^{1 - \alpha} = Q_n(t, g)^\alpha Q_n(t', g)^{1 - \alpha}. \end{aligned}$$

Taking logarithms, dividing by n , and letting $n \rightarrow \infty$ proves convexity.

Next, fixing $t > 0$ and applying (1.2), we find for $s > 0$,

$$Q_n(t + s, g) = \sum_{A \in \mathcal{M}_0^{n, \mathbb{H}}} \sup_{x \in A \cap M'} |J^s T^n|^{t+s} e^{S_n g(x)} \leq C_1^{-s} \Lambda^{-ns} Q_n(t, g),$$

so that $P_*(t + s, g) \leq P_*(t, g) - s \log \Lambda$, that is, $P_*(t, g)$ is strictly decreasing in t . \square

2.3. Proof that $P_*(t, g) \geq P(t, g)$ (Proposition 2.4). If \mathcal{Q} is a partition of M we let $\text{Int } \mathcal{Q}$ denote the set of interiors of elements of \mathcal{Q} . In [BD], we worked with \mathcal{P} , the (finite) partition of M into maximal connected sets on which T and T^{-1} are continuous, noticing that the set $\text{Int } \mathcal{P}$ coincides with \mathcal{M}_{-1}^1 , while the refinements $\mathcal{P}_{-k}^n = \bigvee_{i=-k}^n T^{-i} \mathcal{P}$ may also contain isolated points if three or more scatterers have a common tangential trajectory (see [BD, Fig.1]). (Note that \mathcal{P} is a set-theoretical partition: zero measure sets do not need to be ignored.) We also observed that \mathcal{P} is a generator for any T -invariant Borel probability measure μ , since $\bigvee_{i=-\infty}^{\infty} T^{-i} \mathcal{P}$ separates¹⁷ points in the compact metric space M : if $x \neq y$ there exists $k \in \mathbb{Z}$ such that $T^k(x)$ and $T^k(y)$ lie in different elements of \mathcal{P} . Let $\bar{\mathcal{P}}$ be the partition of M into maximal connected sets on which T is continuous. Then $\mathcal{P} = \bar{\mathcal{P}} \vee T(\bar{\mathcal{P}})$, so $\bar{\mathcal{P}}$ is also a generator for T . We have $\text{Int } \bar{\mathcal{P}} = \mathcal{M}_0^1$. More generally, $\text{Int}(\bigvee_{k=0}^{n-1} T^{-k} \bar{\mathcal{P}}) = \mathcal{M}_0^n$ for $n \geq 1$.

Proof of Proposition 2.4. If a T -invariant probability measure μ gives positive weight to $M \setminus M'$ or, more generally, if $\int_M \log J^u T d\mu = \infty$, then $P_\mu(t, g) = -\infty$, so $P_\mu(t, g) < P_*(t, g)$. We can thus assume without loss of generality that μ is a T -invariant probability measure with $\int_M \log J^u T d\mu < \infty$, in particular $\mu(\mathcal{S}_n) = 0$ for each $n \in \mathbb{Z}$. Then

$$(2.14) \quad H_\mu \left(\bigvee_{k=0}^{n-1} T^{-k} \bar{\mathcal{P}} \right) = H_\mu(\mathcal{M}_0^n),$$

since the boundary of any element of $\bigvee_{k=0}^{n-1} T^{-k} \bar{\mathcal{P}}$ is contained in \mathcal{S}_n .

Since $\bar{\mathcal{P}}$ is a generator, we have $h_\mu(T) = h_\mu(T, \bar{\mathcal{P}})$ for any T -invariant probability measure μ on M (see e.g. [W, Theorem 4.17]). Then, using (2.9), we find, adapting the classical argument (see e.g. [W, Prop 9.10]), that

$$\begin{aligned} h_\mu(T, \bar{\mathcal{P}}) - t \int_M \log J^u T d\mu &= \lim_{n \rightarrow \infty} \frac{1}{n} H_\mu \left(\bigvee_{k=0}^{n-1} T^{-k} \bar{\mathcal{P}} \right) + \lim_{n \rightarrow \infty} \frac{1}{n} \int_M t \sum_{k=0}^{n-1} \log J^s T \circ T^k d\mu \\ &\leq \lim_{n \rightarrow \infty} \frac{1}{n} \left(\sum_{A \in \mathcal{M}_0^n} \mu(A) [-\log \mu(A) + \sup_{A \cap M'} t \log J^s T^n] \right) \\ &\leq \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{A \in \mathcal{M}_0^n} \mu(A) \log \frac{\sup_{A \cap M'} |J^s T^n|^t}{\mu(A)} \leq \lim_{n \rightarrow \infty} \frac{1}{n} \log \sum_{A \in \mathcal{M}_0^n} \sup_{A \cap M'} |J^s T^n|^t, \end{aligned}$$

¹⁷In fact, all points $x \neq y$ may be separated while the definition of a generator in [Pa] allows a zero measure set of pathological pairs.

where we used (2.14) in the third line, and the convexity of the logarithm in the fifth line. Finally, notice that each element of \mathcal{M}_0^n is a union of elements of $\mathcal{M}_0^{n,\mathbb{H}}$, modulo the boundaries of homogeneity strips. But since the distortion bound Lemma 2.2 extends to the boundaries of homogeneity strips, we have

$$\sup_{A \cap M'} |J^s T^n|^t = \sup_{\substack{B \in \mathcal{M}_0^{n,\mathbb{H}} \\ B \subset A}} \sup_{B \cap M'} |J^s T^n|^t \leq \sum_{\substack{B \in \mathcal{M}_0^{n,\mathbb{H}} \\ B \subset A}} \sup_{B \cap M'} |J^s T^n|^t,$$

for each $A \in \mathcal{M}_0^n$. Using this bound in the previous estimate and applying Proposition 2.6 implies $h_\mu(T) - t \int_M \log J^u T d\mu \leq P_*(t)$ for every T -invariant probability measure μ .

If $g \neq 0$, we may write

$$\begin{aligned} h_\mu(T, \bar{P}) + \int (t \log J^s T + g) d\mu &= \lim_{n \rightarrow \infty} \frac{1}{n} H_\mu \left(\bigvee_{k=0}^{n-1} T^{-k} \bar{P} \right) + \lim_{n \rightarrow \infty} \frac{1}{n} \int S_n(t \log J^s T + g) d\mu \\ &\leq \lim_{n \rightarrow \infty} \frac{1}{n} \log \sum_{A \in \mathcal{M}_0^n} \sup_{A \cap M'} |J^s T^n|^t e^{S_n g}, \end{aligned}$$

and this last expression is bounded by $P_*(t, g)$ by the same reasoning as above, using that the analogue of Lemma 2.2 holds for $e^{S_n g}$: Recalling (2.10), for all $n > 0$, all $A \in \mathcal{M}_0^{n,\mathbb{H}}$, and $x, y \in A$, since the diameter of $T^i A$ is bounded by (2.5),

$$(2.15) \quad e^{S_n g(x) - S_n g(y)} \leq 1 + \bar{C} C_* \cdot |\nabla g|_{C^0}.$$

□

3. GROWTH LEMMAS

In this section, after preliminaries in §3.1, introducing in particular the contraction rate θ and sets $\mathcal{G}_n(W)$ appearing when iterating the transfer operator \mathcal{L}_t , we prove a series of growth and complexity lemmas which will allow us to control the sums over $\mathcal{G}_n(W)$ for $W \in \widehat{\mathcal{W}}^s$. This culminates in the lower bound of Proposition 3.14, which implies exact exponential growth (Proposition 3.15) of $Q_n(t, g)$. (This exact exponential growth is essential to control the peripheral spectrum of \mathcal{L}_t .)

Since $J^s T$ is not bounded away from zero, we shall use different strategies for $t \in (0, 1]$ and $t > 1$. Several important growth lemmas are proved for $t \in (0, 1]$ in Sections 3.3 and 3.5. We then use the results for $t \leq 1$ to bootstrap an analogous set of lemmas for $t > 1$ in Sections 3.4, 3.5, and 3.6.

3.1. One-Step Expansion: $\theta(t_1)$. Choice of $q(t_0)$, $k_0(t_0, t_1)$, $\delta_0(t_0, t_1)$. $\mathcal{G}_n(W)$, $\mathcal{I}_n(W)$. We begin by proving an adaptation of the one-step expansion (see e.g. [CM, Lemma 5.56]) for our choice of potential and homogeneity strips. Using the notation from (2.6), recall $t_* > 1$ from (1.8), and the adapted metric from [CM, Section 5.10]:

$$\|dx\|_* = \frac{\mathcal{K} + |\mathcal{V}|}{\sqrt{1 + \mathcal{V}^2}} \|dx\|.$$

Lemma 3.1 (One-Step Expansion). *For $t_1 \in (1, t_*)$, fix¹⁸ $\theta \in (\Lambda^{-1}, \Lambda^{-1/2})$ such that $\theta^{t_1} < e^{P_*(t_1)}$. Then for each $\bar{t}_0 \in (0, 1)$ and $q > 2/\bar{t}_0$, there exist $\bar{k}_0 = \bar{k}_0(\bar{t}_0, t_1, q) \geq 1$ and $\bar{\delta}_0 = \bar{\delta}_0(\bar{t}_0, t_1, q) > 0$ such that*

$$(3.1) \quad \sum_i |J_{V_i} T|^t_{C^0(V_i),*} < \theta^t, \quad \forall W \in \widehat{\mathcal{W}}^s \text{ with } |W| < \bar{\delta}_0, \quad \forall t \geq \bar{t}_0,$$

¹⁸This is possible by definition of t_* and Proposition 2.4. It implies $\theta^t < e^{P_*(t)}$ for $t \leq t_1$ since $P_*(t)$ is decreasing.

where the V_i range over the maximal connected weakly (q, \bar{k}_0) -homogeneous¹⁹ components of $T^{-1}W$, and $|J_{V_i}T|_{C^0(V_i),*}$ denotes the maximum on V_i of the Jacobian of T along V_i for the metric $\|\cdot\|_*$.

Proof. Note that $|J_{V_i}T|_{C^0(V_i),*} \leq \Lambda^{-1}$ and, if $V_i \subset \mathbb{H}_k$, then $|J_{V_i}T|_{C^0(V_i),*} \leq Ck^{-q}$ for some $C > 0$ [CM, eq. (5.36)]. There exists $\bar{\delta} > 0$ such that if $W \in \widehat{\mathcal{W}}^s$ with $|W| \leq \bar{\delta}$, then $T^{-1}W$ has at most $\frac{\tau_{\max}}{\tau_{\min}} + 1$ connected components, and all but at most one component experience nearly tangential collisions (see [CM, Sect. 5.10]).

For $\bar{t}_0 \in (0, 1)$ and $q > 2/\bar{t}_0$, choose $\bar{k}_0 = \bar{k}_0(\bar{t}_0, t_1, q)$ such that

$$(3.2) \quad \Lambda^{-\bar{t}_0} + \frac{\tau_{\max}}{\tau_{\min}} \sum_{|k| \geq \bar{k}_0} C^{\bar{t}_0} k^{-q\bar{t}_0} \leq \Lambda^{-\bar{t}_0} + \frac{\tau_{\max}}{\tau_{\min}} C^{\bar{t}_0} \bar{k}_0^{-1} < \theta^{\bar{t}_0}.$$

For all $W \in \widehat{\mathcal{W}}^s$, we have $|T^{-1}W| \leq C'|W|^{1/2}$ [CM, Exercise 4.50] for some $C' > 0$ independent of W . Next, choose $\bar{\delta}_0(\bar{t}_0, t_1)$ so small that $C'\bar{\delta}_0^{1/2} \leq \bar{k}_0^{-q}$, so that if $|W| \leq |\bar{\delta}_0|$, then each component of $T^{-1}W$ making a nearly tangential collision lies in a union of homogeneity strips \mathbb{H}_k for $k \geq \bar{k}_0$.

Then if $|W| \leq \bar{\delta}_0$, the quantity $\sum_i |J_{V_i}T|_{C^0(V_i),*}^{\bar{t}_0}$ is bounded by the left-hand side of (3.2), proving (3.1) for $t = \bar{t}_0$. Finally, for all $t \geq \bar{t}_0$,

$$(3.3) \quad \sup_{\substack{W \in \widehat{\mathcal{W}}^s \\ |W| \leq \bar{\delta}_0}} \sum_i |J_{V_i}T|_{C^0(V_i),*}^t \leq \Lambda^{-t+\bar{t}_0} \sup_{\substack{W \in \widehat{\mathcal{W}}^s \\ |W| \leq \bar{\delta}_0}} \sum_i |J_{V_i}T|_{C^0(V_i),*}^{\bar{t}_0} \leq \Lambda^{-t+\bar{t}_0} \theta^{\bar{t}_0} \leq \theta^t.$$

□

We now choose the parameters defining $\widehat{\mathcal{W}}^s$, $\widehat{\mathcal{W}}_{\mathbb{H}}^s$ and \mathcal{W}^s , $\mathcal{W}_{\mathbb{H}}^s$ depending on $t_0 \in (0, 1)$, $t_1 \in (1, t_*)$:

Definition 3.2. *Given $t_0 \in (0, 1)$, $t_1 \in (1, t_*)$, we fix $q(t_0) > 1$ such that $qt_0/2 \geq 2$, and fix $\theta = \theta(t_1)$, $k_0 = k_0(t_0, t_1, q) := \bar{k}_0(\frac{t_0}{2}, t_1, q)$, $\delta_0 = \delta_0(t_0, t_1, q) := \bar{\delta}_0(\frac{t_0}{2}, t_1, q)$ as in Lemma 3.1. Reduce δ_0 if needed so that $C_d \delta_0^{\frac{1}{q+1}} \leq 3/4$, with C_d from (2.3). This choice of (t_0, t_1) , θ , q , δ_0 , and k_0 determines the set of stable curves $\widehat{\mathcal{W}}^s$, $\widehat{\mathcal{W}}_{\mathbb{H}}^s$ and stable manifolds \mathcal{W}^s , $\mathcal{W}_{\mathbb{H}}^s$.*

Our proofs use sets $\mathcal{G}_n(W)$, $\mathcal{I}_n(W)$ associated with δ_0 and k_0 , and, for $\delta < \delta_0$, also $\mathcal{G}_n^\delta(W)$, $\mathcal{I}_n^\delta(W)$:

For $W \in \widehat{\mathcal{W}}^s$, we let $\mathcal{G}_1(W)$ denote the maximal, weakly homogeneous, connected components of $T^{-1}W$, with long pieces subdivided to have length between $\delta_0/2$ and δ_0 . Inductively, we define²⁰ $\mathcal{G}_n(W) = \cup_{W_i \in \mathcal{G}_{n-1}(W)} \mathcal{G}_1(W_i)$. Thus $\mathcal{G}_n(W)$ is the countable collection of subcurves of $T^{-n}W$ subdivided according to the extended singularity set $\mathcal{S}_{-n}^{\mathbb{H}}$, and $T^j(V)$ is weakly homogeneous for all $0 \leq j \leq n-1$ and all $V \in \mathcal{G}_n(W)$, in particular $\mathcal{G}_n(W) \subset \widehat{\mathcal{W}}_{\mathbb{H}}^s$. For each $n \geq 1$, let $L_n(W)$ denote the elements of $\mathcal{G}_n(W)$ whose length is at least $\delta_0/3$. Let $\mathcal{I}_n(W)$ denote those elements $W_i \in \mathcal{G}_n(W)$ such that $T^k W_i$ is never contained in an element of $L_{n-k}(W)$ for all $k = 0, \dots, n-1$.

Finally, for $\delta < \delta_0$, define $\mathcal{G}_n^\delta(W)$ like $\mathcal{G}_n(W)$, but subdividing long pieces into pieces of length between $\delta/2$ and δ . Similarly, denote by $L_k^\delta(W)$ those elements of $\mathcal{G}_k^\delta(W)$ having length at least $\delta/3$, and by $\mathcal{I}_n^\delta(W)$ those elements $W_i \in \mathcal{G}_n^\delta(W)$ such that $T^k W_i$ has never been contained in an element of $L_{n-k}^\delta(W)$ for all $k = 0, \dots, n-1$.

3.2. Initial Lemmas for all $t > 0$. We start with two easy lemmas. (In the present paper, the parameter ς appearing in Lemmas 3.3 and 3.4 will be zero or $1/p$, for $p > q + 1$ chosen in (4.1).)

¹⁹By weakly (q, \bar{k}_0) -homogeneous, we mean weakly homogeneous as defined in Sect. 2.1 using the parameters q and \bar{k}_0 for the homogeneity strips. Note that W is not necessarily weakly homogeneous, but each V_i is.

²⁰This definition of $\mathcal{G}_n(W)$ is as in [DZ1, DZ2], but different from [BD] where homogeneity was not required.

Lemma 3.3. Fix $t_0 \in (0, 1)$. There exists $C_0 = C_0(t_0) > 0$ such that for all $g \in C^0$, every $t \geq t_0$, all $t_1 \in (1, t_*)$, and all $0 \leq \varsigma < \min\{t, 1, \frac{2t-t_0}{2-t_0}\}$, we have

$$(3.4) \quad \sum_{W_i \in \mathcal{I}_n(W)} \frac{|W_i|^\varsigma}{|W|^\varsigma} |J_{W_i} T^n|_{C^0(W_i)}^t |e^{S_n g}|_{C^0(W_i)} \leq C_0 \theta^{n(t-\varsigma)} e^{n|g|_{C^0}}, \quad \forall W \in \widehat{\mathcal{W}}^s, \quad \forall n \geq 1.$$

Proof. The case $\varsigma = 0, g = 0$ can be proved by induction on n using (3.1), since elements of $\mathcal{I}_n(W)$ have been short at each intermediate step. This is the same as in [DZ1, Lemma 3.1] (the exponent t changes nothing), and the²¹ constant $C_0^{[\varsigma=0]}$ comes from switching from the metric induced by the adapted norm $\|\cdot\|_*$ to the standard Euclidean norm at the last step.

For $\varsigma > 0, g = 0$, we use a Hölder inequality,

$$\sum_{W_i \in \mathcal{I}_n(W)} \frac{|W_i|^\varsigma}{|W|^\varsigma} |J_{W_i} T^n|_{C^0(W_i)}^t \leq \left(\sum_{W_i \in \mathcal{I}_n(W)} \frac{|W_i|}{|W|} |J_{W_i} T^n|_{C^0(W_i)} \right)^\varsigma \left(\sum_{W_i \in \mathcal{I}_n(W)} |J_{W_i} T^n|_{C^0(W_i)}^{\frac{t-\varsigma}{1-\varsigma}} \right)^{1-\varsigma}.$$

Since $|J_{W_i} T^n|_{C^0(W_i)} \leq e^{C_d \frac{|T^n W_i|}{|W_i|}}$ by (2.3), the first sum is bounded by $e^{C_d \varsigma}$. Then, since $\frac{t-\varsigma}{1-\varsigma} \geq \frac{t_0}{2}$, Definition 3.2 and Lemma 3.1 for $\bar{t}_0 = t_0/2$, together with the case $\varsigma = 0$, imply the second sum is bounded by $(C_0^{[\varsigma=0]})^{1-\varsigma} \theta^{n(t-\varsigma)}$. This completes the proof of the lemma in the case $g = 0$. For nonzero g , use $|e^{S_n g}|_{C^0(W_i)} \leq e^{n|g|_{C^0}}$ for all W_i to bootstrap from the bound for $g = 0$. \square

Lemma 3.4. Fix $t_0 \in (0, 1)$ and $t_1 \in (1, t_*)$. Let $\tilde{t}_1 > t_0$. There exists $C_2 = C_2(t_0, t_1, \tilde{t}_1) > 0$ such that, for all $g \in C^0$ and all $\varsigma \in [0, 1]$, we have

$$(3.5) \quad \sum_{W_i \in \mathcal{G}_n(W)} \frac{|W_i|^\varsigma}{|W|^\varsigma} |J_{W_i} T^n|_{C^0(W_i)}^{t+\varsigma} |e^{S_n g}|_{C^0(W_i)} \leq C_2 Q_n(t, g), \quad \forall W \in \widehat{\mathcal{W}}^s, \quad \forall n \geq 1, \quad \forall t \in [t_0, \tilde{t}_1].$$

Proof. The case $\varsigma = 0$ and $g = 0$ is trivial since by definition each $W_i \in \mathcal{G}_n(W)$ is contained in a single element of $\mathcal{M}_0^{n, \mathbb{H}}$. Since there can be at most $2/\delta_0$ elements of $\mathcal{G}_n(W)$ in one element of $\mathcal{M}_0^{n, \mathbb{H}}$ (with $\delta_0 = \delta_0(t_0, t_1, g)$ from Definition 3.2), the lemma holds with $C_2[0] = 2\delta_0^{-1} e^{\tilde{t}_1 C}$, where C is from Lemma 2.2 (recall also footnote (21)). Next, for $\varsigma > 0$ and $g = 0$, notice that by (2.3),

$$|W_i| |J_{W_i} T^n|_{C^0(W_i)} \leq e^{C_d \delta_0^{1/(q+1)}} |T^n W_i| \leq e^{C_d \delta_0^{1/(q+1)}} |W|,$$

so that the sum for $\varsigma > 0$ is bounded by the sum for $\varsigma = 0$ times $e^{\varsigma C_d \delta_0^{1/(q+1)}}$. If $g \neq 0$, then again using Lemma 2.2 on each W_i , we have

$$|J_{W_i} T^n|_{C^0(W_i)}^t |e^{S_n g}|_{C^0(W_i)} \leq e^{tC} \sup_{W_i \cap M'} \|J^s T^n\|^t |e^{S_n g}|,$$

and the required bound follows with $C_2 = C_2[0] e^{\varsigma C_d \delta_0^{1/(q+1)}}$. \square

3.3. Growth Lemmas for $t \in (0, 1]$. In this section, we prove two growth and complexity lemmas for $t \in (0, 1]$. The first one shows that we can make the contribution from the sum over short pieces small compared to the sum over all pieces in $\mathcal{G}_n(W)$ by choosing a small length scale. Recall the constant C_* from (2.10).

Lemma 3.5. Let $t_0 \in (0, 1)$ and $t_1 \in (1, t_*)$. For any $\varepsilon > 0$, there exist $\delta_1 > 0$ and $n_1 \geq 1$ such that for all $W \in \widehat{\mathcal{W}}^s$ with $|W| \geq \delta_1/3$, all $n \geq n_1$, and all $g \in C^1$ with²²

$$(3.6) \quad 2|g|_{C^0} < -t_0 \log \theta, \quad \text{i.e.} \quad e^{|g|_{C^0} \theta^{t_0/2}} < 1, \quad \text{and} \quad |\nabla g| \leq (C_* \delta_0)^{-1},$$

²¹ The sets $\widehat{\mathcal{W}}^s$ and $\mathcal{I}_n(W)$ become smaller if t_1 is larger; while θ increases if t_1 is larger, this does not affect C_0 .

²² The bound on $|\nabla g|$ bounds the distortion constant of g by 2, so that δ_1 and n_1 may be chosen uniformly in g .

we have

$$\sum_{\substack{W_i \in \mathcal{G}_n^{\delta_1}(W) \\ |W_i| < \delta_1/3}} |J_{W_i} T^n|_{C^0(W_i)}^t |e^{S_n g}|_{C^0(W_i)} \leq \varepsilon \sum_{W_i \in \mathcal{G}_n^{\delta_1}(W)} |J_{W_i} T^n|_{C^0(W_i)}^t |e^{S_n g}|_{C^0(W_i)}, \quad \forall t \in [t_0, 1].$$

In particular, taking $\varepsilon = 1/4$ gives $\delta_1 < \delta_0$ and $n_1 \geq 1$ such that for all $n \geq n_1$, for all $g \in C^0$ satisfying (3.6), for all $W \in \widehat{\mathcal{W}}^s$ with $|W| \geq \delta_1/3$, we have

$$(3.7) \quad \sum_{W_i \in L_n^{\delta_1}(W)} |J_{W_i} T^n|_{C^0(W_i)}^t |e^{S_n g}|_{C^0(W_i)} \geq \frac{3}{4} \sum_{W_i \in \mathcal{G}_n^{\delta_1}(W)} |J_{W_i} T^n|_{C^0(W_i)}^t |e^{S_n g}|_{C^0(W_i)}, \quad \forall t \in [t_0, 1].$$

Proof. First assume $g = 0$. Let $\varepsilon > 0$ and choose $\bar{\varepsilon} > 0$ so that $6C_1^{-1}\bar{\varepsilon}/(1-\bar{\varepsilon}) < \varepsilon$ (where $C_1 \in (0, 1]$ is from (1.2)). Next, choose n_1 such that $C_0\theta^{tn_1}\Lambda^{-n_1(1-t)} < \bar{\varepsilon}$ (where C_0 is from Lemma 3.3 for $\varsigma = 0$). Recalling again that $|T^{-1}U| \leq C|U|^{1/2}$ for any $U \in \widehat{\mathcal{W}}^s$, we may choose $\delta_1 > 0$ such that, if $|U| < \delta_1$, then each homogeneous connected component of $T^{-n}U$ has length shorter than δ_0 for each $n \leq 2n_1$. Then using Lemma 3.3 with $\varsigma = 0$, if $U \in \widehat{\mathcal{W}}^s$ with $|U| \leq \delta_1$,

$$(3.8) \quad \sum_{W_i \in \mathcal{G}_n(U)} |J_{W_i} T^n|_{C^0(W_i)}^t \leq C_0\theta^{tn}, \quad \text{for all } n \leq 2n_1.$$

Now for $n \geq n_1$, write $n = kn_1 + \ell$, for some $0 \leq \ell < n_1$. Let $W \in \widehat{\mathcal{W}}^s$ with $|W| \geq \delta_1/3$. Looking only at times mn_1 , $m = 0, \dots, k-1$, we group elements $W_i \in \mathcal{G}_n(W)$ with $|W_i| < \delta_1/3$ according to the largest m such that $T^{(k-m)n_1+\ell}W_i \subset V_j \in L_{mn_1}^{\delta_1}(W)$. This is similar to using²³ the *most recent long ancestor*, except that we only look at times that are multiples of n_1 . We denote by $\bar{I}_{(k-m)n_1+\ell}^{\delta_1}(V_j)$ the set of $W_i \in \mathcal{G}_n(W)$ identified with $V_j \in L_{mn_1}^{\delta_1}(W)$ in this way. Since $|W| \geq \delta_1/3$, every element of $\mathcal{G}_n^{\delta_1}(W)$ must have a long ancestor.

Note that since $T^{(k-m')n_1+\ell}W_i$ is contained in an element of $\mathcal{G}_{m'k}(W)$ that is shorter than $\delta_1/3$ for $m' < m$, we may apply (3.8) inductively $k-m$ times. Thus,

$$\begin{aligned} \sum_{\substack{W_i \in \mathcal{G}_n^{\delta_1}(W) \\ |W_i| < \delta_1/3}} |J_{W_i} T^n|_{C^0(W_i)}^t &\leq \sum_{m=0}^{k-1} \sum_{V_j \in L_{mn_1}^{\delta_1}(W)} |J_{V_j} T^{mn_1}|_{C^0(V_j)}^t \sum_{W_i \in \bar{I}_{(k-m)n_1+\ell}^{\delta_1}(V_j)} |J_{W_i} T^{(k-m)n_1+\ell}|_{C^0(W_i)}^t \\ &\leq \sum_{m=0}^{k-1} \sum_{V_j \in L_{mn_1}^{\delta_1}(W)} |J_{V_j} T^{mn_1}|_{C^0(V_j)}^t C_0\theta^{tn_1(k-m)}. \end{aligned}$$

Next, notice that for $t \in (0, 1]$, $V \in \widehat{\mathcal{W}}^s$ and each $k \geq 1$, using $|V| = \sum_{W_i \in \mathcal{G}_k^{\delta_1}(V)} |T^k(W_i)|$,

$$(3.9) \quad \begin{aligned} \sum_{W_i \in \mathcal{G}_k^{\delta_1}(V)} |J_{W_i} T^k|_{C^0(W_i)}^t &= \sum_{W_i \in \mathcal{G}_k^{\delta_1}(V)} |J_{W_i} T^k|_{C^0(W_i)} |J_{W_i} T^k|_{C^0(W_i)}^{t-1} \\ &\geq C_1\Lambda^{k(1-t)} \sum_{W_i \in \mathcal{G}_k^{\delta_1}(V)} \frac{|T^k W_i|}{|W_i|} \geq C_1\Lambda^{k(1-t)} |V| \delta_1^{-1}. \end{aligned}$$

Also note that by the proof of Lemma 2.1, we have

$$(3.10) \quad \frac{\sup_{x \in W_i} |J_{W_i} T^n(x)|^t}{\inf_{y \in W_i} |J_{W_i} T^n(y)|^t} \leq 1 + C_d \delta_0^{1/(q+1)} \leq 2,$$

²³The most recent long ancestor for $W_i \in \mathcal{G}_n(W)$ corresponds to the maximal $m \leq n$ such that $T^{-m}W_i \subset V_j$ and $V_j \in L_m(W)$, not to be confused with the first long ancestor, see (4.15).

since $t \leq 1$. Putting these estimates together, we obtain,

$$\begin{aligned}
& \frac{\sum_{\substack{W_i \in \mathcal{G}_n^{\delta_1}(W) \\ |W_i| < \delta_1/3}} |J_{W_i} T^n|^t_{C^0(W_i)}}{\sum_{W_i \in \mathcal{G}_n^{\delta_1}(W)} |J_{W_i} T^n|^t_{C^0(W_i)}} \\
& \leq \sum_{m=0}^{k-1} \frac{2 \sum_{V_j \in L_{mn_1}^{\delta_1}(W)} |J_{V_j} T^{mn_1}|^t_{C^0(V_j)} C_0 \theta^{tn_1(k-m)}}{\sum_{V_j \in L_{mn_1}^{\delta_1}(W)} |J_{V_j} T^{mn_1}|^t_{C^0(V_j)} \sum_{W_i \in \mathcal{G}_{(k-m)n_1+\ell}^{\delta_1}(V_j)} |J_{W_i} T^{(k-m)n_1+\ell}|^t_{C^0(W_i)}} \\
& \leq \sum_{m=0}^{k-1} \frac{2 \sum_{V_j \in L_{mn_1}^{\delta_1}(W)} |J_{V_j} T^{mn_1}|^t_{C^0(V_j)} C_0 \theta^{tn_1(k-m)}}{\sum_{V_j \in L_{mn_1}^{\delta_1}(W)} |J_{V_j} T^{mn_1}|^t_{C^0(V_j)} C_1 \Lambda^{(k-m)n_1(1-t)} |V_j| \delta_1^{-1}} \\
(3.11) \quad & \leq 6C_1^{-1} \sum_{m=0}^{k-1} \bar{\varepsilon}^{k-m} \leq 6C_1^{-1} \frac{\bar{\varepsilon}}{1-\bar{\varepsilon}},
\end{aligned}$$

where in the second inequality we used (3.9) on each $V_j \in L_{mn_1}^{\delta_1}(W)$. This ends the case $g = 0$.

If $g \neq 0$, note that by (2.10) and using (3.6),

$$\frac{\sup_{x \in W_i} e^{S_k g(x)}}{\inf_{y \in W_i} e^{S_k g(y)}} \leq 1 + C_* |\nabla g| d(x, y) \leq 2,$$

for all $W_i \in \mathcal{G}_k^{\delta_0}(W)$ and any $k > 0$. Letting $\varepsilon > 0$, choose $\bar{\varepsilon} > 0$ such that $12C_1^{-1}\bar{\varepsilon}/(1-\bar{\varepsilon}) < \varepsilon$. Then we take n_1 such that $C_0 \theta^{tn_1} e^{2n_1|g|_{C^0}} \Lambda^{-n_1(1-t)} < \bar{\varepsilon}$, and we choose $\delta_1 > 0$ such that, if $U \in \widehat{\mathcal{W}}^s$ satisfies $|U| < \delta_1$, then

$$(3.12) \quad \sum_{W_i \in \mathcal{G}_n(U)} |J_{W_i} T^n|^t_{C^0(W_i)} |e^{S_n g}|_{C^0(W_i)} \leq C_0 \theta^{tn} e^{n|g|_{C^0}}, \quad \text{for all } n \leq 2n_1,$$

which is the analogue of (3.8). (For fixed t_0 , note that n_1 and δ_1 depend only on ε , uniformly in g satisfying (3.6).) The proof above can then be followed line by line, inserting $e^{S_n g}$. Thus (3.9) becomes,

$$(3.13) \quad \sum_{W_i \in \mathcal{G}_k^{\delta_1}(V)} |J_{W_i} T^k|^t_{C^0(W_i)} |e^{S_k g}|_{C^0(W_i)} \geq C_1 \Lambda^{k(1-t)} e^{-k|g|_{C^0}} |V| \delta_1^{-1}.$$

Inserting this lower bound in (3.11) and applying Lemma 3.3 with $\varsigma = 0$ yields,

$$(3.14) \quad \frac{\sum_{\substack{W_i \in \mathcal{G}_n^{\delta_1}(W) \\ |W_i| < \delta_1/3}} |J_{W_i} T^n|^t_{C^0(W_i)} |e^{S_n g}|_{C^0(W_i)}}{\sum_{W_i \in \mathcal{G}_n^{\delta_1}(W)} |J_{W_i} T^n|^t_{C^0(W_i)} |e^{S_n g}|_{C^0(W_i)}}$$

$$\begin{aligned}
& \leq \sum_{m=0}^{k-1} \frac{4 \sum_{V_j \in L_{mn_1}^{\delta_1}(W)} |J_{V_j} T^{mn_1}|_{C^0(V_j)}^t |e^{S_{mn_1}g}|_{C^0(V_j)} C_0 \theta^{tn_1(k-m)} e^{n_1(k-m)|g|_{C^0}}}{\sum_{V_j \in L_{mn_1}^{\delta_1}(W)} |J_{V_j} T^{mn_1}|_{C^0(V_j)}^t |e^{S_{mn_1}g}|_{C^0(V_j)} C_1 \Lambda^{(k-m)n_1(1-t)} e^{-n_1(k-m)|g|_{C^0}} |V_j| \delta_1^{-1}} \\
& \leq 12C_1^{-1} \sum_{m=0}^{k-1} \bar{\varepsilon}^{k-m} \leq 12C_1^{-1} \frac{\bar{\varepsilon}}{1-\bar{\varepsilon}} < \varepsilon,
\end{aligned}$$

where the extra factor of 2 comes from the distortion constant of g . \square

The second lemma proves the analogue of Lemma 3.5 for elements of $\mathcal{M}_0^{n, \mathbb{H}}$, in anticipation of Proposition 3.14. For $A \in \mathcal{M}_0^{n, \mathbb{H}}$, let $B_{n-1}(A)$ denote the element of $\mathcal{M}_{-n+1}^{0, \mathbb{H}} \vee \mathcal{H}$ containing $T^{n-1}A \in \mathcal{M}_{-n+1}^{1, \mathbb{H}}$, recalling that \mathcal{H} is the partition of M into homogeneity strips \mathbb{H}_k . We introduce this additional intersection with \mathcal{H} (omitted from the definition of $\mathcal{M}_{-n+1}^{0, \mathbb{H}}$) since it will be convenient to work with homogeneous partition elements in what follows. For $\delta > 0$, define

$$(3.15) \quad \mathcal{A}_n(\delta) = \{A \in \mathcal{M}_0^{n, \mathbb{H}} : \text{diam}^u(B_{n-1}(A)) \geq \delta/3\}.$$

The following result shows that most of the weights contributing to $Q_n(t, g)$ come from elements of $\mathcal{A}_n(\delta)$ if δ is chosen small enough.

Lemma 3.6. *Let $t_0 > 0$. For any $v \geq 0$, there exist $\delta_2 > 0$ and $c_0 = c_0(v) > 0$ with $c_0(v') \geq c_0(v)$ if $v' \in [0, v]$, such that for any $g \in C^1$ satisfying (3.6) with $|\nabla g|_{C^0} \leq v$,*

$$\sum_{A \in \mathcal{A}_n(\delta_2)} \sup_{x \in A \cap M'} |J^s T^n(x)|^t e^{S_n g(x)} \geq c_0 Q_n(t, g), \quad \forall n \in \mathbb{N}, \quad \forall t \in [t_0, 1].$$

Proof. Assume first $g = 0$. We begin by relating $J^s T^n$ on $A \in \mathcal{M}_0^{n, \mathbb{H}}$ with $J^u T^{-n+1}$ on $T^{n-1}A$. Recalling the definition of $E(y)$ in (1.5), if $x \in A$ and $y = T^{n-1}x$, we have

$$J^s T^n(x) = \frac{(E \cos \varphi) \circ T^{-n}(y)}{(E \cos \varphi)(y)} J^u T^{-n}(y).$$

Here, $J^u T^{-n} = \det(DT^{-n}|_{E^u})$, where E^u is the unstable direction for T (not T^{-1}), so that $J^u T^{-n}$ is a contraction. Next, since²⁴ $J^u T^{-1}(y) = C^{\pm 1} \cos \varphi(y)$, and the function E is uniformly bounded away from 0, we conclude,

$$(3.16) \quad J^s T^n(x) = C^{\pm 1} \cos \varphi(T^{-n}y) J^u T^{-n+1}(T^{-1}y).$$

For brevity, for any set $A \subset M$, we will denote

$$(3.17) \quad |J^s T^n|_A^t := \sup_{x \in A \cap M'} |J^s T^n(x)|^t \quad \text{and similarly,} \quad |J^u T^{-n}|_A^t := \sup_{x \in A \cap M'} |J^u T^{-n}(x)|^t.$$

Next, we consider the evolution of elements of $\mathcal{M}_{-k}^{0, \mathbb{H}}$ under iteration by T^j for $j \geq 1$. If $B \in \mathcal{M}_{-k}^{0, \mathbb{H}}$, then we subdivide $T^j B$ according to singularity curves and homogeneity strips at each iterate, much as we would consider the evolution of an unstable curve U under T^j . We write $T^j B = \cup_{B' \in G_j(B)} B'$, where $G_j(B)$ is the maximal decomposition of $T^j B$ into elements of $\mathcal{M}_{-k-j}^{0, \mathbb{H}} \vee \mathcal{H}$, recalling that \mathcal{H} denotes the partition of M according to homogeneity strips. This last intersection with \mathcal{H} is necessary since we will work with homogeneous elements $B' \subset T^j B$ (to maintain bounded distortion for $J^u T^{-j}$ on B'). Let $L_j^\delta(B)$ denote those elements $B' \in G_j(B)$ with $\text{diam}^u(B') \geq \delta/3$.

²⁴We use the notation $A = C^{\pm 1}B$ to denote $C^{-1}B \leq A \leq CB$.

Now by Definition 3.2 and applying the time reversal of the proof of Lemma 3.3 (with $\varsigma = 0$), there exists $\delta_2 > 0$ such that if $\max\{\text{diam}^u(B), \text{diam}^s(B)\} \leq \delta_2$, then

$$(3.18) \quad \sum_{B' \in G_j(B)} |J^u T^{-j}|_{B'}^t \leq C_0 \theta^{tj}, \quad \text{for all } j \leq n_1,$$

where $n_1 = n_1(1/4)$ is from (3.7) in Lemma 3.5. For convenience, we choose $\delta_2 \leq \delta_1(1/4)$. Also, if $B \in \mathcal{M}_{-k}^{0, \mathbb{H}}$, then $\text{diam}^s(B) \leq C\Lambda^{-k}$ for some uniform constant $C > 0$. We choose $n_2 \geq n_1$ so that $\text{diam}^s(B) \leq \delta_2$ if $B \in \mathcal{M}_{-k}^{0, \mathbb{H}}$ for $k \geq n_2$.

We fix $n \geq n_2 + 1$ and prove the lemma for such n . For $B \in \mathcal{M}_{-n+1}^{0, \mathbb{H}} \vee \mathcal{H}$, let B_{-j} denote the element of $\mathcal{M}_{-n+1+j}^{0, \mathbb{H}} \vee \mathcal{H}$ containing $T^{-j}B$. We call B_{-j} the most recent u -long ancestor of B if j is the minimal integer $k \leq n - n_2$ such that $\text{diam}^u(B_{-k}) \geq \delta_2$. If no such j exists, we say that B has been u -short since time n_2 . (It follows from the definition of n_2 , that $\text{diam}^s(B_{-j}) \leq \delta_2$ for all $j \leq n - n_2 - 1$.) Let $\mathbb{L}_{-n+1+j}^{\delta_2}$ denote those elements of $\mathcal{M}_{-n+1+j}^{0, \mathbb{H}} \vee \mathcal{H}$ which are u -long, and let $\mathbb{S}_{-n+1+j}^{\delta_2}$ denote those elements which are u -short (in the length scale δ_2). Similarly, let $\mathbb{I}_j^{\delta_2}(B_{-j})$ denote the collection of $B \in \mathcal{M}_{-n+1}^{0, \mathbb{H}} \vee \mathcal{H}$ whose most recent u -long ancestor is B_{-j} . Note that $\mathbb{I}_j^{\delta_2}(B_{-j}) \subset G_j(B_{-j})$.

Thus if $k \geq n_2$ and $B' \in \mathcal{M}_{-k}^{0, \mathbb{H}}$, then estimating inductively as in the proof of Lemma 3.3,

$$(3.19) \quad \sum_{B \in \mathbb{I}_j^{\delta_2}(B')} |J^u T^{-j}|_B^t \leq C_0 \delta_2^{-1} \theta^{tj} \text{ for all } j \geq 0,$$

where the factor δ_2^{-1} is due to the fact that B' itself may be u -long, in which case it would be artificially subdivided into $\sim \delta_2^{-1}$ pieces of u -diameter less than δ_2 before being iterated.

Let $\mathcal{A}_n^c(\delta_2) = \mathcal{M}_0^{n, \mathbb{H}} \setminus \mathcal{A}_n(\delta_2)$. By (3.16),

$$\sum_{A \in \mathcal{A}_n^c(\delta_2)} |J^s T^n|_A^t = C^{\pm 1} \sum_{A \in \mathcal{A}_n^c(\delta_2)} |\cos \varphi|_A^t |J^u T^{-n+1}|_{T^{n-1}A}^t.$$

Note that if $B \in \mathcal{M}_{-n+1}^{0, \mathbb{H}} \vee \mathcal{H}$ with $\text{diam}^u(B) < \delta_2/3$, then any $A \in \mathcal{M}_0^{n, \mathbb{H}}$ for which $B = B_{n-1}(A)$ belongs to $\mathcal{A}_n^c(\delta_2)$. Also,

$$T^{n-1}A \in \mathcal{M}_{-n+1}^{1, \mathbb{H}} = \mathcal{M}_{-n+1}^{0, \mathbb{H}} \vee \mathcal{H} \vee \mathcal{M}_0^1$$

so that for fixed B , the number of A such that $B_{n-1}(A) = B$ is at most $\#\mathcal{M}_0^1$. Moreover, since all such A are by definition contained in $T^{-n+1}(B_{n-1}(A)) = T^{-n+1}B$, and $T^{-n+1}B \in \mathcal{M}_0^{n-1, \mathbb{H}} \vee T^{-n+1}\mathcal{H}$, all A corresponding to one B are contained in the same homogeneity strip, so that $|\cos \varphi|_A$ is comparable on all such A . In addition, since $\mathcal{M}_{-n+1}^{0, \mathbb{H}} \vee \mathcal{H}$ is the partition into connected components of $M \setminus \cup_{i=0}^n T^i \mathcal{S}_0^{\mathbb{H}}$, it follows that T^{-k} is smooth on B , and $T^{-k}B$ belongs to a single homogeneity strip for each $0 \leq k \leq n-1$. Then, applying the time reversal of Lemma 2.2 to unstable curves, we conclude that $|J^u T^{-n+1}|_{T^{n-1}A}$ is comparable to $|J^u T^{-n+1}|_B$ for each A such that $B_{n-1}(A) = B$. Thus,

$$(3.20) \quad \sum_{A \in \mathcal{A}_n^c(\delta_2)} |J^s T^n|_A^t = C^{\pm 1} \sum_{B \in \mathbb{S}_{-n+1}^{\delta_2}} |\cos \varphi|_{T^{-n+1}B}^t |J^u T^{-n+1}|_B^t,$$

and similarly,

$$(3.21) \quad \sum_{A \in \mathcal{A}_n(\delta_2)} |J^s T^n|_A^t = C^{\pm 1} \sum_{B \in \mathbb{L}_{-n+1}^{\delta_2}} |\cos \varphi|_{T^{-n+1}B}^t |J^u T^{-n+1}|_B^t.$$

Next, we group elements of $\mathbb{S}_{-n+1}^{\delta_2}$ by most recent u -long ancestor in $\mathcal{M}_{-n+1+j}^{0, \mathbb{H}}$, as described above. By (3.18), there is no need to consider long ancestors for $j < n_1$. Note that if $B' \in \mathcal{M}_{-n+1+j}^{0, \mathbb{H}} \vee \mathcal{H}$

and $B \in \mathbb{I}_j^{\delta_2}(B')$, then $T^{-n+1}B$ lies in the same homogeneity strip as $T^{-n+1+j}B'$, so that $\cos \varphi$ is comparable on each of these sets. Thus, by (3.18)–(3.19),

$$\begin{aligned}
(3.22) \quad & \sum_{B \in \mathbb{S}_{-n+1}^{\delta_2}} |\cos \varphi|_{T^{-n+1}B}^t |J^u T^{-n+1}|_B^t \\
&= \sum_{j=n_1}^{n-n_2-1} \sum_{B' \in \mathbb{L}_{-n+1+j}^{\delta_2}} \sum_{B \in \mathbb{I}_j^{\delta_2}(B')} |\cos \varphi|_{T^{-n+1}B}^t |J^u T^{-j}|_B^t |J^u T^{-n+1+j}|_B^t \\
&\quad + \sum_{B' \in \mathbb{S}_{-n_2}^{\delta_2}} \sum_{B \in \mathbb{I}_{n-n_2}^{\delta_2}(B')} |\cos \varphi|_{T^{-n+1}B}^t |J^u T^{-j}|_B^t |J^u T^{-n+1+j}|_B^t \\
&\leq \sum_{j=n_1}^{n-n_2-1} \sum_{B' \in \mathbb{L}_{-n+1+j}^{\delta_2}} C \delta_2^{-1} \theta^{tj} |\cos \varphi|_{T^{-n+1+j}B'}^t |J^u T^{-n+1+j}|_B^t \\
&\quad + \sum_{B' \in \mathbb{S}_{-n_2}^{\delta_2}} C \theta^{t(n-n_2-1)} |\cos \varphi|_{T^{-n_2}B'}^t |J^u T^{-n_2}|_B^t,
\end{aligned}$$

where the final sum over $B' \in \mathbb{S}_{-n_2}^{\delta_2}$ represents those $B \in \mathbb{S}_{-n+1}^{\delta_2}$ which have had no u -long ancestor since before time n_2 .

To proceed, we will need the following sublemma, linking the contribution from $\mathbb{L}_{-n+1+j}^{\delta_2}$ to the contribution from $\mathbb{L}_{-n+1}^{\delta_2}$.

Sublemma 3.7. *Let $t_0 \in (0, 1)$. There exists $C > 0$ such that for all $t \in [t_0, 1]$, each $n_1 \leq j \leq n - n_2 - 1$, and all $B' \in \mathbb{L}_{-n+1+j}^{\delta_2}$,*

$$|\cos \varphi|_{T^{-n+1+j}B'}^t |J^u T^{-n+1+j}|_B^t \leq C \delta_2^{-1} \Lambda^{j(t-1)} \sum_{B \in L_j(B')} |\cos \varphi|_{T^{-n+1}B}^t |J^u T^{-n+1}|_B^t,$$

where $L_j(B')$ denotes the collection of elements $B \in G_j(B')$ with $\text{diam}^u(B) \geq \delta_2/3$.

Proof. Since $B' \in \mathbb{L}_{-n+1+j}^{\delta_2}$, there exists an unstable curve $U \subset B'$ with $|U| \geq \delta_2/3$. Let $G'_j(B')$ denote those elements $B \in G_j(B')$ such that $T^j U \cap B \neq \emptyset$. Letting $\mathcal{G}_j^{\delta_2}(U)$ denote the j th generation of homogeneous elements of $T^j U$, using the time reversed definition of $\mathcal{G}_j^{\delta_2}(W)$ for stable curves from Section 3.1. If $U_i \in \mathcal{G}_j^{\delta_2}(U)$ has $|U_i| \geq \delta_2/3$, and $B \cap U_i \neq \emptyset$ for some $B \in G'_j(B')$, then necessarily, $\text{diam}^u(B) \geq \delta_2/3$. Let $L'_j(B') \subset L_j(B')$ denote this collection of long elements. Then letting $L_j^{\delta_2}(U) \subset \mathcal{G}_j^{\delta_2}(U)$ denote those elements of $\mathcal{G}_j^{\delta_2}(U)$ with length at least $\delta_2/3$, we estimate,

$$\begin{aligned}
(3.23) \quad & \sum_{B \in L_j(B')} |\cos \varphi|_{T^{-n+1}B}^t |J^u T^{-n+1}|_B^t \\
&\geq C |\cos \varphi|_{T^{-n+1+j}B'}^t |J^u T^{-n+1+j}|_B^t \sum_{B \in L'_j(B')} |J^u T^{-j}|_B^t \\
&\geq C' |\cos \varphi|_{T^{-n+1+j}B'}^t |J^u T^{-n+1+j}|_B^t \delta_2 \sum_{U_i \in L_j^{\delta_2}(U)} |J_{U_i} T^{-j}|_{C^0(U_i)}^t \\
&\geq C'' \delta_2 |\cos \varphi|_{T^{-n+1+j}B'}^t |J^u T^{-n+1+j}|_B^t \frac{3}{4} \sum_{U_i \in \mathcal{G}_j^{\delta_2}(U)} |J_{U_i} T^{-j}|_{C^0(U_i)}^t \\
&\geq C''' \delta_2 |\cos \varphi|_{T^{-n+1+j}B'}^t |J^u T^{-n+1+j}|_B^t \Lambda^{j(1-t)} \sum_{U_i \in \mathcal{G}_j^{\delta_2}(U)} |J_{U_i} T^{-j}|_{C^0(U_i)}^t
\end{aligned}$$

$$\geq C'' \delta_2 |\cos \varphi|_{T^{-n+1+j}B'}^t |J^u T^{-n+1+j}|_{B'}^t \Lambda^{j(1-t)} \frac{|U|}{\delta_2},$$

where in the first inequality we have used the fact that $\cos \varphi$ is comparable on $T^{-n+1}B$ and $T^{-n+1+j}B'$, in the second inequality we have applied the time reversal of Lemma 2.2 and the factor δ_2 appears since there may be up to $\sim \delta_2^{-1}$ elements of $L_j^{\delta_2}(U)$ in each element $B \in L_j'(B')$ (due to artificial subdivisions in the definition of $\mathcal{G}_j^{\delta_2}(U)$), and in the third inequality we have applied the time reversal of Lemma 3.5 and (3.7) from Lemma 3.5 since $\delta_2 \leq \delta_1$ and $j \geq n_1$. Since $|U| \geq \delta_2/3$, this completes the proof of the sublemma. \square

Using the sublemma, we now estimate the right hand side of (3.22), summing over $B' \in \mathbb{L}_{-n+1+j}^{\delta_2}$ and noting that if $B \in L_j(B')$, then $B \in \mathbb{L}_{-n+1}^{\delta_2}$ and each such B is associated with a unique B' :

$$\begin{aligned} \sum_{B \in \mathbb{S}_{-n+1}^{\delta_2}} |\cos \varphi|_{T^{-n+1}B}^t |J^u T^{-n+1}|_B^t &\leq \sum_{B' \in \mathbb{S}_{-n_2}^{\delta_2}} C \theta^{t(n-n_2-1)} |\cos \varphi|_{T^{-n_2}B'}^t |J^u T^{-n_2}|_{B'}^t \\ &\quad + \sum_{j=n_1}^{n-n_2-1} C \delta_2^{-2} \theta^{tj} \Lambda^{j(t-1)} \sum_{B \in \mathbb{L}_{-n+1}^{\delta_2}} |\cos \varphi|_{T^{-n+1}B}^t |J^u T^{-n+1}|_B^t \\ &\leq C_{n_2}(t_0) \theta^{tn} + C \delta_2^{-2} \sum_{B \in \mathbb{L}_{-n+1}^{\delta_2}} |\cos \varphi|_{T^{-n+1}B}^t |J^u T^{-n+1}|_B^t, \end{aligned}$$

for some constant $C_{n_2}(t_0) > 0$ depending only on n_2 and t_0 . To see the estimate on the first term, note that by (3.16), since elements of $\mathbb{S}_{-n_2}^{\delta_2}$ are connected components of $M \setminus \cup_{i=0}^{n_2+1} T^i \mathcal{S}_0^{\mathbb{H}}$, we have

$$\sum_{B' \in \mathbb{S}_{-n_2}^{\delta_2}} |\cos \varphi|_{T^{-n_2}B'}^t |J^u T^{-n_2}|_{B'}^t \leq C \sum_{A \in \mathcal{M}_0^{n_2+1, \mathbb{H}}} |J^s T^{n_2+1}|_A^t \leq C Q_{n_2+1}(t) \leq C Q_{n_2+1}(t_0).$$

Next, note that the sum over $\mathbb{L}_{-n+1}^{\delta_2}$ grows at a rate of at least $C \Lambda^{n(1-t)}$ by the proof of the sublemma. Thus we may choose $n_3 \geq n_2$ large enough that $C_{n_2}(t_0) \theta^{tn} \leq C \Lambda^{n(1-t)}$ for all $n \geq n_3$, which implies,

$$\sum_{B \in \mathbb{S}_{-n+1}^{\delta_2}} |\cos \varphi|_{T^{-n+1}B}^t |J^u T^{-n+1}|_B^t \leq C \delta_2^{-2} \sum_{B \in \mathbb{L}_{-n+1}^{\delta_2}} |\cos \varphi|_{T^{-n+1}B}^t |J^u T^{-n+1}|_B^t.$$

Using this estimate with (3.20), (3.21) and the fact that $\mathcal{A}_n(\delta_2) \cup \mathcal{A}_n^c(\delta_2) = \mathcal{M}_0^{n, \mathbb{H}}$ yields,

$$\begin{aligned} Q_n(t) &= \sum_{A \in \mathcal{A}_n^c(\delta_2)} |J^s T^n|_A^t + \sum_{A \in \mathcal{A}_n(\delta_2)} |J^s T^n|_A^t \\ &\leq C(\delta_2^{-2} + 1) \sum_{B \in \mathbb{L}_{-n+1}^{\delta_2}} |\cos \varphi|_{T^{-n+1}B}^t |J^u T^{-n+1}|_B^t \leq C(\delta_2^{-2} + 1) \sum_{A \in \mathcal{A}_n(\delta_2)} |J^s T^n|_A^t, \end{aligned}$$

completing the proof of the lemma for $n \geq n_3$ and $g = 0$. The statement for general n (and $g = 0$) follows, possibly reducing the constant c_0 , since there are only finitely many n to correct for.

If $g \neq 0$, the proof remains as is until (3.18) with the same choices of n_2 and δ_2 (these choices are independent of g), so that (3.18) holds with $|e^{S_j g}|_{T^{-j}B'}$ inserted in the left-hand side and $e^{j|g|c^0}$ in the right. The analogous modification is made to (3.19). Then (3.20) becomes

$$(3.24) \quad \sum_{A \in \mathcal{A}_n^c(\delta_2)} |J^s T^n|_A^t |e^{S_n g}|_A = C^{\pm 1} \sum_{B \in \mathbb{S}_{-n+1}^{\delta_2}} |\cos \varphi|_{T^{-n+1}B}^t |J^u T^{-n+1}|_B^t |e^{S_{n-1} g}|_{T^{-n+1}B},$$

where C depends on $|\nabla g|_{C^0}$ via (2.15), with the analogous modification to (3.21). Then (3.22) is modified in the obvious way for $n \geq n_2 + 1$,

$$\begin{aligned}
(3.25) \quad & \sum_{B \in \mathbb{S}_{-n+1}^{\delta_2}} |\cos \varphi|_{T^{-n+1}B}^t |J^u T^{-n+1}|_B^t |e^{S_{n-1}g}|_{T^{-n+1}B} \\
& \leq \sum_{j=n_1}^{n-n_2-1} \sum_{B' \in \mathbb{L}_{-n+1+j}^{\delta_2}} C \delta_1^{-1} \theta^{tj} e^{j|g|_{C^0}} |\cos \varphi|_{T^{-n+1+j}B'}^t |J^u T^{-n+1+j}|_{B'}^t |e^{S_{n-1-j}g}|_{T^{-n+1+j}B'} \\
& + \sum_{B' \in \mathbb{S}_{-n_2}^{\delta_2}} C \theta^{t(n-n_2-1)} e^{(n-n_2-1)|g|_{C^0}} |\cos \varphi|_{T^{-n_2}B'}^t |J^u T^{-n_2}|_{B'}^t |e^{S_{n_2}g}|_{T^{-n_2}B'}.
\end{aligned}$$

A suitable analogue of Sublemma 3.7 yields $C > 0$ such that for each $n_1 \leq j \leq n - n_2 - 1$ and $B' \in \mathbb{L}_{-n+1+j}^{\delta_2}$,

$$\begin{aligned}
& |\cos \varphi|_{T^{-n+1+j}B'}^t |J^u T^{-n+1+j}|_{B'}^t |e^{S_{n-1-j}g}|_{T^{-n+1+j}B'} \\
& \leq C \delta_2^{-1} \Lambda^{j(t-1)} e^{j|g|_{C^0}} \sum_{B \in L_j(B')} |\cos \varphi|_{T^{-n+1}B}^t |J^u T^{-n+1}|_B^t |e^{S_{n-1}g}|_{T^{-n+1}B},
\end{aligned}$$

where we have used the lower bound (3.13) rather than (3.9) in (3.23). This provides the contraction required to complete the proof of Lemma 3.6 since $\theta^t e^{|g|_{C^0}} < 1$ by (3.6). \square

3.4. Defining $s_1 > 1$. Growth Lemmas for $t \in (1, s_1)$. In this section, we bootstrap from our results for $t \leq 1$ to conclude a parallel set of results for $t \in (1, s_1)$, for $s_1 > 1$ from Definition 3.9 below. To do this, we will apply Propositions 3.14 and 3.15 from Section 3.5 for $t \leq 1$ whose proofs rely only on the lemmas in Section 3.3. In Section 3.6, we show how to extend this to all $t < t_*$.

The easy lemma below will be crucial to define s_1 :

Lemma 3.8. *We have $P_*(1) = 0$. Moreover, the limit $\chi_1 := \lim_{s \rightarrow 1^-} \frac{P_*(s)}{1-s}$ exists and $\chi_1 \geq \log \Lambda > 0$.*

In fact, $\chi_1 = \int_M \log J^u T \, d\mu_{\text{SRB}}$, which follows from Theorems 1.2 and 2.5.

Proof. Proposition 3.14 for $t = 1$ together with [DZ1, Lemma 3.2] prove that $Q_n(1)$ is uniformly bounded for all n , so that $P_*(1) \leq 0$. Since Proposition 2.4 gives $P_*(1) \geq P(1) = 0$, we have established that $P_*(1) = 0$. Next, the convexity of $P_*(t)$ (Proposition 2.6) on $(0, \infty)$ implies that left (and right) derivatives exist at every $t > 0$. Thus, since $P_*(1) = 0$, the limit below exists

$$(3.26) \quad \lim_{s \rightarrow 1^-} \frac{P_*(s)}{1-s} = \lim_{s \rightarrow 1^-} \frac{P_*(s) - P_*(1)}{1-s}.$$

The proof that $P(t)$ is strictly decreasing in Proposition 2.6 implies $\chi_1 \geq \log \Lambda > 0$. \square

Definition 3.9. *Recalling $\theta(t_1) \in (\Lambda^{-1}, \Lambda^{-1/2})$ from Definition 3.2, we define $s_1 := \frac{\chi_1}{\chi_1 + \log \theta} > 1$.*

Note that s_1 is just the intersection point between the tangent line to $P_*(t)$ at $t = 1$ (which is the largest t where we have established the lower bound (3.9) on the sum over $\mathcal{G}_n(W)$) and the line $y = t \log \theta$. If $t < s_1$ then $\theta^t < e^{P_*(t)}$, which can be viewed as a pressure gap condition. Note finally that establishing Theorem 2.5. in a neighbourhood of $t = 1$ will give $s_1 \leq t_*$.

A key to many results for $0 < t \leq 1$ is the lower bound on the rate of growth given by (3.9) in the proof of Lemma 3.5. Our next lemma obtains this lower bound for $t \geq 1$, interpolating via a Hölder inequality.

Lemma 3.10. *Let $t_0 \in (0, 1)$ and $t_1 \in (1, t_*)$. Let $\bar{t}_1 \geq 1$. For any $\kappa > 0$, there exist $C_\kappa > 0$, $\eta_\kappa > 0$ such that for all $g \in C^0$ and $\delta > 0$, and all $W \in \widehat{\mathcal{W}}^s$ with $|W| \geq \delta/3$,*

$$(3.27) \quad \sum_{W_i \in \mathcal{G}_n^\delta(W)} |J_{W_i} T^n|_{C^0(W_i)}^t |e^{S_n g}|_{C^0(W_i)} \geq C_\kappa \delta^{\frac{1}{\eta_\kappa} - 1} e^{-n(\chi_1 + \kappa)(t-1) - n|g|_{C^0}}, \quad \forall n \geq 1, \quad \forall t \in [1, \bar{t}_1].$$

Moreover, if $|W| \geq \delta_0/3$, then

$$\sum_{W_i \in \mathcal{G}_n^\delta(W)} |J_{W_i} T^n|_{C^0(W_i)}^t |e^{S_n g}|_{C^0(W_i)} \geq C_\kappa \delta^{-1} e^{-n(\chi_1 + \kappa)(t-1) - n|g|_{C^0}}, \quad \forall n \geq 1, \quad \forall t \in [1, \bar{t}_1].$$

Proof. Assume first $g = 0$. For $t \geq 1$, we have for any $s \in (0, 1)$, taking $\eta(s) \in (0, 1]$ such that $\eta t + (1 - \eta)s = 1$, that $\sum_i a_i = \sum_i a_i^{\eta t + (1-\eta)s} \leq (\sum_i a_i^t)^\eta (\sum_i a_i^s)^{1-\eta}$ for any positive numbers a_i . It follows that for any $W \in \widehat{\mathcal{W}}^s$ with $|W| \geq \delta/3$ and all $n \geq 1$,

$$(3.28) \quad \sum_{W_i \in \mathcal{G}_n^\delta(W)} |J_{W_i} T^n|^t \geq \frac{\left(\sum_{W_i \in \mathcal{G}_n^\delta(W)} |J_{W_i} T^n|\right)^{1/\eta}}{\left(\sum_{W_i \in \mathcal{G}_n^\delta(W)} |J_{W_i} T^n|^s\right)^{(1-\eta)/\eta}} \geq \left(\frac{C_1}{3}\right)^{1/\eta} \left(C_2 [0] \frac{\delta_0}{\delta} \frac{2}{c_2} e^{nP_*(s)}\right)^{(\eta-1)/\eta},$$

where we have used (3.9) for the lower bound in the numerator, and Lemma 3.4 with $\varsigma = 0$ and $\tilde{t}_1 = 1$ combined with Proposition 3.15 for the upper bound in the denominator. The factor δ_0/δ comes from the fact that here we use $\mathcal{G}_n^\delta(W)$, while Lemma 3.4 uses $\mathcal{G}_n^{\delta_0}(W)$. Since $\eta = (1-s)/(t-s)$, $e^{nP_*(s)(\eta-1)/\eta} = e^{-n(t-1)P_*(s)/(1-s)}$. For fixed $\kappa > 0$, Lemma 3.8 allows us to choose $s = s(\kappa) \in (0, 1)$ (and hence $\eta_\kappa = \eta(s) > 0$) such that $P_*(s)/(1-s) \leq \chi_1 + \kappa$, completing the proof for $g = 0$ since $\eta(s) > (1-s)/\bar{t}_1$. For $g \neq 0$, (3.27) follows since $|e^{S_n g}|_{C^0(W_i)} \geq e^{-n|g|_{C^0}}$ for each W_i .

For the second inequality of the lemma when $|W| \geq \delta_0/3$, notice that (3.9) gives a lower bound of $\frac{C_1 \delta_0}{3\delta}$ in this case. The rest of the estimate is the same (up to C_κ changing by a power of δ_0). \square

By definition, $\theta^t e^{\chi_1(t-1)} < 1$ if $t < s_1$. Thus for $\bar{t}_1 \in (1, s_1)$ there exists $\kappa_1 = \kappa(\bar{t}_1) > 0$ such that

$$(3.29) \quad \theta^{\bar{t}_1} e^{(\chi_1 + \kappa_1)(\bar{t}_1 - 1)} < 1, \quad \text{and thus } \theta^t e^{(\chi_1 + \kappa_1)(t-1)} < 1, \quad \forall t \leq \bar{t}_1.$$

Our next lemma is the analogue of Lemma 3.5 for $t > 1$.

Lemma 3.11. *Let $t_0 \in (0, 1)$, $t_1 \in (1, t_*)$ and $\bar{t}_1 \in (1, s_1)$. Let $\kappa_1 = \kappa(\bar{t}_1)$ satisfy (3.29). Then for any $\varepsilon > 0$ there exist $\delta_1 > 0$ and $n_1 \geq 1$, such that²⁵ for all $W \in \widehat{\mathcal{W}}^s$ with $|W| \geq \delta_1/3$,*

$$\sum_{\substack{W_i \in \mathcal{G}_n^{\delta_1}(W) \\ |W_i| < \delta_1/3}} |J_{W_i} T^n|_{C^0(W_i)}^t |e^{S_n g}|_{C^0(W_i)} \leq \varepsilon \sum_{W_i \in \mathcal{G}_n^{\delta_1}(W)} |J_{W_i} T^n|_{C^0(W_i)}^t |e^{S_n g}|_{C^0(W_i)}, \quad \forall t \in [1, \bar{t}_1], \quad \forall n \geq n_1,$$

for all $g \in C^1$ satisfying (3.6) and such that, in addition,

$$(3.30) \quad 2|g|_{C^0} < -\bar{t}_1 \log \theta - (\chi_1 + \kappa_1)(\bar{t}_1 - 1), \quad \text{i.e. } \theta^{\bar{t}_1} e^{(\chi_1 + \kappa_1)(\bar{t}_1 - 1) + 2|g|_{C^0}} < 1.$$

Let $[t_0, \bar{t}_1] \subset (0, s_1)$. For all $g \in C^1$ satisfying (3.6) and (3.30), Lemma 3.5 and Lemma 3.11 for $\varepsilon = 1/4$ give $n_1 \geq 1$ and $\delta_1 > 0$ such that for all $n \geq n_1$ and all $W \in \widehat{\mathcal{W}}^s$ with $|W| \geq \delta_1/3$,

$$(3.31) \quad \sum_{W_i \in \mathcal{L}_n^{\delta_1}(W)} |J_{W_i} T^n|_{C^0(W_i)}^t |e^{S_n g}|_{C^0(W_i)} \geq \frac{3}{4} \sum_{W_i \in \mathcal{G}_n^{\delta_1}(W)} |J_{W_i} T^n|_{C^0(W_i)}^t |e^{S_n g}|_{C^0(W_i)}, \quad \forall t \in [t_0, \bar{t}_1].$$

Proof of Lemma 3.11. Assume first that $g = 0$. Define $\rho = \theta^{\bar{t}_1} e^{(\chi_1 + \kappa_1)(\bar{t}_1 - 1)} < 1$ (with $\kappa_1(t_0, t_1, \bar{t}_1)$ from (3.29)). For $\varepsilon > 0$, choose $\bar{\varepsilon} > 0$ such that $\frac{6\delta_0 C_0^2}{C_{\kappa_1}(1-\rho)} \frac{\bar{\varepsilon}}{1-\bar{\varepsilon}} < \frac{\varepsilon}{2}$ and choose m_1 such that $\rho^{m_1} \leq \bar{\varepsilon}$. Next, choose $\delta_1 > 0$ such that (3.8) holds for all $n \leq 2m_1$.

²⁵We take $\delta_1 < \delta_1(\varepsilon)$ and $n_1 \geq n_1(\varepsilon)$ with $\delta_1(\varepsilon)$ and $n_1(\varepsilon)$ from Lemma 3.5.

For $W \in \widehat{W}^s$ with $|W| \geq \delta_1/3$, let $S_n^{\delta_1}(W)$ denote the elements $W_i \in \mathcal{G}_n^{\delta_1}(W)$ such that $|W_i| < \delta_1/3$, and let $L_n^{\delta_1}(W) = \mathcal{G}_n^{\delta_1}(W) \setminus S_n^{\delta_1}(W)$. For $n \geq m_1$, write $n = \ell m_1 + r$ for some $0 \leq r < m_1$. As in the proof of Lemma 3.5, we group elements $W_i \in S_n^{\delta_1}(W)$ according to the largest k such that $T^{(\ell-k)m_1+r}W_i \subset V_j \in L_{km_1}^{\delta_1}(W)$. Such a $k \in [0, \ell-1]$ must exist since $|W| \geq \delta_1/3$ while $|W_i| < \delta_1/3$. By choice of δ_1 and m_1 , we may apply (3.8) for $(\ell-k)m_1+r$ iterates to obtain

$$\sum_{W_i \in S_n^{\delta_1}(W)} |J_{W_i} T^n|_{C^0(W_i)}^t \leq \sum_{k=0}^{\ell-1} \sum_{V_j \in L_{km_1}^{\delta_1}(W)} |J_{V_j} T^{km_1}|_{C^0(V_j)}^t C_0 \theta^{t((\ell-k)m_1+r)}.$$

Next, for each k , we consider each $V_j \in L_{km_1}^{\delta_1}(W)$ as being contained in an element $U_i \in \mathcal{G}_{km_1}^{\delta_0}(W)$. Since $|V_j| \geq \delta_1/3$, there are at most $3\delta_0/\delta_1$ V_j corresponding to each such U_i . Then we group each $U_i \in \mathcal{G}_{km_1}^{\delta_0}(W)$ according to its most recent long ancestor $W_a \in L_j^{\delta_0}(W)$ for some $j \in [0, km_1]$. Note that $j=0$ is possible if $|W| \geq \delta_0/3$. However, if $|W| < \delta_0/3$ and no such time j exists for U_i , then we associate such U_i with index $j=0$ in any case. In either case, $U_i \in \mathcal{I}_{km_1}^{\delta_0}(W)$, where $\mathcal{I}_{km_1}^{\delta_0}(W) = \mathcal{I}_{km_1}(W)$ as defined in Lemma 3.3. With these groupings, we estimate,

$$\begin{aligned} \sum_{V_j \in L_{km_1}^{\delta_1}(W)} |J_{V_j} T^{km_1}|_{C^0(W_i)}^t &\leq \frac{3\delta_0}{\delta_1} \left(\sum_{U_i \in \mathcal{I}_{km_1}^{\delta_0}(W)} |J_{U_i} T^{km_1}|_{C^0(U_i)}^t \right. \\ &\quad \left. + \sum_{j=1}^{km_1} \sum_{W_a \in L_j^{\delta_0}(W)} |J_{W_a} T^j|^t \sum_{U_i \in \mathcal{I}_{km_1-j}^{\delta_0}(W_a)} |J_{U_i} T^{km_1-j}|_{C^0(U_i)}^t \right) \\ &\leq \frac{3\delta_0}{\delta_1} \left(C_0 \theta^{tkm_1} + \sum_{j=1}^{km_1} \sum_{W_a \in L_j^{\delta_0}(W)} |J_{W_a} T^j|^t C_0 \theta^{t(km_1-j)} \right), \end{aligned}$$

where we have applied Lemma 3.3 to each collection $\mathcal{I}_{km_1-j}^{\delta_0}(W)$. Collecting these estimates yields,

$$\sum_{W_i \in S_n^{\delta_1}(W)} |J_{W_i} T^n|_{C^0(W_i)}^t \leq \frac{3\delta_0}{\delta_1} \left(C_0^2 \frac{n}{m_1} \theta^{tn} + \sum_{k=1}^{\ell-1} \sum_{j=1}^{km_1} C_0^2 \theta^{t(n-j)} \sum_{W_a \in L_j^{\delta_0}(W)} |J_{W_a} T^j|_{C^0(W_a)}^t \right).$$

For fixed $k > 0$ and each j for which $L_j^{\delta_0}(W)$ is not empty, we group elements $W_i \in \mathcal{G}_n^{\delta_1}(W)$ according to which $W_a \in \mathcal{G}_j^{\delta_0}(W)$ they descend from. Then we use Lemma 3.10 and the distortion bound (3.10) to estimate a lower bound,

$$\begin{aligned} \sum_{W_i \in \mathcal{G}_n^{\delta_1}(W)} |J_{W_i} T^n|_{C^0(W_i)}^t &\geq \sum_{W_a \in L_j^{\delta_0}(W)} \frac{1}{2} |J_{W_a} T^j|_{C^0(W_a)}^t \sum_{W_i \in \mathcal{G}_{n-j}^{\delta_1}(W_a)} |J_{W_i} T^{n-j}|_{C^0(W_i)}^t \\ &\geq \frac{C_{\kappa_1}}{2\delta_1} e^{-(n-j)(\chi_1 + \kappa_1)(t-1)} \sum_{W_a \in L_j^{\delta_0}(W)} |J_{W_a} T^j|_{C^0(W_a)}^t. \end{aligned}$$

Combining upper and lower bounds yields (using (3.27) for the term²⁶ corresponding to $j = 0$),

$$\begin{aligned} \frac{\sum_{W_i \in \mathcal{S}_n^{\delta_1}(W)} |J_{W_i} T^n|_{C^0(W_i)}^t}{\sum_{W_i \in \mathcal{G}_n^{\delta_1}(W)} |J_{W_i} T^n|_{C^0(W_i)}^t} &\leq \frac{\frac{3\delta_0 n}{\delta_1 m_1} C_0^2 \theta^{tn}}{C_{\kappa_1} \delta_1^{\frac{1}{\eta_{\kappa_1}} - 1} e^{-n(\chi_1 + \kappa_1)(t-1)}} \\ &\quad + \sum_{k=1}^{\ell-1} \sum_{j=1}^{km_1} \frac{\frac{3\delta_0}{\delta_1} C_0^2 \theta^{t(n-j)} \sum_{W_a \in L_j^{\delta_0}(W)} |J_{W_a} T^j|_{C^0(W_a)}^t}{\frac{C_{\kappa_1}}{2\delta_1} e^{-(n-j)(\chi_1 + \kappa_1)(t-1)} \sum_{W_a \in L_j^{\delta_0}(W)} |J_{W_a} T^j|_{C^0(W_a)}^t} \\ &\leq \frac{3\delta_0 n C_0^2}{C_{\kappa_1} m_1} \delta_1^{-\frac{1}{\eta_{\kappa_1}}} \rho^n + \sum_{k=1}^{\ell-1} \sum_{j=1}^{km_1} \frac{6\delta_0 C_0^2}{C_{\kappa_1}} \rho^{n-j} \\ &\leq \frac{3\delta_0 n C_0^2}{C_{\kappa_1} m_1} \delta_1^{-\frac{1}{\eta_{\kappa_1}}} \rho^n + \frac{6\delta_0 C_0^2}{C_{\kappa_1} (1-\rho)} \sum_{i=1}^{\ell-1} \bar{\varepsilon}^i \leq \frac{3\delta_0 n C_0^2}{C_{\kappa_1} m_1} \delta_1^{-\frac{1}{\eta_{\kappa_1}}} \rho^n + \frac{\varepsilon}{2}, \end{aligned}$$

by choice of $\bar{\varepsilon}$, where we have used the fact that $\sum_{k=1}^{\ell-1} \sum_{j=1}^{km_1} \rho^{n-j} \leq \sum_{k=1}^{\ell-1} \frac{\rho^{n-km_1}}{1-\rho} \leq \sum_{i=1}^{\ell-1} \frac{\bar{\varepsilon}^i}{1-\rho}$. Finally, we choose $n_1 \geq m_1$ sufficiently large that the first term is less than $\frac{\varepsilon}{2}$ for all $n \geq n_1$, completing the proof in the case $g = 0$.

If $g \neq 0$, define $\rho_g = \theta^{\bar{t}_1} e^{(\chi_1 + \kappa_1)(\bar{t}_1 - 1) + 2|g|_{C^0}} < 1$. For $\varepsilon > 0$, pick $\bar{\varepsilon} > 0$ such that $\frac{12\delta_0 C_0^2}{C_{\kappa_1} (1-\rho_g)} \frac{\bar{\varepsilon}}{1-\bar{\varepsilon}} < \frac{\varepsilon}{2}$. Then choose m_1 and δ_1 as in the case $g = 0$, but with ρ_g in place of ρ . (The choices m_1 and δ_1 are uniform for g satisfying (3.6) and (3.30).) The argument then follows precisely as above with the inclusion of g as in (3.14) in the proof of Lemma 3.5. The fact that the distortion constant for g is at most 2 is used for the lower bound for each $W_a \in L_j^{\delta_0}(W)$ appearing in the denominator:

$$\begin{aligned} &\sum_{W_i \in \mathcal{G}_{n-j}^{\delta_1}(W_a)} |J_{W_i} T^n|_{C^0(W_i)}^t |e^{S_n g}|_{C^0(W_i)} \\ &\geq \frac{1}{4} |J_{W_a} T^j|_{C^0(W_a)}^t |e^{S_j g}|_{C^0(W_a)} \sum_{W_i \in \mathcal{G}_{n-j}^{\delta_1}(W_a)} |J_{W_i} T^{n-j}|_{C^0(W_i)}^t |e^{S_{n-j} g}|_{C^0(W_i)} \\ &\geq \frac{1}{4} |J_{W_a} T^j|_{C^0(W_a)}^t |e^{S_j g}|_{C^0(W_a)} C_{\kappa_1} \delta_1^{-1} e^{-(n-j)(\chi_1 + \kappa_1)(t-1) - (n-j)|g|_{C^0}}. \end{aligned}$$

□

Our final lemma of this section is the analogue of Lemma 3.6 for $t > 1$. Define $\mathcal{A}_n(\delta)$ as in (3.15).

Lemma 3.12. *Let $t_0 \in (0, 1)$, $t_1 \in (1, t_*)$ and $\bar{t}_1 \in (1, s_1)$. Let $\delta_2 > 0$ be as in Lemma 3.6. There exists a decreasing function $c_0 : [0, \infty) \rightarrow \mathbb{R}^+$ such that for any $v \geq 0$ and any $g \in C^1$ satisfying (3.6), (3.30), and $|\nabla g|_{C^0} \leq v$, we have*

$$\sum_{A \in \mathcal{A}_n(\delta_2)} \sup_{x \in A \cap M'} |J^s T^n(x)|^t e^{S_n g(x)} \geq c_0(v) Q_n(t, g), \quad \forall n \in \mathbb{N}, \quad \forall t \in [1, \bar{t}_1].$$

Proof. The calculations in the proof of Lemma 3.6 for $g = 0$ are valid for all $t > 0$ up through (3.22). To proceed, we replace Sublemma 3.7 by the following.

Sublemma 3.13. *Let $t_0 \in (0, 1)$, $t_1 \in (1, t_*)$ and $\bar{t}_1 \in (1, s_1)$. There exists $C > 0$ such that for all $t \in [1, \bar{t}_1]$, each $n_1 \leq j \leq n - n_2 - 1$ and all $B' \in \mathbb{L}_{-n+1+j}^{\delta_2}$,*

$$|\cos \varphi|_{T^{-n+1+j} B'}^t |J^u T^{-n+1+j}|_{B'}^t \leq C \delta_2^{-1/\eta_{\kappa_1}} e^{j(\chi_1 + \kappa_1)(t-1)} \sum_{B \in L_j(B')} |\cos \varphi|_{T^{-n+1} B}^t |J^u T^{-n+1}|_B^t,$$

where $L_j(B')$ denotes the collection of elements $B \in G_j(B')$ with $\text{diam}^u(B) \geq \delta_2/3$.

²⁶A better estimate is possible in the case $|W| \geq \delta_0/3$, but we will not need this.

Proof. The proof of this sublemma only requires one adjustment to the estimate in (3.23). Using the same notation as in Sublemma 3.7, we have

$$\begin{aligned}
\sum_{B \in L_j(B')} |\cos \varphi|_{T^{-n+1}B}^t |J^u T^{-n+1}|_B^t &\geq C |\cos \varphi|_{T^{-n+1+j}B'}^t |J^u T^{-n+1+j}|_{B'}^t \sum_{B \in L_j^t(B')} |J^u T^{-j}|_B^t \\
&\geq C' |\cos \varphi|_{T^{-n+1+j}B'}^t |J^u T^{-n+1+j}|_{B'}^t \delta_2 \sum_{U_i \in L_j^{\delta_2}(U)} |J_{U_i} T^{-j}|_{C^0(U_i)}^t \\
&\geq C' \delta_2 |\cos \varphi|_{T^{-n+1+j}B'}^t |J^u T^{-n+1+j}|_{B'}^t \frac{3}{4} \sum_{U_i \in \mathcal{G}_j^{\delta_2}(U)} |J_{U_i} T^{-j}|_{C^0(U_i)}^t \\
&\geq C'' \delta_2^{1/\eta_{\kappa_1}} |\cos \varphi|_{T^{-n+1+j}B'}^t |J^u T^{-n+1+j}|_{B'}^t C_{\kappa_1} e^{-j(\chi_1 + \kappa_1)(t-1)},
\end{aligned}$$

where the only new justifications are that we use the time reversal of (3.31) in the third inequality since $\delta_2 \leq \delta_1$ and $|U| \geq \delta_2/3$, and in the fourth inequality, we apply the time reversal of Lemma 3.10 since $j \geq n_1$ with κ_1 from (3.29). \square

Using Sublemma 3.13, we estimate the right hand side of (3.22) as in the proof of Lemma 3.6, summing over $B' \in \mathbb{L}_{-n+1+j}^{\delta_2}$ and recalling that if $B \in L_j(B')$, then $B \in \mathbb{L}_{-n+1}^{\delta_2}$ and each such B is associated with a unique B' :

$$\begin{aligned}
(3.32) \quad \sum_{B \in \mathbb{S}_{-n+1}^{\delta_2}} |\cos \varphi|_{T^{-n+1}B}^t |J^u T^{-n+1}|_B^t &\leq \sum_{B' \in \mathbb{S}_{-n_2}^{\delta_2}} C \theta^{t(n-n_2-1)} |\cos \varphi|_{T^{-n_2}B'}^t |J^u T^{-n_2}|_{B'}^t \\
&\quad + \sum_{j=n_1}^{n-n_2-1} C \delta_2^{-1-1/\eta_{\kappa_1}} \theta^{tj} e^{j(\chi_1 + \kappa_1)(t-1)} \sum_{B \in \mathbb{L}_{-n+1}^{\delta_2}} |\cos \varphi|_{T^{-n+1}B}^t |J^u T^{-n+1}|_B^t \\
&\leq C_{n_2} \theta^{tn} + C \delta_2^{-1-1/\eta_{\kappa_1}} \sum_{B \in \mathbb{L}_{-n+1}^{\delta_2}} |\cos \varphi|_{T^{-n+1}B}^t |J^u T^{-n+1}|_B^t,
\end{aligned}$$

for some constant $C_{n_2} > 0$ depending only on n_2 , where we have used the fact that $\theta^t e^{(\chi_1 + \kappa_1)(t-1)} < 1$ to sum over j .

The sum over $B \in \mathbb{L}_{-n+1}^{\delta_2}$ shrinks at a rate bounded below by $C \delta_2^{1/\eta_{\kappa_1}} e^{-n(\chi_1 + \kappa_1)(1-t)}$ by the proof of Sublemma 3.13. Thus we may choose $n_3 \geq n_2$ large enough that $C_{n_2} \theta^{tn} \leq C \delta_2^{-1} e^{-n(\chi_1 + \kappa_1)(t-1)}$ for all $n \geq n_3$, which implies,

$$\sum_{B \in \mathbb{S}_{-n+1}^{\delta_2}} |\cos \varphi|_{T^{-n+1}B}^t |J^u T^{-n+1}|_B^t \leq C \delta_2^{-1-1/\eta_{\kappa_1}} \sum_{B \in \mathbb{L}_{-n+1}^{\delta_2}} |\cos \varphi|_{T^{-n+1}B}^t |J^u T^{-n+1}|_B^t.$$

The proof of the Lemma 3.6 proceeds without further changes from this point, ending the proof of Lemma 3.12 if $g = 0$.

If $g \neq 0$, choosing δ_2 as in the proof of Lemma 3.6 implies that (3.24) and (3.25) remain as written. The only change required in the proof is to use the lower bound (3.27) with $\kappa = \kappa_1$ to prove the analogue of Sublemma 3.7: There exists $C > 0$ such that for all $n_1 \leq j \leq n - n_2 - 1$ and $B' \in \mathbb{L}_{-n+1+j}^{\delta_2}$,

$$\begin{aligned}
|\cos \varphi|_{T^{-n+1+j}B'}^t |J^u T^{-n+1+j}|_{B'}^t |e^{S_{n-1-j}g}|_{T^{-n+1+j}B'} & \\
\leq C \delta_2^{-1/\eta_{\kappa_1}} e^{j(\chi_1 + \kappa_1)(t-1)} e^{j|g|_{C^0}} \sum_{B \in L_j(B')} |\cos \varphi|_{T^{-n+1}B}^t |J^u T^{-n+1}|_B^t |e^{S_{n-1}g}|_{T^{-n+1}B}. &
\end{aligned}$$

We then proceed as in (3.32) using the contraction provided by (3.30) to sum over j . \square

3.5. Lower Bounds on Complexity. Exact Exponential Growth of $Q_n(t, g)$. In order to conclude that the spectral radius of \mathcal{L}_t on \mathcal{B} is $e^{P_*(t)}$ and to control the peripheral spectrum of \mathcal{L}_t , we shall establish the exact exponential growth of $Q_n(t)$.

The lower bound on the spectral radius of \mathcal{L}_t is a consequence of the following lemma, guaranteeing that the weighted complexity of long elements of $\widehat{\mathcal{W}}^s$ grows at the rate $Q_n(t, g)$.

Proposition 3.14. *Let $t_0 \in (0, 1)$, $t_1 \in (1, t_*)$ and $\bar{t}_1 \in (1, s_1)$. There exists a decreasing function $c_1 : [0, \infty) \rightarrow \mathbb{R}^+$ such that for any $v \geq 0$ and any $W \in \widehat{\mathcal{W}}^s$ with $|W| \geq \delta_1/3$,*

$$(3.33) \quad \sum_{W_i \in \mathcal{G}_n(W)} |J_{W_i} T^n|_{C^0(W_i)}^t |e^{S_n g}|_{C^0(W_i)} \geq c_1(v) Q_n(t, g), \quad \forall n \geq 1, \quad \forall t \in [t_0, \bar{t}_1],$$

for any $g \in C^1$ with $|\nabla g|_{C^0} \leq v$ and such that (3.6) and (3.30) hold.

Proof. As usual we first consider $g = 0$. The main idea of the proof is to show that for each curve $W \in \mathcal{W}^s$ with $|W| \geq \delta_1/3$, the image $T^{-n}W$ intersects a positive fraction of elements of $\mathcal{M}_n^{0, \mathbb{H}}$, weighted by $|J^s T^n|^t$, for n large enough. The mixing property of μ_{SRB} is instrumental here.

To do this, we recall the construction of locally maximal homogeneous Cantor rectangles from [CM, Section 7.12] (and similar to those used in [BD, Section 5.3] where we worked²⁷ with \mathcal{W}^s instead of $\mathcal{W}_{\mathbb{H}}^s$). We call $D \subset M$ a *solid rectangle* if D is a closed, simply connected region whose boundary consists of two homogeneous unstable and two stable manifolds. Given such a rectangle D , the maximal Cantor rectangle $R(D)$ in D is the union of all points in D whose homogeneous stable and unstable manifolds completely cross D . Note that $R(D)$ is closed and contains the boundary of D [CM, Section 7.11], but is not simply connected due to the effect of the singularities, which create, for any $\varepsilon > 0$, a dense set of points with stable and unstable manifolds shorter than ε .

In what follows, we restrict to Cantor rectangles with sufficiently high density, i.e.,

$$(3.34) \quad \inf_{x \in R} \frac{m_{W^u}(W^u(x) \cap R)}{m_{W^u}(W^u(x) \cap D(R))} \geq 0.99,$$

where m_{W^u} denotes arclength measure along an unstable manifold. We say that a homogeneous stable curve $W \in \widehat{\mathcal{W}}_{\mathbb{H}}^s$ *properly crosses* a maximal homogeneous Cantor rectangle $R = R(D)$ satisfying (3.34) if W crosses both unstable sides of D , and, in addition, for every $x \in R$, the point $W \cap W^u(x)$ divides the curve $W^u(x) \cap D(R)$ in a ratio between 0.1 and 0.9, and on either side of $W \cap W^u(x)$, the density of R in $W^u(x) \cap D(R)$ is at least 0.9. Reversing the roles of stable and unstable manifolds, we obtain the analogous definition of an unstable curve properly crossing a Cantor rectangle.

By [CM, Lemma 7.87], we choose a finite number of locally maximal homogeneous Cantor rectangles $\mathcal{R}(\delta_2) = \{R_1, \dots, R_k\}$ satisfying (3.34) and its analogue along stable manifolds, with the property that any homogeneous stable or unstable curve of length at least $\delta_2/3$ properly crosses at least one of them. Let δ'_2 be the minimum diameter of the rectangles in $\mathcal{R}(\delta_2)$ and note that δ'_2 is a function only of δ_2 .

Now fix $n \geq 1$ and let $\mathcal{A}_n^i \subset \mathcal{A}_n(\delta_2)$ denote those elements $A \in \mathcal{A}_n(\delta_2)$ such that $B_{n-1}(A)$ contains an homogeneous unstable curve of length at least $\delta_2/3$ that properly crosses R_i . Due to Lemma 3.6 for $t \leq 1$ and Lemma 3.12 for $t > 1$, there exists i^* such that

$$(3.35) \quad \sum_{A \in \mathcal{A}_n^{i^*}} \sup_{x \in A \cap M'} |J^s T^n(x)|^t \geq \frac{c_0}{k} Q_n(t).$$

Fix an arbitrary homogeneous $W \in \widehat{\mathcal{W}}^s$ with $|W| \geq \delta_1/3$, and let $R_j \in \mathcal{R}(\delta_2)$ denote the Cantor rectangle that is properly crossed by W (recalling that $\delta_2 \leq \delta_1$). By the mixing property of μ_{SRB} and [CM, Lemma 7.90], there exists $N_1 = N_1(\delta_2) \geq 1$ such that $T^{-N_1}R_i$ has a homogeneous

²⁷The construction in [CM, Section 7.12] uses $\mathcal{W}_{\mathbb{H}}^s$, but since each V in \mathcal{W}^s are unions of manifolds W_i in $\mathcal{W}_{\mathbb{H}}^s$, if the W_i cross properly, so does V .

connected component that properly crosses R_{i^*} , for all $i = 1, \dots, k$. In particular, $T^{-N_1}R_j$ properly crosses R_{i^*} , so an element of $\mathcal{G}_{N_1}(W)$ properly crosses R_{i^*} .

Let $W_1 \in \mathcal{G}_{N_1}(W)$ denote the component of $T^{-N_1}W$ that properly crosses R_{i^*} and note that W_1 crosses $B_{n-1}(A)$ for all $A \in \mathcal{A}_n^{i^*}$. Since W_1 is homogeneous and $N_1 \geq 1$, W_1 cannot cross a singularity line in $T^{-1}\mathcal{S}_0$ (since then the curve would have been subdivided at time $N_1 - 1$), and so for each such A , W_1 crosses an element $B'_A \in \mathcal{M}_{-n+1}^{1, \mathbb{H}}$, $B'_A \subset B_{n-1}(A)$. Let $V'_A = W_1 \cap B'_A$ and let $V_A = T^{-n+1}V'_A$. Then V_A is a homogeneous component belonging to an element of $\mathcal{G}_{n-1+N_1}(W)$. By Lemma 2.2, recalling the notation $|J^s T^k|_{A'}^t = \sup_{x \in A' \cap M'} |J^s T^k(x)|^t$ from (3.17),

$$|J_{V_A} T^{n-1}|_{C^0(V_A)}^t = e^{\pm tC} |J^s T^{n-1}|_{T^{-n+1}B_{n-1}(A)}^t,$$

since $T^{-n+1}B_{n-1}(A) \in \mathcal{M}_0^{n-1, \mathbb{H}} \vee T^{-n+1}\mathcal{H}$. By definition, $T^{-n+1}B_{n-1}(A)$ contains A . Thus,

$$(3.36) \quad |J^s T^{n-1}|_A^t \leq e^{tC} |J_{V_A} T^{n-1}|_{C^0(V_A)}^t.$$

Next, we wish to compare $J^s T$ on $T^{n-1}A$ with $J_{V'_A} T$. Since $V'_A \subset W_1 \subset T^{-N_1}W$, we have that TV'_A is a stable curve, and so is TW_1 , so that $J_{V'_A} T = e^{\pm C_d} J_{W_1} T = e^{\pm C_d} k^{-q}$, where k is the index of the homogeneity strip containing W_1 . But since $|W_1| \geq \delta'_2$ (since W_1 properly crosses R_{i^*}), we have $k \leq (\delta'_2)^{-1/(q+1)}$ and so $J_{W_1} T \geq C(\delta'_2)^{q/(q+1)}$. Since $J^s T \leq e^{C_d}$, we have, using (3.36), that $|J^s T^n|_A^t \leq C(\delta'_2)^{-tq/(q+1)} |J_{V_A} T^n|_{C^0(V_A)}^t$. Then summing over $A \in \mathcal{A}_n^{i^*}$, we obtain,

$$(3.37) \quad \sum_{A \in \mathcal{A}_n^{i^*}} |J^s T^n|_A^t \leq C(\delta'_2)^{-tq/(q+1)} \sum_{V_i \in \mathcal{G}_n(TW_1)} |J_{V_i} T^n|_{C^0(V_i)}^t.$$

Next, we express the sum over $\mathcal{G}_{n+N_1-1}(W)$ in two ways. On the one hand, by Lemma 3.4,

$$(3.38) \quad \begin{aligned} \sum_{V_j \in \mathcal{G}_{n+N_1-1}(W)} |J_{V_j} T^{n+N_1-1}|_{C^0(V_j)}^t &\leq \sum_{W_i \in \mathcal{G}_n(W)} |J_{W_i} T^n|_{C^0(W_i)}^t \sum_{V_j \in \mathcal{G}_{N_1-1}(W_i)} |J_{V_j} T^{N_1-1}|_{C^0(V_j)}^t \\ &\leq C_2[0] Q_{N_1-1}(t) \sum_{W_i \in \mathcal{G}_n(W)} |J_{W_i} T^n|_{C^0(W_i)}^t. \end{aligned}$$

On the other hand, letting W'_1 be the element of $\mathcal{G}_{N_1-1}(W)$ containing TW_1 ,

$$(3.39) \quad \begin{aligned} \sum_{V_j \in \mathcal{G}_{n+N_1-1}(W)} |J_{V_j} T^{n+N_1-1}|_{C^0(V_j)}^t &\geq e^{-tC_d} |J_{W'_1} T^{N_1-1}|_{C^0(W'_1)}^t \sum_{V_i \in \mathcal{G}_n(W'_1)} |J_{V_i} T^n|_{C^0(V_i)}^t \\ &\geq e^{-tC_d} C'(\delta'_2)^{t \left(\frac{2q+1}{q+1} \right)^{N_1-1}} \sum_{V_i \in \mathcal{G}_n(W'_1)} |J_{V_i} T^n|_{C^0(V_i)}^t, \end{aligned}$$

where the lower bound on $|J_{W'_1} T^{N_1-1}|_{C^0(W'_1)}^t$ comes from the fact that $|W'_1| \geq \delta'_2$ and for a stable curve V such that V and $T^{-1}V$ are both homogeneous, $|T^{-1}V| \leq C|V|^{\frac{q+1}{2q+1}}$, and this bound can be iterated $N_1 - 1$ times as in [BD, eq. (5.3)].

Combining (3.37), (3.38) and (3.39), and recalling (3.35) yields,

$$\sum_{W_i \in \mathcal{G}_n(W)} |J_{W_i} T^n|_{C^0(W_i)}^t \geq (C_2[0])^{-1} Q_{N_1-1}(t)^{-1} C''(\delta'_2)^{t \left(\frac{2q+1}{q+1} \right)^{N_1}} \frac{c_0}{k} Q_n(t),$$

which completes the proof of the proposition if $g = 0$.

If $g \neq 0$, starting as above, we choose the finite family of Cantor rectangles $\mathcal{R}(\delta_2)$ in the same way, and find an index i^* such that the analogue of (3.35)

$$\sum_{A \in \mathcal{A}_n^{i^*}} \sup_{x \in A \cap M'} |J^s T^n(x)|^t e^{S_n g(x)} \geq \frac{c_0}{k} Q_n(t, g),$$

holds, using Lemma 3.6 if $t \leq 1$ and Lemma 3.12 if $t > 1$. Fixing $W \in \mathcal{W}^s$, choosing N_1 as above, and using the same notation introduced there, we obtain the modification of (3.36),

$$\|J^s T^{n-1}|^t e^{S_{n-1}g}|_A \leq e^{tC} (1 + \bar{C} C_* |\nabla g|_{C^0}) \|J_{V_A} T^{n-1}|^t e^{S_{n-1}g}|_{C^0(V_A)},$$

applying (2.15). Next, (3.37) needs only the multiplication by $e^{S_n g}$ to each term on both sides, up to replacing the constant C by $C(1 + \bar{C} C_* \cdot |\nabla g|_{C^0})$. The upper bound (3.38) requires only a change of constant to $C_2[0]Q_{N_1-1}(t, g)$, using Lemma 3.4 with $\varsigma = 0$, while the lower bound (3.39) requires the added factor $e^{-(N_1-1)|g|_{C^0}}$ on the right hand side. Since N_1 is fixed (depending only on $\mathcal{R}(\delta_2)$), these bounds are combined as in the case $g = 0$ to complete the proof of the proposition. \square

The following important consequence of Proposition 3.14 will be used to characterize the peripheral spectrum of \mathcal{L}_t .

Proposition 3.15 (Exact Exponential Growth of $Q_n(t, g)$). *Let $t_0 \in (0, 1)$, $t_1 \in (1, t_*)$ and $\bar{t}_1 \in (1, s_1)$. There exists a decreasing function $c_2 : [0, \infty) \rightarrow \mathbb{R}^+$ such that for any $v \geq 0$ and any $g \in C^1$ with $|\nabla g|_{C^0} \leq v$ and such that (3.6) and (3.30) hold, we have*

$$(3.40) \quad e^{nP_*(t, g)} \leq Q_n(t, g) \leq \frac{2}{c_2(v)} e^{nP_*(t, g)}, \quad \forall t \in [t_0, \bar{t}_1], \quad \forall n \geq 1.$$

Proof. The lower bound follows immediately from submultiplicativity of $Q_n(t, g)$ (obtained in the proof of Proposition 2.6 for any $t > 0$ and $g \in C^1$) since then $P_*(t, g) = \inf_n \frac{1}{n} \log Q_n(t, g)$.

To obtain the upper bound for $g = 0$, we first prove the following supermultiplicative property: There exists $c_2 > 0$ such that for all $t \in [t_0, \bar{t}_1]$ and for any $j, n \geq 1$,

$$(3.41) \quad Q_{n+j}(t) \geq c_2 Q_n(t) Q_j(t).$$

Let $W \in \widehat{\mathcal{W}}^s$ with $|W| \geq \delta_1/3$. For $n, j \geq 1$, by Lemma 3.4 with $\varsigma = 0$,

$$\sum_{W_i \in \mathcal{G}_{n+j}^{\delta_1}(W)} |J_{W_i} T^{n+j}|_{C^0(W_i)}^t \leq C_2[0] Q_{n+j}(t).$$

On the other hand, if $n \geq n_1$, then using Lemma 2.1,

$$\begin{aligned} \sum_{W_i \in \mathcal{G}_{n+j}^{\delta_1}(W)} |J_{W_i} T^{n+j}|_{C^0(W_i)}^t &\geq C \sum_{V_k \in \mathcal{G}_n^{\delta_1}(W)} |J_{V_k} T^n|_{C^0(V_k)}^t \sum_{W_i \in \mathcal{G}_j^{\delta_1}(V_k)} |J_{W_i} T^j|_{C^0(W_i)}^t \\ &\geq C \sum_{V_k \in \mathcal{L}_n^{\delta_1}(W)} |J_{V_k} T^n|_{C^0(V_k)}^t \sum_{W_i \in \mathcal{G}_j^{\delta_1}(V_k)} |J_{W_i} T^j|_{C^0(W_i)}^t \\ &\geq C \sum_{V_k \in \mathcal{L}_n^{\delta_1}(W)} |J_{V_k} T^n|_{C^0(V_k)}^t c_1 Q_j(t) \\ &\geq C c_1 Q_j(t)^{\frac{3}{4}} \sum_{V_k \in \mathcal{G}_n^{\delta_1}(W)} |J_{V_k} T^n|_{C^0(V_k)}^t \geq C' c_1^2 Q_j(t) Q_n(t), \end{aligned}$$

where in the third and fifth inequalities, we have used Proposition 3.14 and in the fourth inequality we have applied (3.31). This proves (3.41) for $n \geq n_1$, and the case $n \leq n_1$ follows by adjusting the constant c_2 . (Note that c_2 is uniform in t .) The proof of the upper bound on $Q_n(t)$ then proceeds precisely as in the proof of [BD, Proposition 4.6]. The case of nonzero g is identical. \square

3.6. Growth Lemmas and Exact Exponential Growth for $t \in (s_1, t_*)$. The main result of this section is Proposition 3.18 which extends Propositions 3.14 and 3.15 to all $t < t_*$. The constant $t_* > 1$ is defined by (1.8), while $s_1 > 1$ is introduced in Definition 3.9. What we have proved up to now suffices to establish all the results of Sections 4–6 for $t \in (0, s_1)$. In particular Theorem 2.5 holds in a neighbourhood of $t = 1$, so we know that $s_1 \leq t_*$.

Recall that $t_1 \in (1, t_*)$ is fixed in Definition 3.2, determining $\theta \in (\Lambda^{-1}, \Lambda^{-1/2})$ and our main statements are for $t \in [t_0, t_1]$. If $s_1 \geq t_1$, there is nothing to do. Otherwise, $\theta^{s_1} < e^{P(s_1)} \leq e^{P_*(s_1)}$ by Proposition 2.4. Since $P_*(t)$ is convex and decreasing, the left-hand slopes are lower semi-continuous, so we may choose $\bar{t}_1 \in (1, s_1)$ so that the intersection point $s_2(\bar{t}_1)$ between the tangent line to $P_*(t)$ (from the left) at $t = \bar{t}_1$ and the line $t \log \theta$ satisfies $s_2 > s_1$. Indeed, we have

$$(3.42) \quad s_2 = s_2(\bar{t}_1) := \frac{P_*(\bar{t}_1) + \chi_2 \bar{t}_1}{\chi_2 + \log \theta}, \quad \text{where } \chi_2 = \chi_2(\bar{t}_1) := \lim_{s \rightarrow \bar{t}_1^-} \frac{P_*(s) - P_*(\bar{t}_1)}{\bar{t}_1 - s} \geq \log \Lambda,$$

where (by convexity of $P_*(t)$) the limit defining χ_2 exists and $P_*(t)$ lies above its tangents, so that $\theta^t < e^{P_*(t)}$ for all $t < s_2$.

Our next lemma is an analogue of Lemma 3.10, interpolating now from \bar{t}_1 to s_2 .

Lemma 3.16. *Fix $t_0 \in (0, 1)$ and $t_1 \in (1, t_*)$, and let $\bar{t}_1 \in (1, s_1)$ and $s_2(\bar{t}_1) > s_1$ be as above. For any $\bar{t}_2 \in (s_1, s_2)$ and any $\kappa > 0$, there exist $C_\kappa > 0$, $\eta_\kappa > 0$ such that for all $g \in C^0$, all $\delta > 0$, all $W \in \widehat{\mathcal{W}}^s$ with $|W| \geq \delta/3$, and all $n \geq 1$,*

$$(3.43) \quad \sum_{W_i \in \mathcal{G}_n^\delta(W)} |J_{W_i} T^n|^t |e^{S_n g}|_{C^0(W_i)} \geq C_\kappa \delta^{\frac{1}{\eta_\kappa} - 1} e^{-n(\chi_2 + \kappa)(t - \bar{t}_1) + nP_*(\bar{t}_1) - n|g|_{C^0}}, \quad \forall t \in [\bar{t}_1, \bar{t}_2].$$

Moreover, if $|W| \geq \delta_0/3$, then

$$\sum_{W_i \in \mathcal{G}_n^\delta(W)} |J_{W_i} T^n|^t |e^{S_n g}|_{C^0(W_i)} \geq C_\kappa \delta^{-1} e^{-n(\chi_2 + \kappa)(t-1) - n|g|_{C^0}}, \quad \forall n \geq 1, \quad \forall t \in [\bar{t}_1, \bar{t}_2].$$

Proof. We adapt the proof of Lemma 3.10. First assume $g = 0$. For $t \geq s_1$, let $s \in (1, \bar{t}_1)$, and $\eta(s) \in (0, 1]$ such that $\eta t + (1 - \eta)s = \bar{t}_1$. Then again using the Hölder inequality, for any $W \in \widehat{\mathcal{W}}^s$ with $|W| \geq \delta/3$ and all $n \geq 1$,

$$(3.44) \quad \sum_{W_i \in \mathcal{G}_n^\delta(W)} |J_{W_i} T^n|^t \geq \frac{\left(\sum_{W_i \in \mathcal{G}_n^\delta(W)} |J_{W_i} T^n|^{\bar{t}_1} \right)^{1/\eta}}{\left(\sum_{W_i \in \mathcal{G}_n^\delta(W)} |J_{W_i} T^n|^s \right)^{(1-\eta)/\eta}} \geq \frac{(c_1 e^{nP_*(\bar{t}_1)})^{1/\eta}}{(C_2[0] \frac{\delta_0}{\delta} \frac{2}{c_2} e^{nP_*(s)})^{(1-\eta)/\eta}},$$

where we have used Propositions 3.14 and 3.15 for the lower bound in the numerator, and Lemma 3.4 with $\varsigma = 0$ and Proposition 3.15 for the upper bound in the denominator. Since $\eta = (\bar{t}_1 - s)/(t - s)$,

$$e^{-n(P_*(s) - P_*(\bar{t}_1))/\eta} e^{nP_*(s)} = e^{-n(t-s) \frac{P_*(s) - P_*(\bar{t}_1)}{\bar{t}_1 - s}} e^{nP_*(s)}.$$

For fixed $\kappa > 0$, by (3.42), we may choose $s = s(\kappa) \in (1, \bar{t}_1)$ and $\eta_\kappa > 0$ such that $(t - s) \frac{P_*(s) - P_*(\bar{t}_1)}{\bar{t}_1 - s} \leq (t - \bar{t}_1)(\chi_2 + \kappa)$, completing the proof for $g = 0$ since $P_*(s) \geq P_*(\bar{t}_1)$. For $g \neq 0$, the lemma follows, again using the bound $|e^{S_n g}|_{C^0(W_i)} \geq e^{-n|g|_{C^0}}$. \square

By definition, $\theta^t e^{\chi_2(t - \bar{t}_1) - P_*(\bar{t}_1)} < 1$ if $t < s_2$. Thus for $\bar{t}_2 \in (s_1, s_2)$, there exists $\kappa_2 = \kappa(\bar{t}_2) > 0$ such that

$$(3.45) \quad \theta^{\bar{t}_2} e^{(\chi_2 + \kappa_2)(\bar{t}_2 - \bar{t}_1) - P_*(\bar{t}_1)} < 1, \quad \text{and thus } \theta^t e^{(\chi_2 + \kappa_2)(t - \bar{t}_1) - P_*(\bar{t}_1)} < 1, \quad \forall t \leq \bar{t}_2.$$

Our next lemma extends Lemma 3.11 for $t \in [\bar{t}_1, \bar{t}_2]$.

Lemma 3.17. *Let $t_0 \in (0, 1)$ and $t_1 \in (1, t_*)$. Let $\bar{t}_2 \in (s_1, s_2)$, and let $\kappa_2 = \kappa(\bar{t}_2)$ satisfy (3.45). Then for any $\varepsilon > 0$ there exist $\delta_1 > 0$ and $n_1 \geq 1$, such that for all $W \in \widehat{\mathcal{W}}^s$ with $|W| \geq \delta_1/3$, and for all $n \geq n_1$,*

$$\sum_{\substack{W_i \in \mathcal{G}_n^{\delta_1}(W) \\ |W_i| < \delta_1/3}} |J_{W_i} T^n|^t |e^{S_n g}|_{C^0(W_i)} \leq \varepsilon \sum_{W_i \in \mathcal{G}_n^{\delta_1}(W)} |J_{W_i} T^n|^t |e^{S_n g}|_{C^0(W_i)}, \quad \forall t \in [1, \bar{t}_2],$$

for all $g \in C^1$ satisfying (3.6), (3.30) and such that, in addition,

$$(3.46) \quad 2|g|_{C^0} < -\bar{t}_2 \log \theta - (\chi_2 + \kappa_2)(\bar{t}_2 - \bar{t}_1) + P_*(\bar{t}_1), \quad \text{i.e.} \quad \theta^{\bar{t}_2} e^{(\chi_2 + \kappa_2)(\bar{t}_2 - \bar{t}_1) - P_*(\bar{t}_1) + 2|g|_{C^0}} < 1.$$

Proof. The proof of Lemma 3.17 proceeds with the analogous modifications used in the proof of Lemma 3.11, using the lower bound (3.43) in place of (3.27). The proof goes through due to the contraction provided by (3.45) and (3.46). \square

Let $[t_0, \bar{t}_2] \subset (0, s_2)$. For all $g \in C^1$ satisfying (3.6), (3.30), (3.46), Lemmas 3.5, 3.11 and 3.17 for $\varepsilon = 1/4$ give $n_1 \geq 1$ and $\delta_1 > 0$ such that for all $n \geq n_1$ and all $W \in \widehat{\mathcal{W}}^s$ with $|W| \geq \delta_1/3$,

$$(3.47) \quad \sum_{W_i \in \mathcal{L}_n^{\delta_1}(W)} |J_{W_i} T^n|_{C^0(W_i)}^t e^{S_n g}|_{C^0(W_i)} \geq \frac{3}{4} \sum_{W_i \in \mathcal{G}_n^{\delta_1}(W)} |J_{W_i} T^n|_{C^0(W_i)}^t e^{S_n g}|_{C^0(W_i)}, \quad \forall t \in [t_0, \bar{t}_2].$$

At this point it is clear that Lemma 3.12 (with the same constant $\delta_2 > 0$, but possibly smaller $c_0 > 0$), and Propositions 3.14 and 3.15 (with possibly smaller constants $c_1, c_2 > 0$) hold with \bar{t}_1 replaced by $\bar{t}_2 \in (s_1, s_2)$.

The interpolation can now be continued inductively. Suppose we have created a sequence $1 < \bar{t}_1 < s_1 < \bar{t}_2 < s_2 < \dots < \bar{t}_n < s_n < t_1 < t_*$ so that Propositions 3.14 and 3.15 hold with \bar{t}_1 replaced by \bar{t}_n . Then since $s_n < t_1$, we have $\theta^{s_n} < e^{P_*(s_n)}$ and we may define

$$\chi_{n+1} = \lim_{s \rightarrow \bar{t}_n} \frac{P_*(s) - P_*(\bar{t}_n)}{\bar{t}_n - s} \geq \log \Lambda, \quad \text{and} \quad s_{n+1} = \frac{P_*(\bar{t}_n) + \chi_n \bar{t}_n}{\chi_n + \log \theta} > s_n,$$

where $s_{n+1} > s_n$ by choice of \bar{t}_n . Following the proof of Lemma 3.16 with $\bar{t}_1, \bar{t}_2, \chi_2$ replaced by $\bar{t}_n, \bar{t}_{n+1}, \chi_{n+1}$, it follows that the conclusion of the lemma holds for all $t \in [\bar{t}_n, \bar{t}_{n+1}]$. Analogous modifications to Lemma 3.17 imply that Lemma 3.12 and the propositions of Section 3.5 hold with \bar{t}_1 replaced by $\bar{t}_{n+1} \in (s_n, s_{n+1})$.

Finally, the sequence (s_n) cannot accumulate on any $s_\infty \leq t_1$. For if it does, then by definition of θ , it follows that $\theta^{s_\infty} < e^{P_*(s_\infty)}$, so we may repeat the construction above, finding a point of intersection $s' > s_\infty$ between $t \log \theta$ and the left hand tangent to $P_*(t)$ at some $\bar{t}_n < s_\infty$. It follows that this sequence of interpolations can be chosen so that $t_1 < \bar{t}_n < t_*$ for some $n \geq 1$. At this point we stop, and since we have made only finitely many choices of the required constants, we have extended the analogues of Propositions 3.14 and 3.15 to all $t_1 < t_*$:

Proposition 3.18. *Let $t_0 \in (0, 1)$ and $t_1 \in (1, t_*)$. There exist decreasing functions $c_i : [0, \infty) \rightarrow \mathbb{R}^+$, $i = 1, 2$, such that for any $v \geq 0$ and any $g \in C^1$ with $|\nabla g|_{C^0} \leq v$ and such that $|g|_{C^0}$ is sufficiently small (depending on the number of interpolations required to reach t_1),*

a) *for any $W \in \widehat{\mathcal{W}}^s$ with $|W| \geq \delta_1/3$,*

$$\sum_{W_i \in \mathcal{G}_n(W)} |J_{W_i} T^n|_{C^0(W_i)}^t e^{S_n g}|_{C^0(W_i)} \geq c_1(v) Q_n(t, g), \quad \forall n \geq 1, \quad \forall t \in [t_0, t_1];$$

b) *for all $n \geq 1$, we have $e^{nP_*(t, g)} \leq Q_n(t, g) \leq \frac{2}{c_2(v)} e^{nP_*(t, g)}$, $\forall t \in [t_0, t_1]$.*

4. SPECTRAL PROPERTIES OF \mathcal{L}_t (THEOREM 4.1)

4.1. Definition of Norms and Spaces \mathcal{B} and \mathcal{B}_w . For fixed $t_0 > 0$ and $t_1 \in (\max\{t_0, 1\}, t_*)$, we choose $\theta(t_1) \in (\Lambda^{-1}, \Lambda^{-1/2})$ satisfying $\theta^{t_1} < e^{P_*(t_1)}$, $q > \min\{1, 2/t_0\}$, $k_0 = k_0(t_0, t_1)$ (for the homogeneity strips (2.1)), and $\delta_0 = \delta_0(t_0, t_1)$ from Definition 3.2. These choices affect the definitions of \mathcal{W}^s and $\mathcal{W}_{\mathbb{H}}^s$, as well as conditions (4.1) and (4.2) below on the parameters $\alpha, \beta, \gamma, p, \varepsilon_0$, determining spaces $\mathcal{B} = \mathcal{B}(t_0, t_1)$ and $\mathcal{B}_w = \mathcal{B}_w(t_0, t_1)$ on which \mathcal{L}_t will be bounded for all $t \geq t_0$. An additional condition on the parameter p depending on $t_1 < t_*$ will be needed to obtain the Lasota–Yorke bound (4.8) (see Lemma 4.7) and thus the spectral gap of \mathcal{L}_t on \mathcal{B} for all $t \in [t_0, t_1]$.

First we define notions of distance²⁸ between stable curves and test functions as follows.

Since the slopes of stable curves are uniformly bounded away from the vertical, we view each $W \in \widehat{\mathcal{W}}^s$ as the graph of a function of the r -coordinate over an interval I_W ,

$$W = \{G_W(r) : r \in I_W\} = \{(r, \varphi_W(r)) : r \in I_W\}.$$

By the uniform bound on the curvature of $W \in \widehat{\mathcal{W}}^s$, we have $B := \sup_{W \in \widehat{\mathcal{W}}^s} |\varphi_W''| < \infty$.

Next, given $W_1, W_2 \in \widehat{\mathcal{W}}^s$ with functions $\varphi_{W_1}, \varphi_{W_2}$, we define

$$d_{\mathcal{W}^s}(W_1, W_2) = |I_{W_1} \triangle I_{W_2}| + |\varphi_{W_1} - \varphi_{W_2}|_{C^1(I_{W_1} \cap I_{W_2})},$$

if W_1 and W_2 lie in the same homogeneity strip, and $d_{\mathcal{W}^s}(W_1, W_2) = 3B + 1$ otherwise.

Finally, if $d_{\mathcal{W}^s}(W_1, W_2) < 3B + 1$, then for $\psi_1 \in C^0(W_1), \psi_2 \in C^0(W_2)$, define

$$d(\psi_1, \psi_2) = |\psi_1 \circ G_{W_1} - \psi_2 \circ G_{W_2}|_{C^0(I_{W_1} \cap I_{W_2})},$$

while if $d_{\mathcal{W}^s}(W_1, W_2) \geq 3B + 1$ and $\psi_1 \in C^0(W_1), \psi_2 \in C^0(W_2)$, we set $d(\psi_1, \psi_2) = \infty$.

We next define the norms, introducing parameters α, β, γ, p , and ε_0 . Choose²⁹

$$(4.1) \quad \alpha \in \left(0, \frac{1}{q+1}\right], \quad p > q + 1, \quad \beta \in \left(\frac{1}{p}, \alpha\right), \quad \gamma \in \left(0, \min\left\{\frac{1}{p}, \alpha - \beta, \frac{1}{6q+7}\right\}\right).$$

(This implies $\alpha \leq 1/3$, $\gamma < 1/p$, and $\min(\beta, t) > \frac{1}{p}$.) Finally for $C_{vert} < \infty$ to be determined in (4.21), let ε_0 satisfy

$$(4.2) \quad 0 < C_{vert} \varepsilon_0^{1/(q+1)} \leq \frac{3}{4}.$$

For $f \in C^1(M)$, recalling $C^\eta(W)$ and \mathcal{W}_H^s from Section 2.1, define the weak norm of f by³⁰

$$|f|_w = \sup_{W \in \mathcal{W}_H^s} \sup_{|\psi|_{C^\alpha(W)} \leq 1} \int_W f \psi \, dm_W,$$

define the stable norm of f by³¹

$$(4.3) \quad \|f\|_s = \sup_{W \in \mathcal{W}_H^s} \sup_{|\psi|_{C^\beta(W)} \leq |W|^{-1/p}} \int_W f \psi \, dm_W,$$

and the unstable norm of f by

$$\|f\|_u = \sup_{\varepsilon \leq \varepsilon_0} \sup_{\substack{W_1, W_2 \in \mathcal{W}_H^s \\ d_{\mathcal{W}^s}(W_1, W_2) \leq \varepsilon}} \sup_{\substack{|\psi_i|_{C^\alpha(W_i)} \leq 1 \\ d(\psi_1, \psi_2) = 0}} \varepsilon^{-\gamma} \left| \int_{W_1} f \psi_1 \, dm_{W_1} - \int_{W_2} f \psi_2 \, dm_{W_2} \right|.$$

Finally, define the strong norm of f to be

$$\|f\|_{\mathcal{B}} = \|f\|_s + c_u \|f\|_u$$

for a constant $c_u = c_u(\beta, \gamma, p) > 0$ (so that c_u depends on $[t_0, t_1]$) to be chosen in (4.9). Define \mathcal{B} to be the completion of $C^1(M)$ in the $\|\cdot\|_{\mathcal{B}}$ norm, and \mathcal{B}_w to be the completion of $C^1(M)$ in the $|\cdot|_w$ norm.

²⁸The triangle inequality is not satisfied, but this is of no consequence for our purposes.

²⁹The condition $\gamma \leq \frac{1}{6q+7}$ is used in Lemma 4.3.

³⁰Using weakly homogeneous curves implies that Lebesgue measure belongs to \mathcal{B} , see Remark 4.5.

³¹The weight $|W|^{-1/p}$ in (4.3) is used both in the proof of compactness (Proposition 4.2) and to control the estimate over unmatched pieces in the Lasota-Yorke inequality (Section 4.3.3).

4.2. Statement of the Spectral Result. Embeddings. The equilibrium measure in Theorem 1.1 and its properties will be obtained by letting the transfer operator \mathcal{L}_t act on \mathcal{B} .

Theorem 4.1 (Spectrum of \mathcal{L}_t on \mathcal{B}). *For each $t_0 \in (0, 1)$ and $t_1 \in (1, t_*)$ there exists a Banach space $\mathcal{B} = \mathcal{B}(t_0, t_1)$ such that for each $t \in [t_0, t_1]$, the operator \mathcal{L}_t is bounded on \mathcal{B} with spectral radius equal to $e^{P_*(t)}$ and, recalling $\theta(t_1)$ from Lemma 3.1, essential spectral radius not larger than*

$$\max\{\Lambda^{(-\beta+1/p)}, \theta^{(t-1/p)} e^{-P_*(t)}, \Lambda^{-\gamma}\} e^{P_*(t)} < e^{P_*(t)}.$$

Moreover \mathcal{L}_t has a spectral gap: the only eigenvalue of modulus $e^{P_*(t)}$ is $e^{P_*(t)}$ and it is simple.

Let ν_t denote the unique element of \mathcal{B} with $\nu_t(1) = 1$ satisfying $\mathcal{L}_t \nu_t = e^{P_*(t)} \nu_t$, and let $\tilde{\nu}_t$ denote the maximal eigenvector for the dual, $\mathcal{L}_t^* \tilde{\nu}_t = e^{P_*(t)} \tilde{\nu}_t$. Then the distribution μ_t defined by $\mu_t(\psi) = \frac{\tilde{\nu}_t(\psi \nu_t)}{\tilde{\nu}_t(\nu_t)}$ is in fact a T -invariant probability measure. This measure is mixing, correlations for C^α observables decay exponentially with rate ν for any

$$(4.4) \quad \nu > \nu_0(t) := \sup\{|\lambda| \mid \lambda \in \text{sp}(e^{-P_*(t)} \mathcal{L}_t) \setminus \{1\}\},$$

and correlations for Hölder observables of arbitrary exponent decay exponentially.

Recall that $\theta < 1/\sqrt{\Lambda} < 1$. Note that since $qt_0 > 1$ while $\beta < 1/(q+1)$ and $\gamma \leq \min\{1/(q+1), 1/(q+1) - \beta\}$, our bound on the essential spectral radius tends to $e^{P_*(t_0)}$ as $t_0 \rightarrow 0$. Similarly, as $t_1 \rightarrow t_*$ we need to let $p \rightarrow \infty$ to ensure $\theta^{t_1-1/p} < e^{P_*(t_1)}$ (see Lemma 4.7) and our bound on the essential spectral radius tends to $e^{P_*(t_*)}$ as $t_1 \rightarrow t_*$.

As usual, Hennion's theorem is the key to proving the above theorem. It requires two ingredients: the compact embedding proposition below and the Lasota–Yorke estimates in Proposition 4.6.

Proposition 4.2 (Embeddings). *For any $t_0 \in (0, 1)$ and $t_1 \in (1, t_*)$, the continuous inclusions*

$$C^1(M) \subset \mathcal{B} \subset \mathcal{B}_w \subset (C^1(M))^*$$

hold, so that $C^1(M) \subset (\mathcal{B}_w)^* \subset \mathcal{B}^* \subset (C^1(M))^*$. In addition, the inclusions $C^1(M) \subset \mathcal{B}$ and $\mathcal{B} \subset \mathcal{B}_w$ are injective, and the embedding of the unit ball of \mathcal{B} in \mathcal{B}_w is compact.

The embedding $\mathcal{B}_w \subset (C^1(M))^*$ is understood in the following sense: For $f \in C^1(M)$, we identify f with the measure $f d\mu_{\text{SRB}} \in (C^1(M))^*$. Then, for $f \in \mathcal{B}_w$ there exists $C_f < \infty$ such that, letting $f_n \in C^1(M)$ be a sequence converging to f in the \mathcal{B}_w norm, for every $\psi \in C^1(M)$ the limit $f(\psi) := \lim_{n \rightarrow \infty} \int f_n \psi d\mu_{\text{SRB}}$ exists and satisfies $|f(\psi)| \leq C_f |\psi|_{C^1(M)}$. See Lemma 4.14 for a strengthening of this embedding.

Proof of Proposition 4.2. The proof of the claims in the first sentence is the same as the proof of [BD, Prop. 4.2, Lemma 4.4]. The injectivity of the first inclusion is obvious, while the injectivity of the second follows from our definition of $C^\beta(W)$: if $|f|_w = 0$ then $\|f\|_u = 0$ since the class of test functions is the same, but also $\|f\|_s = 0$ since $C^1(W)$ is dense in $C^\beta(W)$, proving injectivity. The proof of the compact embedding follows exactly the lines of that of [BD, Prop. 6.1], using $\widehat{\mathcal{W}}^s$. The only differences are that, in the unstable norm, $|\psi|_{C^\beta(W)} \leq |\log|W||^\gamma$ there is replaced by $|\psi|_{C^\beta(W)} \leq |W|^{-1/p}$, while the logarithmic modulus of continuity $|\log \epsilon|^{-\zeta}$ there is replaced by a Hölder modulus of continuity ϵ^γ . \square

To show that the transfer operator \mathcal{L}_t is bounded on \mathcal{B} , we require the following lemma.

Lemma 4.3. *For any $f \in C^1(M)$ and any $t \geq t_0$, the image $\mathcal{L}_t f$ belongs to the closure of $C^1(M)$ in the strong norm $\|\cdot\|_{\mathcal{B}}$, for $\mathcal{B} = \mathcal{B}(t_0, t_1)$.*

We prove Lemma 4.3 in Section 4.4.

Remark 4.4 (Lemma 4.9 in [BD]). *We remark that the proof of [BD, Lemma 4.9], which is the analogue of the present Lemma 4.3, was omitted there. The reference given there to [DZ1, Lemma 3.8] is not correct since $J^s T$ is not piecewise Hölder. However, its statement is correct as the proof in Section 4.4 and Remark 4.11 demonstrate.*

Remark 4.5 (Lebesgue measure belongs to \mathcal{B}). *Since we identify $f \in C^1(M)$ with the measure $f d\mu_{SRB}$, Lebesgue measure is identified with the function $f = 1/\cos \varphi$, which is not in $C^1(M)$. However, it follows from [DZ2, Lemma 3.5], that $1/\cos \varphi$ can be approximated by C^1 functions in the \mathcal{B} norm, so that Lebesgue measure belongs to \mathcal{B} . (The proof requires that our norms integrate on weakly homogeneous stable manifolds, rather than on all $W \in \mathcal{W}^s$ as was done in [BD].)*

4.3. Lasota–Yorke Inequalities. Using the exact bounds for $Q_n(t)$ from Proposition 3.15, we prove the following proposition (under more general conditions than Theorem 4.1).

Proposition 4.6. *Fix $t_0 \in (0, 1)$ and $t_1 \in (1, t_*)$ and let $\mathcal{B} = \mathcal{B}(t_0, t_1)$, $\theta = \theta(t_1)$. Fix $t_2 \in (t_0, \infty)$. For any $t \in [t_0, t_2]$, the operator \mathcal{L}_t extends continuously to \mathcal{B}_w and \mathcal{B} , and setting $C_n = \max_{1 \leq j \leq n} Q_j(t)$, there exists $C = C(t_0, t_2) < \infty$ such that for every $n \geq 0$,*

$$(4.5) \quad |\mathcal{L}_t^n f|_w \leq C Q_n(t) |f|_w, \quad \forall f \in \mathcal{B}_w,$$

$$(4.6) \quad \|\mathcal{L}_t^n f\|_s \leq C Q_n(t) \left[(\Lambda^{-(\beta+1/p)n} + \theta^{(t-1/p)n} Q_n(t)^{-1}) \|f\|_s + C C_n |f|_w \right], \quad \forall f \in \mathcal{B},$$

$$(4.7) \quad \|\mathcal{L}_t^n f\|_u \leq C Q_n(t) \left[n \Lambda^{-\gamma n} \|f\|_u + Q_n(t - 1/p) Q_n(t)^{-1} \|f\|_s \right], \quad \forall f \in \mathcal{B}.$$

Moreover, if $t_2 = t_1$, then, up to taking p large enough, for any

$$\sigma \in (\max\{\Lambda^{-\beta+1/p}, \Lambda^{-\gamma}, \theta^{t-1/p} e^{-P_*(t)}\}, 1),$$

there exists $c_u = c_u(t_0, t_1) > 0$, and $\bar{C}_n > 0$, such that, for all $f \in \mathcal{B}$,

$$(4.8) \quad \|\mathcal{L}_t^n f\|_{\mathcal{B}} \leq C e^{P_*(t)n} \left[\sigma^n \|f\|_{\mathcal{B}} + \bar{C}_n |f|_w \right], \quad \forall n \geq 1.$$

Proving (4.8) will use the following lemma:

Lemma 4.7. *For any $t_1 \in (1, t_*)$ there exists $p > 1$ such that $\theta^{t-1/p} < e^{P_*(t)}$ for all $t \in (1/p, t_1]$.*

Proof. If $t \in (0, 1]$ then $P_*(t) \geq 0$ so that $\theta^{t-1/p} < 1 \leq e^{P_*(t)}$ for all $\theta < 1$, all $p > 1$ and all $t \in (1/p, 1]$. For $t \in (1, t_1]$, since the slopes of $P_*(t)$ are at most $-\log \Lambda$ by the proof of Proposition 2.6, we have $\frac{P_*(t_1) - P_*(t)}{t_1 - t} < -\log \Lambda < \log \theta$, so that $\theta^t e^{-P_*(t)} < \theta^{t_1} e^{-P_*(t_1)}$. The choice of $\theta = \theta(t_1)$ in Definition 3.2 gives $\theta^{t_1} e^{-P_*(t_1)} < 1$. Choosing $p > 1$ such that $\theta^{t_1-1/p} e^{-P_*(t_1)} < 1$ ends the proof. \square

Proof of Proposition 4.6. We first show that (4.5), (4.6), and (4.7) imply that if $t_2 = t_1 < t_*$ and p is large enough, then \mathcal{L}_t satisfies the Lasota–Yorke inequality (4.8) for $f \in \mathcal{B}(t_0, t_1)$: Choosing p according to Lemma 4.7, observe that $\theta^{t-1/p} e^{-P_*(t)} < 1$ implies $\theta^{t-1/p} Q_n(t)^{-1} \leq \theta^{(t-1/p)n} e^{-P_*(t)n} < 1$ for all $n \geq 1$, since $Q_n(t) \geq e^{P_*(t)n}$ by Proposition 3.15. Next, recalling that $P_*(t)$ is convex and strictly decreasing by Proposition 2.6, and fixing

$$\varepsilon_1 := P_*(t - 1/p) - P_*(t) \in (0, P_*(t_0 - 1/p) - P_*(t_0)),$$

we find, using both the lower and upper bounds from Proposition 3.18(b),

$$Q_n(t - 1/p) Q_n(t)^{-1} \leq \frac{2}{c_2} e^{P_*(t-1/p)n} e^{-P_*(t)n} \leq \frac{2}{c_2} e^{\varepsilon_1 n}, \quad \forall n \geq 1.$$

Next, fix $1 > \sigma > \max\{\Lambda^{-\beta+1/p}, \Lambda^{-\gamma}, \theta^{t-1/p} e^{-P_*(t)}\}$ and choose $N \geq 1$ such that

$$\frac{2C}{c_2} \max\{N \Lambda^{-\gamma N}, 2(\Lambda^{-(\beta-1/p)N} + \theta^{(t-1/p)N} e^{-P_*(t)N})\} \leq \sigma^N.$$

Choosing $c_u > 0$ to satisfy

$$(4.9) \quad c_u \leq \frac{c_2^2 \sigma^N}{8C e^{2(P_*(t_0-1/p)-P_*(t_0))N}},$$

we estimate, using once more the upper bound for $Q_n(t)$ from Proposition 3.18(b),

$$\begin{aligned} \|\mathcal{L}_t^N f\|_{\mathcal{B}} &= \|\mathcal{L}_t^N f\|_s + c_u \|\mathcal{L}_t^N f\|_u \\ &\leq e^{P_*(t)N} \left[\frac{\sigma^N}{2} \|f\|_s + c_u \sigma^N \|f\|_u + \frac{4c_u}{c_2^2} e^{\varepsilon_1 N} \|f\|_s \right] + CC_N |f|_w \\ &\leq e^{P_*(t)N} \left[\sigma^N \|f\|_{\mathcal{B}} + e^{-P_*(t)N} CC_N |f|_w \right]. \end{aligned}$$

Iterating this equation and using the first claim of (4.5) (recalling one more time the upper bound for $Q_n(t)$ from Proposition 3.18(b)) yields (4.8) for $n = \ell N$, with $\ell \geq 1$. The general case follows since (4.6) and (4.7) imply $\|\mathcal{L}_t^k f\|_{\mathcal{B}} \leq \bar{C} \|f\|_{\mathcal{B}}$ for $k \leq N$.

By Lemma 4.3, it suffices to prove the bounds (4.5), (4.6), and (4.7) for $f \in C^1(M)$, and they also imply that \mathcal{L}_t extends to a bounded operator on \mathcal{B} and \mathcal{B}_w . This is similar to the proof of [DZ1, Proposition 2.3] and is the content of Sections 4.3.1–4.3.3. \square

4.3.1. *Proof of Weak Norm Bound (4.5).* Let $f \in C^1(M)$, $W \in \mathcal{W}^s$ and $\psi \in C^\alpha(W)$ such that $|\psi|_{C^\alpha(W)} \leq 1$. Then for $n \geq 0$, we have

$$(4.10) \quad \begin{aligned} \int_W \mathcal{L}_t^n f \psi \, dm_W &= \sum_{W_i \in \mathcal{G}_n(W)} \int_{W_i} f \psi \circ T^n |J^s T^n|^t \, dm_{W_i} \\ &\leq \sum_{W_i \in \mathcal{G}_n(W)} |f|_w |\psi \circ T^n|_{C^\alpha(W)} |J^s T^n|^t|_{C^\alpha(W)}. \end{aligned}$$

The contraction along stable manifolds implies for $x, y \in W_i \in \mathcal{G}_n(W)$, recalling (2.2),

$$(4.11) \quad |\psi(T^n x) - \psi(T^n y)| \leq H_W^\alpha(\psi) d(T^n x, T^n y)^\alpha \leq H_W^\alpha(\psi) |J^s T^n|_{C^0(W_i)}^\alpha d(x, y)^\alpha.$$

This implies $H_{W_i}^\alpha(\psi \circ T^n) \leq |J^s T^n|_{C^0(W_i)}^\alpha H_W^\alpha(\psi)$ and $|\psi \circ T^n|_{C^\alpha(W_i)} \leq C_1^{-1} |\psi|_{C^\alpha(W)}$, with C_1 from (1.2).

Moreover, since $\alpha \leq 1/(q+1)$, the distortion bound of Lemma 2.1 implies

$$(4.12) \quad \|J^s T^n\|^t|_{C^\alpha(W_i)} \leq (1 + 2^t C_d) |J^s T^n\|^t|_{C^0(W_i)}, \quad \forall W_i \in \mathcal{G}_n(W).$$

Using (4.11) and (4.12) in (4.10), we obtain,

$$\int_W \mathcal{L}_t^n f \psi \, dm_W \leq \sum_{W_i \in \mathcal{G}_n(W)} |f|_w C_1^{-1} (1 + 2^t C_d) |J^s T^n\|^t|_{C^0(W_i)} \leq C |f|_w Q_n(t),$$

where in the last inequality, we have used Lemma 3.4 with $\varsigma = 0$. Taking the suprema over $\psi \in C^\alpha(W)$ with $|\psi|_{C^\alpha(W)} \leq 1$ and $W \in \mathcal{W}^s$ yields (4.5).

4.3.2. *Proof of Stable Norm Bound (4.6).* Let $f \in C^1(M)$, $W \in \mathcal{W}^s$, and $\psi \in C^\beta(W)$ be such that $|\psi|_{C^\beta(W)} \leq |W|^{-1/p}$. For $n \geq 0$ and $W_i \in \mathcal{G}_n(W)$, we define the average $\bar{\psi}_i = |W_i|^{-1} \int_{W_i} \psi \circ T^n \, dm_{W_i}$. Then as in (4.10), we write,

$$(4.13) \quad \begin{aligned} \int_W \mathcal{L}_t^n f \psi \, dm_W &= \sum_{W_i \in \mathcal{G}_n(W)} \int_{W_i} f (\psi \circ T^n - \bar{\psi}_i) |J^s T^n|^t \, dm_{W_i} \\ &\quad + \sum_{W_i \in \mathcal{G}_n(W)} \bar{\psi}_i \int_{W_i} f |J^s T^n|^t \, dm_{W_i}. \end{aligned}$$

Note that by (4.11),

$$|\psi \circ T^n - \bar{\psi}_i|_{C^\beta(W_i)} \leq 2|J^s T^n|_{C^0(W_i)}^\beta |\psi|_{C^\beta(W)} \leq 2|J^s T^n|_{C^0(W_i)}^\beta |W|^{-1/p}.$$

Therefore, replacing α by β in (4.12), the definition of the strong stable norm gives

$$(4.14) \quad \begin{aligned} & \sum_{W_i \in \mathcal{G}_n(W)} \int_{W_i} f |\psi \circ T^n - \bar{\psi}_i| |J^s T^n|^t dm_{W_i} \\ & \leq \sum_{W_i \in \mathcal{G}_n(W)} 2(1 + 2^t C_d) \|f\|_s \frac{|W_i|^{1/p}}{|W|^{1/p}} |J^s T^n|_{C^0(W_i)}^{t+\beta} \\ & \leq 2(1 + 2^t C_d) C_1^{-1} \Lambda^{-n(\beta-1/p)} \|f\|_s C_2[0] Q_n(t), \end{aligned}$$

where in the second inequality we have used Lemma 3.4 with $\varsigma = 1/p$ (recall $\beta > 1/p$).

For the second sum in (4.13), note that $|\bar{\psi}_i| \leq |W|^{-1/p}$. If $|W| \geq \delta_0/3$, then we simply estimate

$$\sum_{W_i \in \mathcal{G}_n(W)} \bar{\psi}_i \int_{W_i} f |J^s T^n|^t dm_{W_i} \leq \frac{3}{\delta_0^{1/p}} \|f\|_w (1 + 2^t C_d) \sum_{W_i \in \mathcal{G}_n(W)} |J^s T^n|_{C^0(W_i)}^t \leq C \|f\|_w Q_n(t),$$

by Lemma 3.4 with $\varsigma = 0$.

If $|W| < \delta_0/3$, we handle the estimate differently, splitting the sum into two parts as follows. We decompose the elements of $\mathcal{G}_n(W)$ by *first long ancestor* as follows: Recalling the sets $\mathcal{I}_n(W)$ defined in §3.1, we call $V_j \in \mathcal{G}_k(W)$ the first long ancestor of $W_i \in \mathcal{G}_n(W)$ if

$$(4.15) \quad T^{n-k} W_i \subset V_j, \quad |V_j| \geq \delta_0/3, \quad \text{and } TV_j \text{ is contained in an element of } \mathcal{I}_{k-1}(W).$$

We denote by $P_k(W)$ the set of such $V_j \in \mathcal{G}_k(W)$ that are long for the first time at time k . Note that W_i has no long ancestor if and only if $W_i \in \mathcal{I}_n(W)$.

Grouping the terms in the second sum in (4.13) by whether they belong to $\mathcal{I}_n(W)$ or not, we apply the weak norm to those elements that have a first long ancestor, and the strong stable norm to those that do not. Thus,

$$(4.16) \quad \begin{aligned} & \sum_{W_i \in \mathcal{I}_n(W)} \bar{\psi}_i \int_{W_i} f |J^s T^n|^t dm_{W_i} \leq \sum_{W_i \in \mathcal{I}_n(W)} |\bar{\psi}_i| \|f\|_s |W_i|^{1/p} |J^s T^n|_{C^\beta(W_i)}^t \\ & \leq (1 + 2^t C_d) \|f\|_s \sum_{W_i \in \mathcal{I}_n(W)} \frac{|W_i|^{1/p}}{|W|^{1/p}} |J^s T^n|_{C^0(W_i)}^t \leq (1 + 2^t C_d) \|f\|_s C_0 \theta^{n(t-1/p)}, \end{aligned}$$

where in the last estimate we applied Lemma 3.3 with $\varsigma = 1/p$ since³² $1/p \leq \min\{1/2, t/2\}$.

For the terms that have a first long ancestor in $P_k(W)$, we again apply Lemma 3.3 from time 0 (since $|W| < \delta_0/3$) to time k , setting $\mathcal{G}_0(V) = \{V\}$,

$$\begin{aligned} & \sum_{k=1}^n \sum_{V_j \in P_k(W)} \sum_{W_i \in \mathcal{G}_{n-k}(V_j)} \bar{\psi}_i \int_{W_i} f |J^s T^n|^t dm_{W_i} \\ & \leq \sum_{k=1}^n \sum_{V_j \in P_k(W)} \|f\|_w |V_j|^{-1/p} (1 + 2^t C_d) \frac{|V_j|^{1/p}}{|W|^{1/p}} |J^s T^k|_{C^0(V_j)}^t \sum_{W_i \in \mathcal{G}_{n-k}(V_j)} |J^s T^{n-k}|_{C^0(W_i)}^t \\ & \leq \sum_{k=1}^n \sum_{V_j \in P_k(W)} \|f\|_w 3\delta_0^{-1/p} (1 + 2^t C_d) \frac{|V_j|^{1/p}}{|W|^{1/p}} |J^s T^k|_{C^0(V_j)}^t C C_2[0] Q_{n-k}(t) \\ & \leq \sum_{k=1}^n \|f\|_w 3\delta_0^{-1/p} (1 + 2^t C_d) C C_2[0] C_0 \theta^{k(t-1/p)} Q_{n-k}(t), \end{aligned}$$

³²This bound holds since $p > q + 1$ in the definition of the norms, yet $q \geq 2/t$ from (2.1).

applying Lemma 3.4 for $\varsigma = 0$ in the second inequality and Lemma 3.3 (for $\varsigma = 1/p$) in the third.

Putting these estimates together with (4.14) in (4.13) yields³³

$$\int_W \mathcal{L}_t^n f \psi \, dm_W \leq C Q_n(t) (\Lambda^{-n(\beta-1/p)} + \theta^{n(t-1/p)} Q_n(t)^{-1}) \|f\|_s + C \max_{0 \leq j \leq n} Q_j(t) \|f\|_w,$$

and taking the appropriate suprema proves (4.6) (C_n depends on t only through $[t_0, t_1]$).

4.3.3. Proof of Unstable Norm Bound (4.7). Let $\varepsilon < \varepsilon_0$ and let $W^1, W^2 \in \mathcal{W}^s$ with $d_{\mathcal{W}^s}(W^1, W^2) \leq \varepsilon$. For $n \geq 1$ and $\ell = 1, 2$, we partition $T^{-n}W^\ell$ into matched pieces U_j^ℓ and unmatched pieces V_i^ℓ like in [DZ1] as follows.

To each homogeneous connected component V of $T^{-n}W^1$, we associate a family of vertical segments $\{\gamma_x\}_{x \in V}$ of length at most $C_1^{-1} \Lambda^{-n} \varepsilon$ such that if γ_x is not cut by an element of \mathcal{S}_n^{H} , its image $T^n \gamma_x$ will have length $C\varepsilon$ and will intersect W^2 . According to [CM, Sect. 4.4], for such a segment, $T^i \gamma_x$ will be an unstable curve for $i = 1, \dots, n$ and so will remain uniformly transverse to the stable cone and undergo the minimum expansion given by (1.2).

Repeating this procedure for each connected component of $T^{-n}W^1$, we obtain a partition of W^1 into subintervals for which $T^n \gamma_x$ is not cut and intersects W^2 and subintervals for which this is not the case. This also defines an analogous partition on W^2 and on the images $T^{-n}W^1$ and $T^{-n}W^2$. We call two curves in $T^{-n}W^1$ and $T^{-n}W^2$ *matched* if they are connected by the foliation γ_x and their images under T^n are connected by $T^n \gamma_x$. We further subdivide the matched pieces if necessary to ensure that they have length $\leq \delta_0$ and that they remain homogeneous stable curves. Thus there are at most two matched pieces U_j^ℓ corresponding to each element of $\mathcal{G}_n(W^\ell)$. The rest of the connected components of $T^{-n}W^\ell$ we call *unmatched* and denote them by V_i^ℓ . Once again, there are at most two unmatched pieces V_i^ℓ corresponding to each element of $\mathcal{G}_n(W^\ell)$.

Recalling the notation of Section 4.1, we have constructed a pairing on matched pieces U_j^ℓ defined over a common r -interval I_j such that for each j ,

$$(4.17) \quad U_j^\ell = G_{U_j^\ell}(I_j) = \{(r, \varphi_{U_j^\ell}(r)) : r \in I_j\}, \quad \ell = 1, 2.$$

Now let $\psi_\ell \in C^\alpha(W^\ell)$ with $|\psi_\ell|_{C^\alpha(W^\ell)} \leq 1$ and $d(\psi_1, \psi_2) = 0$. Decomposing W^1 and W^2 into matched and unmatched pieces as above, we write,

$$(4.18) \quad \left| \int_{W^1} \mathcal{L}_t^n f \psi_1 - \int_{W^2} \mathcal{L}_t^n f \psi_2 \right| \leq \sum_j \left| \int_{U_j^1} f \psi_1 \circ T^n |J^s T^n|^t - \int_{U_j^2} f \psi_2 \circ T^n |J^s T^n|^t \right| \\ + \sum_{\ell, i} \left| \int_{V_i^\ell} f \psi_\ell \circ T^n |J^s T^n|^t \right|.$$

We estimate the unmatched pieces first. For this we use the fact that unmatched pieces V_i^ℓ occur either because $T^n V_i^\ell$ is near the endpoints of W^ℓ or because a vertical segment $T^n \gamma_x$ intersects \mathcal{S}_n^{H} . In either case, due to the uniform transversality of the stable and unstable cones, we have $|T^n V_i^\ell| \leq C\varepsilon$ for some uniform constant $C > 0$, independent of n and W^ℓ , since $d_{\mathcal{W}^s}(W^1, W^2) \leq \varepsilon$. Thus, we estimate the sum over unmatched pieces using the strong stable norm,

$$(4.19) \quad \sum_{\ell, i} \left| \int_{V_i^\ell} f \psi_\ell \circ T^n |J^s T^n|^t \right| \leq \sum_{\ell, i} \|f\|_s |V_i^\ell|^{1/p} |\psi_\ell \circ T^n|_{C^\beta(V_i^\ell)} (1 + 2^t C_d) |J^s T^n|_{C^0(V_i^\ell)}^t \\ \leq \|f\|_s C_1^{-1} (1 + 2^t C_d) \sum_{\ell, i} |T^n V_i^\ell|^{1/p} |J^s T^n|_{C^0(V_i^\ell)}^{t-1/p} \\ \leq 4C_2[0] C_1^{-1} (1 + 2^t C_d) \|f\|_s \varepsilon^{1/p} Q_n(t-1/p),$$

³³It is in fact possible to show $\max_{0 \leq j \leq n} Q_j(t) \leq \max\{Q_n(t), Q_n(1)\}$, but we shall not use this.

where C_1 is from (1.2) and we have used (4.12) in the first inequality, (4.11) in the second, and Lemma 3.4 (for $\varsigma = 0$) in the third since there are at most two unmatched pieces corresponding to each element of $\mathcal{G}_n(W^\ell)$.

To perform the estimate over matched pieces in (4.18), we will change variables to define the relevant test functions on the same curve. To this end, recalling (4.17), define on each U_j^1 ,

$$\tilde{\psi}_2 = \psi_2 \circ T^n \circ G_{U_j^2} \circ G_{U_j^1}^{-1}, \quad \text{and} \quad \tilde{J}^s T^n = J^s T^n \circ G_{U_j^2} \circ G_{U_j^1}^{-1}.$$

Sublemma 4.8. *There exists $C > 0$, independent of t, n, W^1 , and W^2 such that*

- a) $d_{\mathcal{W}^s}(U_j^1, U_j^2) \leq Cn\Lambda^{-n}\varepsilon =: \varepsilon_1$, for all j ;
- b) $|\psi_1 \circ T^n |J^s T^n|^t - \tilde{\psi}_2 |\tilde{J}^s T^n|^t|_{C^\beta(U_j^1)} \leq C2^t |J^s T^n|_{C^0(U_j^1)}^t \varepsilon^{\alpha-\beta}$, for all j .

Proof. Part (a) of the sublemma is [DZ1, Lemma 4.2]. To prove part (b), note that due to the uniform bound on slopes of stable curves, it follows

$$(4.20) \quad 1 \leq JG_W(r) := \sqrt{1 + (\varphi'_W(r))^2} \leq \sqrt{1 + (\mathcal{K}_{\max} + \tau_{\min}^{-1})^2} =: C_g < \infty.$$

Therefore $1 \leq |JG_{U_j^\ell}|_{C^0(I_j)} \leq C_g$, and we have

$$\begin{aligned} & |\psi_1 \circ T^n |J^s T^n|^t - \tilde{\psi}_2 |\tilde{J}^s T^n|^t|_{C^\beta(U_j^1)} \\ & \leq C_g |(\psi_1 \circ T^n |J^s T^n|^t) \circ G_{U_j^1} - (\psi_2 |\tilde{J}^s T^n|^t) \circ G_{U_j^2}|_{C^\beta(I_j)} \\ & \leq C_g |\psi_2 \circ T^n|_{C^\beta(U_j^2)} ||J^s T^n|^t \circ G_{U_j^1} - |J^s T^n|^t \circ G_{U_j^2}|_{C^\beta(I_j)} \\ & \quad + C_g ||J^s T^n|^t|_{C^\beta(U_j^1)} |\psi_1 \circ T^n \circ G_{U_j^1} - \psi_2 \circ T^n \circ G_{U_j^2}|_{C^\beta(I_j)} \\ & \leq C_g C_1^{-1} ||J^s T^n|^t \circ G_{U_j^1} - |J^s T^n|^t \circ G_{U_j^2}|_{C^\beta(I_j)} \\ & \quad + C_g (1 + 2^t C_d) ||J^s T^n|^t|_{C^0(U_j^1)} |\psi_1 \circ T^n \circ G_{U_j^1} - \psi_2 \circ T^n \circ G_{U_j^2}|_{C^\beta(I_j)}, \end{aligned}$$

where we have used (4.11) and (4.12) for the final inequality. We first observe that

$$|\psi_1 \circ T^n \circ G_{U_j^1} - \psi_2 \circ T^n \circ G_{U_j^2}|_{C^\beta(I_j)} \leq C\varepsilon^{\alpha-\beta},$$

by [DZ1, Lemma 4.4]. For brevity, set $J_\ell = J^s T^n \circ G_{U_j^\ell}$. By³⁴ [DZ1, eq. (4.16)], we have

$$(4.21) \quad \left| 1 - \frac{J_1(r)}{J_2(r)} \right| \leq C_{vert} \varepsilon^{1/(q+1)}, \quad \forall r \in I_j,$$

for some constant $C_{vert} > 0$ depending only on the uniform angle between the vertical direction and the stable and unstable cones. Thus, since $\varepsilon_0 > 0$ satisfies (4.2) and $\varepsilon \leq \varepsilon_0$, this implies that $\frac{1}{4} \leq \frac{J_1(r)}{J_2(r)} \leq \frac{7}{4}$. Then, estimating as in Lemma 2.1, we have

$$|J_1^t(r) - J_2^t(r)| \leq 2^t |J_1^t|_{C^0(I_j)} C_{vert} \varepsilon^{1/(q+1)}.$$

Following [DZ1, eq. (4.17) and (4.18)], yields,

$$H^\beta(J_1^t - J_2^t) \leq C2^t |J_1^t|_{C^0(I_j)} \sup_{r,s \in I_j} \min\{\varepsilon^{1/(q+1)} |r-s|^{-\beta}, |r-s|^{1/(q+1)-\beta}\},$$

where $H^\beta(\cdot)$ is the Hölder constant with exponent β on I_j . This bound is maximized when $\varepsilon = |r-s|$, which yields $H^\beta(J_1^t - J_2^t) \leq C2^t |J_1^t|_{C^0(I_j)} \varepsilon^{1/(q+1)-\beta}$. Putting these estimates together yields,

$$||J^s T^n|^t \circ G_{U_j^1} - |J^s T^n|^t \circ G_{U_j^2}|_{C^\beta(I_j)} \leq C2^t |J^s T^n|_{C^0(U_j^1)}^t \varepsilon^{1/(q+1)-\beta}.$$

³⁴The case $q = 2$ is treated there, the general case is similar.

Together with the previous estimate on ψ_ℓ , this completes the proof of the sublemma since $\alpha \leq 1/(q+1)$. \square

Returning to (4.18), we split the estimate over matched pieces as follows.

$$(4.22) \quad \left| \int_{U_j^1} f \psi_1 \circ T^n |J^s T^n|^t - \int_{U_j^2} f \psi_2 \circ T^n |J^s T^n|^t \right| \leq \left| \int_{U_j^1} f (\psi_1 \circ T^n |J^s T^n|^t - \tilde{\psi}_2 |\tilde{J}^s T^n|^t) \right| \\ + \left| \int_{U_j^1} f \tilde{\psi}_2 |\tilde{J}^s T^n|^t - \int_{U_j^2} f \psi_2 \circ T^n |J^s T^n|^t \right|.$$

We estimate the first term on the right side using the strong stable norm and Lemma 4.8(b),

$$\left| \int_{U_j^1} f (\psi_1 \circ T^n |J^s T^n|^t - \tilde{\psi}_2 |\tilde{J}^s T^n|^t) \right| \leq \|f\|_s \delta_0^{1/p} C 2^t |J^s T^n|_{C^0(U_j^1)}^t \varepsilon^{\alpha-\beta}.$$

Then, noting that $d(\psi_1 \circ T^n |J^s T^n|^t, \tilde{\psi}_2 |\tilde{J}^s T^n|^t) = 0$ by definition, and the C^α norm of each test function is bounded by $C 2^t |J^s T^n|_{C^0(I_j)}^t$, using (4.11) and (4.12), we estimate the second term on the right side of (4.22) using the strong unstable norm:

$$(4.23) \quad \left| \int_{U_j^1} f \tilde{\psi}_2 |\tilde{J}^s T^n|^t - \int_{U_j^2} f \psi_2 \circ T^n |J^s T^n|^t \right| \leq \|f\|_u d_{\mathcal{W}^s}(U_j^1, U_j^2)^\gamma C 2^t |J^s T^n|_{C^0(U_j^1)}^t \\ \leq C' \|f\|_u n^\gamma \Lambda^{-n\gamma} \varepsilon^\gamma |J^s T^n|_{C^0(U_j^1)}^t,$$

where we used Lemma 4.8(a) in the second inequality. Putting these estimates into (4.22), then combining with (4.19) in (4.18), and summing over j (since there are at most two matched pieces corresponding to each element of $\mathcal{G}_n(W^1)$), yields,

$$(4.24) \quad \left| \int_{W^1} \mathcal{L}_t^n f \psi_1 - \int_{W^2} \mathcal{L}_t^n f \psi_2 \right| \\ \leq C \left(\|f\|_u n^\gamma \Lambda^{-n\gamma} \varepsilon^\gamma Q_n(t) + \|f\|_s (\varepsilon^{1/p} Q_n(t-1/p) + \varepsilon^{\alpha-\beta} Q_n(t)) \right).$$

Dividing through by ε^γ and taking the appropriate suprema over W^ℓ and ψ_ℓ proves (4.7) since $\gamma \leq \min\{1/p, \alpha - \beta\}$.

4.4. Proof of Lemma 4.3. ($\mathcal{L}_t(C^1) \subset \mathcal{B}$). We assume $0 \leq t < 1$. The proof for $t \geq 1$ is similar, but simpler, since $\mathcal{L}_t f$ is bounded when $t \geq 1$. Without loss of generality, we also assume that $t_0 \leq 1/2$, so that, by Definition 3.2, $q \geq 8$ and $p > 9$.

We introduce a mollification in order to approximate $\mathcal{L}_t f$ by functions in $C^1(M)$: Let $\rho: \mathbb{R}^2 \rightarrow \mathbb{R}$ be a C^∞ nonnegative, rotationally symmetric function supported on the unit disk with $\int_{\mathbb{R}^2} \rho d^2 z = 1$ and $|\rho|_{C^1} \leq 2$. For $f \in C^1(M)$ and $\eta > 0$, define

$$g_\eta(x) = \int_{B_\eta(x)} \eta^{-2} \rho \left(\frac{d(x, z)}{\eta} \right) \mathcal{L}_t f(z) d^2 z,$$

where $B_\eta(x)$ is the ball of radius η centered at x . Viewing M as a subset of \mathbb{R}^2 , we set $\mathcal{L}_t f \equiv 0$ outside M so that the integral is well-defined even when $B_\eta(x) \not\subset M$. We first develop bounds on $|g_\eta|_{C^0(M)}$ and $|g_\eta|_{C^1(M)}$, for any $t \geq 0$.

Since $t < 1$, the operator $\mathcal{L}_t f$ is unbounded in neighborhoods of $T\mathcal{S}_0$, so the bounds on g_η will be greatest in such neighborhoods. Suppose x and η are such that $B_\eta(x) \cap T\mathcal{S}_0 \neq \emptyset$ and note that there can be at most $\tau_{\max}/\tau_{\min} + 1$ connected components of $B_\eta(x) \setminus T\mathcal{S}_0$. Fix one such component with boundary $S \in T\mathcal{S}_0$ such that S is the accumulation of the sequence of sets, $B_\eta(x) \cap T\mathbb{H}_k$, $k \geq k_0$. On each such set, $|J^s T|^{1-t} = C^{\pm 1} k^{-q(1-t)}$. Also, due to the uniform transversality of $T\mathcal{S}_0^{\text{H}}$ with the stable cone, we have $\text{diam}^s(B_\eta(x) \cap T\mathbb{H}_k) \leq C k^{-2q-1}$, and $\text{diam}^u(B_\eta(x) \cap T\mathbb{H}_k) \leq C \eta$. Moreover,

since the boundary of $T\mathbb{H}_k$ has distance approximately k^{-2q} from S , we have $B_\eta(x) \cap T\mathbb{H}_k = \emptyset$ unless $k \geq C\eta^{-1/(2q)}$. Assembling these facts, we estimate,

$$|g_\eta| \leq C \sum_{k \geq C\eta^{-1/(2q)}} \int_{B_\eta(x) \cap T\mathcal{S}_0} \eta^{-2} \mathcal{L}_t f d^2 z \leq C|f|_\infty \sum_{k \geq C\eta^{-1/(2q)}} \eta^{-1} k^{-q-1-qt}.$$

We conclude that, for any $0 \leq t < 1$,³⁵

$$(4.25) \quad |g_\eta|_{C^0(M)} \leq C|f|_\infty \eta^{\frac{t}{2} - \frac{1}{2}}, \quad \text{and similarly,} \quad |g_\eta|_{C^1(M)} \leq C|f|_\infty \eta^{\frac{t}{2} - \frac{3}{2}}.$$

To prove Lemma 4.3, we must control $g_\eta - \mathcal{L}_t f$ integrated along stable manifolds. To this end, we will need the following two lemmas. (The first one is classical and the second uses bounds on the auxiliary foliation constructed in [BDL, Section 6].)

Lemma 4.9. *Let $W \in \mathcal{W}^s$ be weakly homogeneous and for $\eta > 0$ let $W_u(\eta) \subset W$ denote the set of points in W whose unstable manifold extends at least length η on both sides of W . Then $m_W(W \setminus W_u(\eta)) \leq C\eta$ for some constant $C > 0$ independent of W and η .*

Proof. This is precisely [CM, Theorem 5.66]. See also the corrected proof in [BDT]. \square

Lemma 4.10. *There exist constants $C, C_s > 0$ such that for any weakly homogeneous unstable curve U and any $\varrho > 0$, there exists a set $U' \subset U$ with $m_U(U \setminus U') \leq C\varrho$ such that*

$$\left| \frac{J^s T(x)}{J^s T(y)} - 1 \right| \leq C_s \left(\varrho^{-\frac{q}{q+1}} k_U^{-q} d(x, y) + d(x, y)^{1/(q+1)} \right), \quad \forall x, y \in U',$$

where k_U is the index of the homogeneity strip containing U .

Proof. Fixing a length $\varrho < k_U^{-q-1}$, we define a foliation of stable curves transverse to U , following the procedure³⁶ in [BDL, Sections 6.1, 6.2]: Choose $n \in \mathbb{N}$ arbitrarily large and define a smooth “seeding” foliation of homogeneous stable curves transverse to connected components of $T^n U$; elements of the seeding foliation are then pulled back under T^{-n} and those that are not cut form a foliation of homogeneous stable curves of length at least ϱ and transverse to U . Letting $U'_n \subset U$ denote the set covered by this surviving foliation, we have $m_U(U \setminus U'_n) \leq C\varrho$, for some $C > 0$ independent of n [BDL, Section 6.1]. Moreover, expressing the foliation in local coordinates (s, u) adapted to the stable and unstable directions defines a function $G(s, u)$ such that each stable curve can be expressed as $\{(s, G(s, u))\}_{s \in [-\varrho, \varrho]}$, and $G(0, u) = u$, so that the unstable manifold U corresponds to the vertical segment $\{(0, u)\}_{u \in [0, |U|]}$. It follows that the slope $\mathcal{V}(u)$ of the tangent vector to the foliation at $(0, u)$ is just $\partial_s G(0, u)$. By [BDL, Lemma 6.5], $\partial_u \partial_s G \in C^0$ with $|\partial_u \partial_s G|_\infty \leq C\varrho^{-q/(q+1)} k_U^{-q}$ (where we have adapted the exponent according to the spacing of our homogeneity strips).

Note that the foliation of stable curves constructed in this way has tangent vectors in $DT^{-n}C^s$. Since the bounds on $m_U(U \setminus U'_n)$ and $|\partial_u \partial_s G|_\infty$ are independent of n , we conclude there exists a set $U' \subset U$ with $m_U(U \setminus U') \leq C\varrho$ such that the stable manifolds passing through U' have length at least ϱ and satisfy $|\partial_u \partial_s G|_\infty \leq C\varrho^{-q/(q+1)} k_U^{-q}$ (see also [BDL, Remark 1.1]).

Finally, for $u, v \in U'$ we estimate as in (2.7) (with $n = 1$), using (2.8) for $\log \frac{\cos \varphi(u)}{\cos \varphi(v)}$ and $|\mathcal{V}(u) - \mathcal{V}(v)| \leq C\varrho^{-q/(q+1)} k_U^{-q} d(u, v)$ from the construction in [BDL]. Putting these estimates together proves the lemma. \square

We record for future use that for any measurable set $V \subseteq W \in \mathcal{W}^s$,

$$(4.26) \quad \begin{aligned} \int_V \mathcal{L}_t f \psi dm_V &= \int_{T^{-1}V} f |J^s T|^t \psi \circ T dm_{T^{-1}V} \\ &\leq |f|_\infty |\psi|_\infty |T^{-1}V|^{1-t} |V|^t \leq C|f|_\infty |\psi|_\infty |V|^{(1+t)/2}, \end{aligned}$$

³⁵For $t = 0$, any choice of $q > 1$ gives the same bound.

³⁶[BDL] constructs this as a foliation of unstable curves transverse to a stable curve. By the time reversal property of the billiard, the same construction holds with stable and unstable directions exchanged.

where $|V|$ denotes the arc length measure of V , and we have used the Hölder inequality for the first inequality and the bound $|T^{-1}V| \leq C|V|^{1/2}$ in the second.

Approximating the strong stable norm. Fix $\eta > 0$, and let $W \in \mathcal{W}^s$ and $\psi \in C^\beta(W)$ with $|\psi|_{C^\beta(W)} \leq |W|^{-1/p}$. If $|W| \leq \eta$, then using (4.25) and (4.26), we write, simply, using $p > 9$,

$$(4.27) \quad \int_W (\mathcal{L}_t f - g_\eta) \psi \, dm_W \leq C|f|_\infty |W|^{-1/p} (|W|^{\frac{1+t}{2}} + |W|\eta^{\frac{t}{2}-\frac{1}{2}}) \leq C|f|_\infty \eta^{\frac{t}{2}+\frac{1}{3}}.$$

In what follows, we assume $|W| > \eta$. Let W_η^- denote the curve W minus the η -neighborhood of its boundary. Treating the integral over the two components of $W \setminus W_\eta^-$ in the same way as (4.27), we estimate, using that $m_W(W \setminus W_\eta^-) \leq 2\eta$,

$$(4.28) \quad \int_{W \setminus W_\eta^-} (\mathcal{L}_t f - g_\eta) \psi \, dm_W \leq C|f|_\infty \eta^{\frac{t}{2}+\frac{1}{3}}.$$

Next, since W intersects at most $N = \tau_{\max}/\tau_{\min} + 1$ elements of $T\mathcal{S}_0$, the set $W \cap (\cup_{k \geq \eta^{-1/(2q+1)}} T\mathbb{H}_k)$ comprises at most N intervals of length $C\eta^{2q/(2q+1)}$. We estimate as in (4.27) using $V = W \cap (\cup_{k \geq \eta^{-1/(2q+1)}} T\mathbb{H}_k)$ in (4.26), and that $p > q + 1 \geq 9$

$$(4.29) \quad \int_{W \cap (\cup_{k \geq \eta^{-1/(2q+1)}} T\mathbb{H}_k)} (\mathcal{L}_t f - g_\eta) \psi \, dm_W \leq C|f|_\infty \eta^{\frac{t}{2}+\frac{3}{10}}.$$

Finally, we estimate $\mathcal{L}_t f - g_\eta$ on those portions of W_η^- that intersect $T\mathbb{H}_k$ for $k \leq \eta^{-1/(2q+1)}$. Let x be such a point in W_η^- . Due to the restriction on k , the ball $B_\eta(x)$ lies in a bounded number of homogeneity strips, so we may use bounded distortion in conjunction with Lemma 4.10 to bound the difference in each such interval. Let $S_\eta = W \setminus W_u(\eta)$ denote the exceptional set of points in Lemma 4.9. We write $A_\eta(x)$ for the subset of $B_\eta(x)$ foliated by unstable manifolds of length at least 2η , and let $E_\eta(x) = B_\eta(x) \setminus A_\eta(x)$. Then,

$$(4.30) \quad \begin{aligned} \mathcal{L}_t f(x) - g_\eta(x) &= \int_{B_\eta(x)} \eta^{-2} \rho\left(\frac{d(x,z)}{\eta}\right) (\mathcal{L}_t f(x) - \mathcal{L}_t f(z)) d^2 z \\ &= \int_{A_\eta(x)} \eta^{-2} \rho\left(\frac{d(x,z)}{\eta}\right) (\mathcal{L}_t f(x) - \mathcal{L}_t f(z)) d^2 z \\ &\quad + \int_{E_\eta(x)} \eta^{-2} \rho\left(\frac{d(x,z)}{\eta}\right) (\mathcal{L}_t f(x) - \mathcal{L}_t f(z)) d^2 z. \end{aligned}$$

We first estimate the integral over $E_\eta(x)$ using the bound $\mathcal{L}_t f(z) \leq C\mathcal{L}_t f(x)$ for $z \in B_\eta(x)$, since $B_\eta(x)$ lies in a bounded number of homogeneity strips. Then, using the fact that the unstable foliation is absolutely continuous, we disintegrate as follows,

$$(4.31) \quad \int_{E_\eta(x)} \eta^{-2} \rho\left(\frac{d(x,z)}{\eta}\right) (\mathcal{L}_t f(x) - \mathcal{L}_t f(z)) d^2 z \leq C\mathcal{L}_t f(x) \eta^{-1} |S_\eta \cap B_\eta(x)|.$$

Next, we estimate the integral over $A_\eta(x)$. Since each point $y \in A_\eta(x) \cap W_\eta^-$ has an unstable manifold U_y extending a length at least η on either side of W , we set $\varrho = \eta^{1+\frac{1}{2q}}$ and denote by $A'_\eta(x)$ those points contained in sets $U'_y \subset U_y$ satisfying Lemma 4.10. It follows from that lemma and the absolute continuity of the unstable foliation that

$$(4.32) \quad \int_{A_\eta(x) \setminus A'_\eta(x)} \eta^{-2} \rho\left(\frac{d(x,z)}{\eta}\right) (\mathcal{L}_t f(x) - \mathcal{L}_t f(z)) d^2 z \leq C\mathcal{L}_t f(x) \eta^{\frac{1}{2q}},$$

where we have again used the bound $\mathcal{L}_t f(z) \leq C\mathcal{L}_t f(x)$ on $B_\eta(x)$.

For $z \in A'_\eta(x)$, we bound the difference $\mathcal{L}_t f(x) - \mathcal{L}_t f(z)$ as follows. Let $y = [x, z]$ denote the point of intersection between the stable manifold of x (which is W) and the unstable manifold of z ,

which is U_y . By definition, $z \in U'_y$ and it is always the case that $y \in U'_y$ since the stable manifold of y , W , has length at least $\eta > \varrho$. Thus,

$$\begin{aligned} & |\mathcal{L}_t f(x) - \mathcal{L}_t f(z)| \\ & \leq \left| \frac{f(T^{-1}x)}{|J^s T|^{1-t}(T^{-1}x)} - \frac{f(T^{-1}y)}{|J^s T|^{1-t}(T^{-1}y)} \right| + \left| \frac{f(T^{-1}y)}{|J^s T|^{1-t}(T^{-1}y)} - \frac{f(T^{-1}z)}{|J^s T|^{1-t}(T^{-1}z)} \right| \\ & \leq \mathcal{L}_t 1(x) [|f|_{C^1} d(T^{-1}x, T^{-1}y) + |f|_{C^0} C d(T^{-1}x, T^{-1}y)^{1/(q+1)} \\ & \quad + |f|_{C^1} d(T^{-1}y, T^{-1}z) + |f|_{C^0} C_s (\eta^{-\frac{2q+1}{2q+2}} d(T^{-1}y, T^{-1}z) + d(T^{-1}y, T^{-1}z)^{1/(q+1)})], \end{aligned}$$

where we have used Lemma 2.1 along W and Lemma 4.10 along U_y with $\varrho = \eta^{\frac{2q+1}{2q}}$. Next, $d(T^{-1}y, T^{-1}z) \leq C d(y, z) \leq C\eta$, while for $x \in T\mathbb{H}_k$,

$$d(T^{-1}x, T^{-1}y) \leq C k^q d(x, y) \leq C \eta^{\frac{q+1}{2q+1}}$$

since $k \leq \eta^{-1/(2q+1)}$. Putting these estimates together we obtain,

$$|\mathcal{L}_t f(x) - \mathcal{L}_t f(z)| \leq |f|_{C^1} \mathcal{L}_t 1(x) C \eta^{\frac{1}{2q+2}} \quad \text{for } z \in A'_\eta(x),$$

and combining this with (4.31) and (4.32) in (4.30) yields,

$$(4.33) \quad |\mathcal{L}_t f(x) - g_\eta(x)| \leq C |f|_{C^1} \mathcal{L}_t 1(x) \eta^{\frac{1}{2q+2}} + C |f|_{C^0} \mathcal{L}_t 1(x) \eta^{-1} |S_\eta \cap B_\eta(x)|.$$

We must integrate this bound over $W_\eta^- \cap (\cup_{k \leq \eta^{-1/(2q+1)}} T\mathbb{H}_k)$. We estimate the integral of the first term in (4.33) simply using (4.26),

$$(4.34) \quad C |f|_{C^1} \eta^{\frac{1}{2q+2}} \int_{W_\eta^- \cap (\cup_{k \leq \eta^{-1/(2q+1)}} T\mathbb{H}_k)} \mathcal{L}_t 1 \psi \, dm_W \leq C |f|_{C^1} \eta^{\frac{1}{2q+2}}.$$

Finally, to bound the second term in (4.33), we write $I_\eta(x) = B_\eta(x) \cap W$ and

$$|S_\eta \cap B_\eta(x)| = \int_{I_\eta(x)} 1_{S_\eta}(z) dm_W(z) = \int_{-\eta}^{\eta} 1_{S_\eta}(x + G_W(x; r)) JG_W(x; r) dr,$$

where $G_W(x; r)$ denotes the (local) graph of the function defining W in a neighborhood of x , as in (4.17), and we have centered the local r -interval at $r = 0$. Then,

$$\begin{aligned} & \int_{W_\eta^- \cap (\cup_{k \leq \eta^{-1/(2q+1)}} T\mathbb{H}_k)} \mathcal{L}_t f(x) \frac{\psi(x)}{\eta} \int_{-\eta}^{\eta} 1_{S_\eta}(x + G_W(x; r)) JG_W(x; r) dr \, dm_W(x) \\ & \leq |f|_{C^0} \frac{|W|^{-1/p}}{\eta} \int_{-\eta}^{\eta} \int_{W_\eta^-} \mathcal{L}_t 1(x) 1_{S_\eta}(x + G_W(x; r)) JG_W(x; r) \, dm_W(x) dr \\ (4.35) \quad & \leq C |f|_{C^0} \frac{|W|^{-1/p}}{\eta} \int_{-\eta}^{\eta} |S_\eta|^{(1+t)/2} dr \leq C |f|_{C^0} \eta^{\frac{1}{3} + \frac{t}{2}}, \end{aligned}$$

where we have used (4.20) and the fact that translations of W_η^- up to length η are subsets of W in order to apply (4.26) for the second inequality, and Lemma 4.9, with $|W| \geq \eta$ and $p > 9$ for the final inequality.

Finally, using (4.34) and (4.35) in (4.33), and adding the contributions from (4.28) and (4.29) in addition to (4.27) yields,

$$(4.36) \quad \int_W (\mathcal{L}_t f - g_\eta) \psi \, dm_W \leq C |f|_{C^1} \eta^{\frac{1}{2q+2}},$$

for some $C > 0$ independent of W , since $\min\{\frac{t}{2} + \frac{3}{10}, \frac{1}{2q+2}\} = \frac{1}{2q+2}$ whenever $q > 1$ and $t > 0$. Taking the appropriate suprema over ψ and W yields the required estimate $\|\mathcal{L}_t f - g_\eta\|_s \leq C |f|_{C^1} \eta^{\frac{1}{2q+2}}$.

Approximating the unstable norm. Let $\varepsilon \leq \varepsilon_0$ and $W_1, W_2 \in \mathcal{W}^s$ with $d_{\mathcal{W}^s}(W_1, W_2) \leq \varepsilon$. Let $\psi_i \in C^\alpha(W_i)$ with $|\psi_i|_{C^\alpha(W_i)} \leq 1$, $i = 1, 2$, and $d(\psi_1, \psi_2) = 0$. We must estimate,

$$\int_{W_1} (\mathcal{L}_t f - g_\eta) \psi_1 dm_{W_1} - \int_{W_2} (\mathcal{L}_t f - g_\eta) \psi_2 dm_{W_2}.$$

We consider two cases.

Case 1: $\eta^{\frac{1}{2q+2}} < \varepsilon^{2\gamma}$. We apply (4.36) to each term separately and obtain

$$\varepsilon^{-\gamma} \left| \int_{W_1} (\mathcal{L}_t f - g_\eta) \psi_1 dm_{W_1} - \int_{W_2} (\mathcal{L}_t f - g_\eta) \psi_2 dm_{W_2} \right| \leq C|f|_{C^1} \eta^{\frac{1}{4q+4}}.$$

Case 2: $\eta^{\frac{1}{2q+2}} \geq \varepsilon^{2\gamma}$. In this case, we write

$$(4.37) \quad \begin{aligned} & \int_{W_1} (\mathcal{L}_t f - g_\eta) \psi_1 dm_{W_1} - \int_{W_2} (\mathcal{L}_t f - g_\eta) \psi_2 dm_{W_2} \\ &= \int_{W_1} \mathcal{L}_t f \psi_1 dm_{W_1} - \int_{W_2} \mathcal{L}_t f \psi_2 dm_{W_2} + \int_{W_2} g_\eta \psi_2 dm_{W_2} - \int_{W_1} g_\eta \psi_1 dm_{W_1}. \end{aligned}$$

We estimate the difference involving $\mathcal{L}_t f$ using the estimates in Section 4.3.3, but using the fact that $f \in C^1(M)$ to obtain stronger bounds. In particular, the integral over unmatched pieces from (4.19) is bounded by $C|f|_{C^0} \varepsilon$. The bound on the first term of (4.22) remains the same, but the bound on the second term from (4.23) is improved to $C|f|_{C^1} \varepsilon$. Putting these estimates together as in (4.24) and dividing³⁷ by ε^γ implies,

$$(4.38) \quad \varepsilon^{-\gamma} \left| \int_{W_1} \mathcal{L}_t f \psi_1 dm_{W_1} - \int_{W_2} \mathcal{L}_t f \psi_2 dm_{W_2} \right| \leq C|f|_{C^1} \varepsilon^{\alpha-\beta-\gamma} \leq C|f|_{C^1} \eta^{\frac{\alpha-\beta-\gamma}{4\gamma(q+1)}}.$$

Next, we turn to the second difference in (4.37). Using the notation of Section 4.3.3, we split the integrals up into one integral over the common r -interval $I_1 \cap I_2$ and at most two integrals over $I_1 \triangle I_2$. The (at most two) curves $V_i^\ell \subset W_\ell$ corresponding to intervals in $I_1 \triangle I_2$ have length bounded by $C\varepsilon$ by definition of $d_{\mathcal{W}^s}(W_1, W_2)$. Thus using (4.25), we have

$$(4.39) \quad \int_{V_i^\ell} g_\eta \psi_i dm_{W_\ell} \leq C|f|_{C^0} \eta^{\frac{t}{2}-\frac{1}{2}} \varepsilon \leq C|f|_{C^0} \eta^{\frac{t}{2}-\frac{1}{2}+\frac{1-\gamma}{4\gamma(q+1)}} \varepsilon^\gamma.$$

On the curves U_1, U_2 , which are the graphs of the functions $\varphi_{U_1}, \varphi_{U_2}$ over $I_1 \cap I_2$, we have,

$$(4.40) \quad \int_{U_1} g_\eta \psi_1 dm_{W_1} - \int_{U_2} g_\eta \psi_2 dm_{W_2} \leq |JG_{U_1}(g_\eta \psi_1) \circ G_{U_1} - JG_{U_2}(g_\eta \psi_2) \circ G_{U_2}|_{C^0(I_1 \cap I_2)},$$

where $G_{U_\ell}(r) = (r, \varphi_{U_\ell}(r))$. Then estimating as in the proof of Sublemma 4.8, we have

$$(4.41) \quad |JG_{U_1}(g_\eta \psi_1) \circ G_{U_1} - JG_{U_2}(g_\eta \psi_2) \circ G_{U_2}|_{C^0(I_1 \cap I_2)} \leq C|g_\eta|_{C^1(M)} \varepsilon \leq C|f|_\infty \eta^{\frac{t}{2}-\frac{3}{2}+\frac{1-\gamma}{4\gamma(q+1)}} \varepsilon^\gamma,$$

where we have used the fact that $d(\psi_1, \psi_2) = 0$ and $|\varphi'_{U_1} - \varphi'_{U_2}| \leq \varepsilon$. Putting these estimates together with (4.38) in (4.37) yields,

$$(4.42) \quad \begin{aligned} & \varepsilon^{-\gamma} \left| \int_{W_1} (\mathcal{L}_t f - g_\eta) \psi_1 dm_{W_1} - \int_{W_2} (\mathcal{L}_t f - g_\eta) \psi_2 dm_{W_2} \right| \\ & \leq C|f|_{C^1} \eta^{\frac{\alpha-\beta-\gamma}{4\gamma(q+1)}} + C|f|_{C^0} \eta^{\frac{t}{2}-\frac{3}{2}+\frac{1-\gamma}{4\gamma(q+1)}}, \end{aligned}$$

and we use $\gamma \leq \frac{1}{6q+7}$ from (4.1) to deduce that $-\frac{3}{2} + \frac{1-\gamma}{4\gamma(q+1)} \geq 0$. This completes Case 2, which, together with Case 1, implies the required bound $\|\mathcal{L}_t f - g_\eta\|_u \leq |f|_{C^1(M)} \eta^\delta$, for some $\delta > 0$, ending the proof of Lemma 4.3.

³⁷We use here the strict inequality $\gamma < \alpha - \beta$.

Remark 4.11 (Adapting the proof of Lemma 4.3 to the case $t = 0$). *Homogeneity strips are not used in [BD], so one requires a nonhomogeneous version of Lemma 4.9, but it is not hard to show directly that there exists $C > 0$ such that $m_W(W \setminus W_u(\eta)) \leq C\sqrt{\eta}$ for any $W \in \mathcal{W}^s$ and $\eta > 0$, and this weaker bound suffices (see discussion of (4.35) below). Lemma 4.10 can be kept unchanged as it is only needed on unstable manifolds contained in a single homogeneity strip.*

We show how to adapt the proof of Lemma 4.3 to the norm from [BD, §4.1] with $q = 2$, and parameters β , γ , and ς : Eq (4.27) and (4.28) get better since the test function satisfies $|\psi| \leq |\log |W||^\gamma$, so we find $\eta^{1/2}|\log \eta|^\gamma$. Similarly, (4.29) has the bound $\eta^{3/10}|\log \eta|^\gamma$. Eq (4.30)–(4.34) remain as written. Eq (4.35) proceeds as above until the last line, at which point we use $|S_\eta| \leq C\sqrt{\eta}$, so that the final bound becomes $C|f|_\infty \eta^{1/4}|\log \eta|^\gamma$. Thus we arrive at (4.36) with a bound $C|f|_{C^1} \eta^{1/6}$. The factor $|\log \eta|^\gamma$ can be absorbed by the various exponents, all being greater than $1/6$. So there is no extra restriction the parameter γ from [BD] from the stable norm estimate.

For the unstable norm estimate, one distinguishes between the case $\eta^{1/6} < |\log \varepsilon|^{-2\varsigma}$, which yields a bound with $\eta^{1/12}$, and the case $\eta^{1/6} \geq |\log \varepsilon|^{-2\varsigma}$, which implies that $\varepsilon \leq \exp(-\eta^{-\frac{\alpha-\beta}{12\varsigma}})$, which is superexponentially small in η , so that (4.37) remains the same, while (4.38) is bounded by $\varepsilon^{\alpha-\beta}|\log \varepsilon|^\varsigma \leq \exp(-\eta^{-\frac{\alpha-\beta}{24\varsigma}})$. Similarly, (4.39) is bounded by $|\log \varepsilon|^{-\varsigma}$ times a factor superexponentially small in η . (We have a power of ε which is factored into $|\log \varepsilon|^{-\varsigma}$ times $\varepsilon^{1-\delta}$, for any δ .) The same is true of (4.40)–(4.41). Finally, in (4.42), we end up with $\exp(-\eta^{-\frac{\alpha-\beta}{24\varsigma}})$ plus $\eta^{-3/2} \exp(-\eta^{-\frac{1}{24\varsigma}})$, and this goes to 0 as η goes to 0, for any $\varsigma > 0$ (in particular, there is no extra condition on ς from this estimate).

4.5. Spectral Gap for \mathcal{L}_t . Constructing μ_t (Proof of Theorem 4.1). We harvest the results from the previous subsections to show Theorem 4.1 at the end of this section. Our first result follows from Proposition 4.6 and the exact growth for $Q_n(t)$ (Propositions 3.15 and 3.18).

Proposition 4.12 (Quasi-compactness). *Let $t_0 \in (0, 1)$ and $t_1 \in (1, t_*)$. Then we can choose parameters for \mathcal{B} such that for any $t \in [t_0, t_1]$, the operator \mathcal{L}_t acting on \mathcal{B} is quasi-compact: its spectral radius is $e^{P_*(t)}$ and its essential spectral radius is at most $\sigma e^{P_*(t)}$, where*

$$\sigma := \max\{\Lambda^{-\beta+1/p}, \theta^{t-1/p} e^{-P_*(t)}, \Lambda^{-\gamma}\} < 1.$$

Moreover, the peripheral spectrum of \mathcal{L}_t contains no Jordan blocks.

Proof. Since $t_0 > 0$ and $t_1 < t_*$, we can choose $p > 1$ such that $p > 2/t_0 \geq 2/t$ and (by Lemma 4.7) $\theta^{(t-1/p)} e^{-P_*(t)} < 1$ for any $t \in [t_0, t_1]$. Then (4.5) and Proposition 3.18(b) imply that the spectral radius of \mathcal{L}_t on \mathcal{B}_w is at most $e^{P_*(t)}$. Combining (4.8) from Proposition 4.6 with Hennion's theorem and compactness of the unit ball of \mathcal{B} in \mathcal{B}_w from Proposition 4.2, the essential spectral radius of \mathcal{L}_t on \mathcal{B} is at most $\sigma e^{P_*(t)} < e^{P_*(t)}$. Hence the spectral radius of \mathcal{L}_t on \mathcal{B} is at most $e^{P_*(t)}$.

Next, notice that by Lemma 2.1 and our choice of δ_1 in (3.47), we have for $W \in \mathcal{W}^s$ with $|W| \geq \delta_1/3$,

$$\begin{aligned} \int_W \mathcal{L}_t^n 1 \, dm_W &= \sum_{W_i \in \mathcal{G}_n^{\delta_1}(W)} \int_{W_i} |J^s T^n|^t \, dm_{W_i} \geq \sum_{W_i \in L_n^{\delta_1}(W)} \frac{1}{3} \delta_1 2^{-t} |J^s T^n|_{C^0(W_i)}^t \\ (4.43) \quad &\geq \frac{1}{4} \delta_1 2^{-t} \sum_{W_i \in \mathcal{G}_n^{\delta_1}(W)} |J^s T^n|_{C^0(W_i)}^t \geq \frac{1}{4} \delta_1 2^{-t} c_1 Q_n(t), \end{aligned}$$

where for the final inequality we have applied Proposition 3.18(b). Then, since $Q_n(t) \geq e^{nP_*(t)}$ by the lower bound in Proposition 3.18(b), we conclude

$$\|\mathcal{L}_t^n\|_{\mathcal{B}} \geq \|\mathcal{L}_t^n 1\|_{\mathcal{B}} (\|1\|_{\mathcal{B}})^{-1} \geq (\|1\|_{\mathcal{B}})^{-1} C \delta_1 e^{P_*(t)n} \implies \lim_{n \rightarrow \infty} \|\mathcal{L}_t^n\|_{\mathcal{B}}^{1/n} \geq e^{P_*(t)}.$$

Thus the spectral radius of \mathcal{L}_t on \mathcal{B} is in fact $e^{P_*(t)}$ and \mathcal{L}_t is quasi-compact on \mathcal{B} .

Finally, to prove there are no Jordan blocks in the peripheral spectrum, assume to the contrary that there exist $f_0, f_1 \in \mathcal{B}$, $f_0 \neq 0$, and $\lambda \in \mathbb{C}$, $|\lambda| = e^{P_*(t)}$, such that $\mathcal{L}_t f_0 = \lambda f_0$ and $\mathcal{L}_t f_1 = \lambda f_1 + f_0$. Then $\mathcal{L}_t^n f_1 = \lambda^n f_1 + n\lambda^{n-1} f_0$, so that

$$n|f_0|_w \leq e^{P_*(t)}|f_1|_w + e^{-(n-1)P_*(t)}|\mathcal{L}_t^n f_1|_w,$$

and dividing by n , letting $n \rightarrow \infty$ and applying (4.5) and Proposition 3.18(b) yields $|f_0|_w = 0$. The injectivity of \mathcal{B}_w into \mathcal{B} given by Proposition 4.2 implies $f_0 = 0$ in \mathcal{B} , a contradiction. \square

For $\varpi \in [0, 1)$, let \mathbb{V}_ϖ denote the eigenspace of \mathcal{L}_t on \mathcal{B} corresponding to the eigenvalue $e^{P_*(t)}e^{2\pi i\varpi}$. Due to Proposition 4.12, we have the following decomposition of \mathcal{L}_t on \mathcal{B} ,

$$(4.44) \quad \mathcal{L}_t = \sum_{\varpi} e^{P_*(t)+2\pi i\varpi} \Pi_\varpi + R_t,$$

where the sum is over finitely many ϖ due to the quasi-compactness of \mathcal{L}_t , and $\Pi_\varpi^2 = \Pi_\varpi$, $\Pi_\varpi \Pi_{\varpi'} = R_t \Pi_\varpi = \Pi_\varpi R_t = 0$ for $\varpi \neq \varpi' \pmod{2\pi}$, and the spectral radius of R_t is strictly less than $e^{P_*(t)}$.

Lemma 4.13. Define $\nu_t = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} e^{-P_*(t)k} \mathcal{L}_t^k 1$.

- a) Then $\nu_t \neq 0$ is a nonnegative Radon measure, and $e^{P_*(t)}$ is in the spectrum of \mathcal{L}_t .
- b) All elements of $\mathbb{V} = \bigoplus_{\varpi} \mathbb{V}_\varpi$ are complex measures, absolutely continuous with respect to ν_t .

Lemma 4.13 is standard, adapting what has been done in the SRB case. We give a proof for completeness.

Proof. (b) The lack of Jordan blocks enables us to define spectral projectors by

$$(4.45) \quad \Pi_\varpi : \mathcal{B} \rightarrow \mathbb{V}_\varpi, \quad \Pi_\varpi f = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} e^{-P_*(t)k - 2\pi i\varpi k} \mathcal{L}_t^k f,$$

where convergence in the \mathcal{B} norm is guaranteed by Propositions 4.6 and 3.18(b). Moreover, since $C^1(M)$ is dense in \mathcal{B} and \mathbb{V}_ϖ is finite-dimensional, for each $\nu \in \mathbb{V}_\varpi$, there exists $\bar{f}_\nu \in C^1(M)$ such that $\Pi_\varpi \bar{f}_\nu = \nu$.

Taking $\nu \in \mathbb{V}_\varpi$ and $\psi \in C^\alpha(M)$, using (4.45) and recalling our identification of $\bar{f}_\nu \in C^1$ with the measure $\bar{f}_\nu d\mu_{\text{SRB}}$, we have

$$|\nu(\psi)| \leq \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} e^{-P_*(t)k} |\mathcal{L}_t^k \bar{f}_\nu(\psi)| \leq |\bar{f}_\nu|_\infty \Pi_0 1(|\psi|) \leq |\bar{f}_\nu|_\infty |\psi|_\infty \Pi_0 1(1).$$

The last two inequalities show respectively that ν is a complex Radon measure, and is absolutely continuous with respect to ν_t , with density $f_\nu \in L^\infty(\nu_t)$. It may be that $f_\nu \neq \bar{f}_\nu$.

- (a) Item (b) implies also that ν_t is a nonnegative Radon measure since $\bar{f}_{\nu_t} = 1$ and Π_0 is nonnegative. Also, if $\nu_t = 0$, then all elements of \mathbb{V}_ϖ are 0, contradicting the fact that the spectral radius of \mathcal{L}_t is $e^{P_*(t)}$. Thus $\nu_t \neq 0$ and $e^{P_*(t)}$ is in the spectrum of \mathcal{L}_t since $\mathcal{L}_t \nu_t = e^{P_*(t)} \nu_t$. \square

The dual operator \mathcal{L}_t^* acting on \mathcal{B}^* has the same spectrum as \mathcal{L}_t on \mathcal{B} . Define

$$(4.46) \quad \tilde{\nu}_t := \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} e^{-P_*(t)k} (\mathcal{L}_t^*)^k d\mu_{\text{SRB}},$$

which converges in the dual norm due to the absence of Jordan blocks. By the analogous arguments to item (b) of Lemma 4.13, the distribution $\tilde{\nu}_t \neq 0$ is a nonnegative Radon measure, and every other eigenvector corresponding to the peripheral spectrum is a Radon measure, absolutely continuous with respect to $\tilde{\nu}_t$, with bounded density.

With $\tilde{\nu}_t$, we will define our candidate μ_t for the equilibrium state in Proposition 4.16. For this (and in (6.23)), we shall use the following lemma (proved exactly as in [BD, Lemma 4.4]) which gives more precise information about the inclusion $\mathcal{B}_w \subset (C^1(M))^*$ in Proposition 4.2. Recalling (2.2), let $H_{\mathcal{W}_{\mathbb{H}}^s}^\alpha(\psi) = \sup_{W \in \mathcal{W}_{\mathbb{H}}^s} H_W^\alpha(\psi)$.

Lemma 4.14. *There exists $C > 0$ such that for all $f \in \mathcal{B}_w$ and $\psi \in C^\alpha(\mathcal{W}_{\mathbb{H}}^s)$,*

$$|f(\psi)| \leq C|f|_w(|\psi|_\infty + H_{\mathcal{W}_{\mathbb{H}}^s}^\alpha(\psi)).$$

In addition, we shall use the following extension of the above lemma, which uses that μ_{SRB} has smooth conditional measures on *stable* manifolds.

Lemma 4.15. *There exists $C > 0$ such that for all $f \in \mathcal{B}_w$, $\psi \in C^\alpha(\mathcal{W}_{\mathbb{H}}^s)$, and $n \geq 0$,*

$$|\mathcal{L}_t^n(f\psi)(1)| \leq CQ_n(t)|f|_w|\psi|_{C^\alpha(\mathcal{W}_{\mathbb{H}}^s)}.$$

Proof. We proceed similarly to the proof of [BD, Lemma 4.4]. By density of C^1 in \mathcal{B}_w , it suffices to prove the inequality for $f \in C^1$. Then according to our convention, for $\psi \in C^\alpha(\mathcal{W}_{\mathbb{H}}^s)$,

$$\mathcal{L}_t^n(f\psi)(1) = \int_M \mathcal{L}_t^n(f\psi) d\mu_{\text{SRB}}.$$

To estimate the integral, we disintegrate μ_{SRB} into a family of conditional probability measures on stable manifolds as follows. Fix a foliation of $\mathcal{F} = \{W_\xi\}_{\xi \in \Xi} \subset \mathcal{W}_{\mathbb{H}}^s$ of maximal, homogeneous local stable manifolds belonging to $\mathcal{W}_{\mathbb{H}}^s$. The conditional measures are defined by $\mu_{\text{SRB}}^\xi = |W_\xi|^{-1} \rho_\xi dm_{W_\xi}$, where ρ_ξ satisfies [CM, Cor 5.30],

$$(4.47) \quad 0 < c_\rho \leq \inf_{\xi \in \Xi} \inf_{W_\xi} \rho_\xi \leq \sup_{\xi \in \Xi} |\rho_\xi|_{C^\alpha(W_\xi)} \leq C_\rho < \infty.$$

We denote the factor measure on the index set Ξ by $\hat{\mu}_{\text{SRB}}$. Then,

$$\begin{aligned} \left| \int_M \mathcal{L}_t^n(f\psi) d\mu_{\text{SRB}} \right| &= \left| \int_{\Xi} \int_{W_\xi} \mathcal{L}_t^n(f\psi) \rho_\xi dm_{W_\xi} |W_\xi|^{-1} d\hat{\mu}_{\text{SRB}}(\xi) \right| \\ &= \left| \int_{\Xi} \sum_{W_i^\xi \in \mathcal{G}_n(W_\xi)} \int_{W_i} f\psi \rho_\xi \circ T^n |J^s T^n|^t dm_{W_i^\xi} |W_\xi|^{-1} d\hat{\mu}_{\text{SRB}}(\xi) \right| \\ &\leq CC_\rho |f|_w |\psi|_{C^\alpha(\mathcal{W}_{\mathbb{H}}^s)} \left| \int_{\Xi} \sum_{W_i^\xi \in \mathcal{G}_n(W_\xi)} |J^s T^n|_{C^0(W_i^\xi)}^t |W_\xi|^{-1} d\hat{\mu}_{\text{SRB}}(\xi) \right| \\ &\leq CC_\rho C_2 Q_n(t) |f|_w |\psi|_{C^\alpha(\mathcal{W}_{\mathbb{H}}^s)} \int_{\Xi} |W_\xi|^{-1} d\hat{\mu}_{\text{SRB}}(\xi), \end{aligned}$$

where in the last line we have used Lemma 3.4 with $\varsigma = 0$. The remaining integral is finite by [CM, Exercise 7.22] since the family $(W_\xi, d\mu_{\text{SRB}}^\xi, \hat{\mu}_{\text{SRB}})_{\xi \in \Xi}$ is a standard family. \square

Proposition 4.16 (Constructing μ_t). *For $\nu \in \mathcal{B}$ and $\tilde{\nu} \in \mathcal{B}^*$ we set $\langle \nu, \tilde{\nu} \rangle := \tilde{\nu}(\nu)$.*

- a) *The measure $\tilde{\nu}_t \in \mathcal{B}^*$ is in fact an element of \mathcal{B}_w^* .*
- b) *We have $\langle \nu_t, \tilde{\nu}_t \rangle \neq 0$, and the distribution μ_t defined for $\psi \in C^\alpha(\mathcal{W}_{\mathbb{H}}^s)$ by*

$$\mu_t(\psi) := \frac{\langle \psi \nu_t, \tilde{\nu}_t \rangle}{\langle \nu_t, \tilde{\nu}_t \rangle}$$

is a T -invariant probability measure.

Proof. a) Let $g_n = n^{-1} \sum_{k=0}^{n-1} e^{-P_*(t)k} (\mathcal{L}_t^*)^k d\mu_{\text{SRB}}$. By definition, $\|g_n - \tilde{\nu}_t\|_{\mathcal{B}^*} \rightarrow 0$ as $n \rightarrow \infty$. Thus for $f \in \mathcal{B}$, we have

$$|\langle f, \tilde{\nu}_t \rangle| \leq |\langle f, \tilde{\nu}_t - g_n \rangle| + |\langle f, g_n \rangle| \leq |\langle f, \tilde{\nu}_t - g_n \rangle| + C|f|_w,$$

where for the last inequality, we used the bound,

$$|\langle f, (\mathcal{L}_t^k)^* d\mu_{\text{SRB}} \rangle| = |\langle \mathcal{L}_t^k f, d\mu_{\text{SRB}} \rangle| \leq C|\mathcal{L}_t^k f|_w \leq C'e^{P_*(t)k}|f|_w,$$

by Lemma 4.14 and Proposition 4.6. Taking $n \rightarrow \infty$ yields the bound $|\langle f, \tilde{\nu}_t \rangle| \leq C|f|_w$ for all $f \in \mathcal{B}$ and since \mathcal{B} is dense in \mathcal{B}_w , the distribution $\tilde{\nu}_t$ extends to a bounded linear operator on \mathcal{B}_w , as required.

b) First we show the expression $\langle \psi \nu_t, \tilde{\nu}_t \rangle$ is well-defined for $\psi \in C^\alpha(\mathcal{W}_{\mathbb{H}}^s)$. According to our convention, for $f \in C^1(M)$, we define for $n \geq 0$,

$$\langle f, \psi (\mathcal{L}_t^n)^* d\mu_{\text{SRB}} \rangle = \int \mathcal{L}_t^n(f\psi) d\mu_{\text{SRB}} \leq CQ_n(t)|f|_w|\psi|_{C^\alpha(\mathcal{W}_{\mathbb{H}}^s)},$$

by Lemma 4.15. Thus $\psi (\mathcal{L}_t^n)^* d\mu_{\text{SRB}}$ extends to a bounded linear functional on \mathcal{B}_w . Applying Proposition 3.18(b) and (4.46), we obtain

$$(4.48) \quad \psi \tilde{\nu}_t \in \mathcal{B}_w^* \quad \text{with} \quad |\langle f, \psi \tilde{\nu}_t \rangle| \leq C'|f|_w|\psi|_{C^\alpha(\mathcal{W}_{\mathbb{H}}^s)}, \quad \forall f \in \mathcal{B}_w.$$

(We do not claim or need that $\psi f \in \mathcal{B}_w$, i.e. that ψf can be approached by a sequence of C^1 functions in the weak norm.) Thus $\langle \psi \nu_t, \tilde{\nu}_t \rangle := \langle \nu_t, \psi \tilde{\nu}_t \rangle$ is well-defined. Remark that the above argument also shows that $\mu_t(f\psi) = \langle f \nu_t, \psi \tilde{\nu}_t \rangle$ for all $f \in C^1(M)$, $\psi \in C^\alpha(\mathcal{W}_{\mathbb{H}}^s)$.

Next, suppose $\langle \nu_t, \tilde{\nu}_t \rangle = 0$. Then for any $f \in C^1(M)$, and $n \geq 1$, using (4.45),

$$(4.49) \quad \begin{aligned} \langle f, \tilde{\nu}_t \rangle &= \frac{1}{n} \sum_{k=0}^{n-1} e^{-P_*(t)k} \langle f, (\mathcal{L}_t^*)^k \tilde{\nu}_t \rangle = \frac{1}{n} \sum_{k=0}^{n-1} e^{-P_*(t)k} \langle \mathcal{L}_t^k f, \tilde{\nu}_t \rangle \\ &\xrightarrow{n \rightarrow \infty} \langle \Pi_0(f), \tilde{\nu}_t \rangle = c_t(f) \langle \nu_t, \tilde{\nu}_t \rangle = 0. \end{aligned}$$

By density of $C^1(M)$ in \mathcal{B} , this implies that $\tilde{\nu}_t = 0$ as an element of \mathcal{B}^* , a contradiction. Thus $\langle \nu_t, \tilde{\nu}_t \rangle \neq 0$, and indeed $c_t(f) = \frac{\langle f, \tilde{\nu}_t \rangle}{\langle \nu_t, \tilde{\nu}_t \rangle}$, so that μ_t is a well-defined element of $\mathcal{B}^* \subset (C^1(M))^*$. It is then easy to see that μ_t is a nonnegative distribution and thus a Radon measure. The fact that μ_t is T -invariant is an exercise, using that ν_t and $\tilde{\nu}_t$ are eigenvectors of \mathcal{L}_t and \mathcal{L}_t^* . \square

Following [BD, Definition 7.5], we remark that elements of \mathcal{B} and \mathcal{B}_w can be viewed both as distributions on M , as well as families of leafwise distributions on stable manifolds. In particular, for $f \in C^1(M)$, $W \in \mathcal{W}^s$, the map defined by

$$\mathcal{D}_{W,f}(\psi) := \int_W f \psi dm_W, \quad \psi \in C^\alpha(W),$$

can be viewed as a distribution of order α on W . Since $|\mathcal{D}_{W,f}(\psi)| \leq |f|_w|\psi|_{C^\alpha(W)}$, the map \mathcal{D}_W can be extended to all $f \in \mathcal{B}_w$. We will use the notation $\int_W \psi f$ for this extension and call the associated family of distributions the leafwise distributions $(f, W)_{W \in \mathcal{W}^s}$ corresponding to f . If f satisfies $\int \psi f \geq 0$ for all $\psi \geq 0$, then the leafwise distribution is a leafwise measure.

Recalling the disintegration of the measure μ_{SRB} from the proof of Lemma 4.15, we state an analogue of [BD, Lemma 7.7]:

Lemma 4.17 (ν_t as a leafwise measure). *Let ν_t^ξ and $\hat{\nu}_t$ denote the conditional measure on W_ξ and the factor measure on Ξ , respectively, obtained by disintegrating ν_t on the foliation of stable manifolds \mathcal{F} . For all $\psi \in C^\alpha(M)$,*

$$\int_{W_\xi} \psi d\nu_t^\xi = \frac{\int_{W_\xi} \psi \rho_\xi \nu_t}{\int_{W_\xi} \rho_\xi \nu_t} \quad \forall \xi \in \Xi, \quad \text{and} \quad d\hat{\nu}_t(\xi) = |W_\xi|^{-1} \left(\int_{W_\xi} \rho_\xi \nu_t \right) d\hat{\mu}_{\text{SRB}}(\xi).$$

Moreover, viewed as a leafwise measure, $\nu_t(W) > 0$ for all $W \in \mathcal{W}_{\mathbb{H}}^s$.

Proof. We begin by showing that $\nu_t(W) > 0$ for all $W \in \mathcal{W}_{\mathbb{H}}^s$. If $|W| \geq \delta_1/3$ (recalling that our choice of δ_1 in (3.31) is uniform for $t \in [t_0, t_1]$), then the positivity follows immediately from the uniform lower bound (4.43). So assume $W \in \mathcal{W}_{\mathbb{H}}^s$ with $|W| < \delta_1/3$.

First, we claim that there exists $n_W = \mathcal{O}(\log |W|)$ such that at least one element of $\mathcal{G}_{n_W}^{\delta_1}(W)$ has length at least $\delta_1/3$. For any $n \geq 1$, if no elements $W_i \in \mathcal{G}_n^{\delta_1}(W)$ have length at least $\delta_1/3$, then $\mathcal{G}_n^{\delta_1}(W) = \mathcal{I}_n^{\delta_1}(W)$ so that by Lemma 3.3 with $\varsigma = 0$, $\sum_{W_i \in \mathcal{G}_n^{\delta_1}(W)} |J^s T^n|_{C^0(W_i)} \leq C_0 \theta^n$, while by (3.9), $\sum_{W_i \in \mathcal{G}_n^{\delta_1}(W)} |J^s T^n|_{C^0(W_i)} \geq C_1 |W| \delta_1^{-1}$. This can continue only so long as

$$C_1 |W| \delta_1^{-1} \leq C_0 \theta^n \implies n \leq \frac{\log \left(\frac{C_1 |W|}{C_0 \delta_1} \right)}{\log \theta} =: n_W.$$

Next, letting $V \in \mathcal{G}_{n_W}^{\delta_1}(W)$ be such that $|V| \geq \delta_1/3$, we estimate as in (3.39), using the fact that since V and $T^{n_W} V$ are both homogeneous, $|V| \leq C |T^{n_W} V|^{\left(\frac{q+1}{2q+1}\right)^{n_W}}$ for some $C \geq 1$, so that

$$|J^s T^{n_W}|_{C^0(V)}^t \geq e^{-tC_d} \frac{|T^{n_W} V|^t}{|V|^t} \geq e^{-tC_d} (\delta_1/(3C))^t \left(\frac{2q+1}{q+1}\right)^{n_W t}.$$

Finally, recalling our choice of n_1 from (3.47), and using the fact that $|V| \geq \delta_1/3$, we estimate,³⁸

$$\begin{aligned} \frac{1}{n} \sum_{k=0}^{n-1} e^{-P_*(t)k} \int_W \mathcal{L}_t^k 1 \, dm_W &\geq e^{-P_*(t)n_W} \frac{1}{n} \sum_{k=n_1+n_W}^{n-1} e^{-P_*(t)(k-n_W)} \int_V \mathcal{L}_t^{k-n_W} 1 |J^s T^{n_W}|^t \, dm_V \\ (4.50) \quad &\geq e^{-P_*(t)n_W} e^{-tC_d} \left(\frac{\delta_1}{3C}\right)^t \left(\frac{2q+1}{q+1}\right)^{n_W t} \frac{1}{n} \sum_{k=n_1+n_W}^{n-1} e^{-P_*(t)(k-n_W)} \int_V \mathcal{L}_t^{k-n_W} 1 \, dm_V \\ &\geq e^{-P_*(t)n_W} e^{-tC_d} \left(\frac{\delta_1}{3C}\right)^t \left(\frac{2q+1}{q+1}\right)^{n_W t} \frac{n-1-n_1-n_W}{n} \frac{1}{4} \delta_1 2^{-t} c_1, \end{aligned}$$

where in the last line we have applied (4.43) and Proposition 3.18(b).

These lower bounds depend only on $|W|$ and carry over to $\nu_t(W)$ since they are uniform in n .

With the lower bounds established, the remainder of the proof follows precisely as in [BD, Lemma 7.7], disintegrating the measure $\left(\frac{1}{n} \sum_{k=0}^{n-1} e^{-kP_*(t)} \mathcal{L}_t^k 1\right) d\mu_{\text{SRB}}$ on the foliation of stable manifolds \mathcal{F} from (4.47), using that convergence in \mathcal{B} to ν_t implies convergence of the integral on each $W_\xi \in \mathcal{F}$. The lower bounds on $\nu_t(W)$ imply that the ratio $\frac{\int_{W_\xi} \psi \rho_\xi \nu_t}{\int_{W_\xi} \rho_\xi \nu_t}$ is well-defined for each $W_\xi \in \mathcal{F}$. □

In view of (4.52) in the proof of Lemma 4.18 below (and also (6.25)), it is convenient to define \mathcal{L}_t acting explicitly on distributions. For any point $x \in M$ that has a stable manifold of zero length, we define $W^s(x) = \{x\}$, and extend \mathcal{W}^s to a larger collection $\widetilde{\mathcal{W}}^s$ including these singletons. For $\alpha \leq 1$, let

$$C^\alpha(\widetilde{\mathcal{W}}^s) := \{\psi \text{ bounded and measurable} \mid |\psi|_{C^\alpha(\widetilde{\mathcal{W}}^s)} := \sup_{W \in \widetilde{\mathcal{W}}^s} |\psi|_{C^\alpha(W)} < \infty\}.$$

Let $C_{\cos}^\alpha(\widetilde{\mathcal{W}}^s)$ denote the set of measurable functions ψ such that $\psi \cos \varphi \in C^\alpha(\widetilde{\mathcal{W}}^s)$. It follows from the uniform hyperbolicity of T that if $\psi \in C^\alpha(\widetilde{\mathcal{W}}^s)$, then $\psi \circ T \in C^\alpha(\widetilde{\mathcal{W}}^s)$ (see (4.11)). Also, as in the proof of Lemma 2.2, by [CM, eq. (5.14)], we have $J^s T(x) \approx \cos \varphi(x)$ for $x \in M'$. We extend $J^s T$ to all $x \in M$ by defining it to be 1 on $M \setminus M'$. Then using (2.3), we have $\psi \circ T / |J^s T|^{1-t} \in C_{\cos}^\alpha(\widetilde{\mathcal{W}}^s)$

³⁸In fact, estimating more carefully for $t \leq 1$, one can obtain the more precise lower bound $C' \delta_1^{1-t} |W|^{C'' P_*(t)} |W|^t$ for some $C', C'' > 0$, but we will not need this here.

whenever $\psi \in C^\alpha(\widetilde{\mathcal{W}}^s)$ and $\alpha \leq 1/(q+1)$. Using these facts, for a distribution $\mu \in (C_{\cos}^\alpha(\widetilde{\mathcal{W}}^s))^*$, define $\mathcal{L}_t : (C_{\cos}^\alpha(\widetilde{\mathcal{W}}^s))^* \rightarrow (C^\alpha(\widetilde{\mathcal{W}}^s))^*$ by

$$(4.51) \quad \mathcal{L}_t \mu(\psi) = \mu \left(\frac{\psi \circ T}{|J^s T|^{1-t}} \right), \quad \text{for all } \psi \in C^\alpha(\widetilde{\mathcal{W}}^s).$$

To reconcile this definition with (1.10), for $f \in C^\alpha(\mathcal{W}^s)$, we identify f with the measure $fd\mu_{\text{SRB}}$. Such a measure belongs to $(C_{\cos}^\alpha(\widetilde{\mathcal{W}}^s))^*$ since $1/\cos \varphi \in L^1(\mu_{\text{SRB}})$. With this convention, the measure $\mathcal{L}_t f$ has density with respect to μ_{SRB} given by (1.10). Finally, note that $\mathcal{B} \subset (C_{\cos}^\alpha(\widetilde{\mathcal{W}}^s))^*$, due to Lemma 4.14 and Remark 4.5.

We are finally ready to prove that \mathcal{L}_t enjoys a spectral gap, using Lemma 4.17 (which exploited that μ_{SRB} has smooth *stable* conditional densities, a very nongeneric property in the setting of hyperbolic dynamics).

Lemma 4.18 (Spectral Gap). *\mathcal{L}_t has a spectral gap on \mathcal{B} , i.e., $e^{P_*(t)}$ is a simple eigenvalue and all other eigenvalues of \mathcal{L}_t have modulus strictly less than $e^{P_*(t)}$.*

Proof. Step 1: the spectrum of $e^{-P_*(t)}\mathcal{L}_t$ consists of finitely many cyclic groups; in particular, each ϖ in (4.44) is rational. To prove this, suppose $\nu \in \mathbb{V}_\varpi$, $\nu \neq 0$, and $\psi \in C^\alpha(M)$. Then by Lemma 4.13(b) and viewing ν as a distribution in the sense of (4.51)

$$(4.52) \quad \begin{aligned} \int_M \psi f_\nu d\nu_t &= \nu(\psi) = e^{-P_*(t)-2\pi i\varpi} \mathcal{L}_t \nu(\psi) = e^{-P_*(t)-2\pi i\varpi} \nu \left(\frac{\psi \circ T}{|J^s T|^{1-t}} \right) \\ &= e^{-P_*(t)-2\pi i\varpi} \nu_t \left(f_\nu \frac{\psi \circ T}{|J^s T|^{1-t}} \right) = e^{-P_*(t)-2\pi i\varpi} \mathcal{L}_t \nu_t (\psi f_\nu \circ T^{-1}) \\ &= e^{-2\pi i\varpi} \nu_t (\psi f_\nu \circ T^{-1}), \end{aligned}$$

so that $f_\nu \circ T^{-1} = e^{2\pi i\varpi} f_\nu$, ν_t -almost everywhere.

Defining $\nu_{k,t} = (f_\nu)^k \nu_t$, for $k \in \mathbb{N}$, we claim that $e^{P_*(t)+2\pi i\varpi k}$ belongs to the spectrum of \mathcal{L}_t and $\nu_{k,t} \in \mathbb{V}_{\varpi k}$. The claim completes the proof of Step 1 since the peripheral spectrum is finite, forcing $\varpi k = 0 \pmod{1}$ for some $k \geq 1$, so that ϖ must be rational.

To prove the claim, set $f_\nu = 0$ outside the support of ν_t , and define the measure $\langle f_\nu \nu_t, \cdot \tilde{\nu}_t \rangle = \langle \nu, \cdot \tilde{\nu}_t \rangle$. We claim that this measure is not identically zero. If it were, then for any $\psi \in \mathcal{B}^*$, making the dual argument to (4.49),

$$\langle \nu, \psi \rangle = \langle \Pi_\varpi \nu, \psi \rangle = \langle \nu, \Pi_\varpi^* \psi \rangle = \langle \nu, \tilde{f}_\varpi \tilde{\nu}_t \rangle \tilde{c}_\varpi(\psi) = 0,$$

where we have used that every eigenvector corresponding to the peripheral spectrum of \mathcal{L}_t^* is absolutely continuous with respect to $\tilde{\nu}_t$, i.e. $\tilde{\nu}_\varpi = \tilde{f}_\varpi \tilde{\nu}_t$, as explained after (4.46). Thus $\nu = 0$, a contradiction.

Since $\langle f_\nu \nu_t, \cdot \tilde{\nu}_t \rangle$ is not identically zero, it follows that $\langle (f_\nu)^k \nu_t, \cdot \tilde{\nu}_t \rangle$ is not identically zero. Thus there exists $\psi \in C^\alpha(M)$ such that $\langle (f_\nu)^k \nu_t, \psi \tilde{\nu}_t \rangle \neq 0$.

For $\varepsilon > 0$, choose $g \in C^1(M)$ such that $\mu_t(|g - (f_\nu)^k|) < \varepsilon$. Note that $g\nu_t \in \mathcal{B}$ by [DZ2, Lemma 5.3]. We will show that $\Pi_{\varpi k}(g\nu_t) \neq 0$. For $\psi \in C^\alpha(M)$ and each $j \geq 0$,

$$\begin{aligned} e^{-P_*(t)j-2\pi i\varpi k j} \langle \mathcal{L}_t^j(g\nu_t), \psi \tilde{\nu}_t \rangle &= e^{-P_*(t)j-2\pi i\varpi k j} \langle g\nu_t, \psi \circ T^j (\mathcal{L}_t^*)^j \tilde{\nu}_t \rangle \\ &= e^{-2\pi i\varpi k j} \langle \nu_t, \tilde{\nu}_t \rangle \mu_t(g \psi \circ T^j), \end{aligned}$$

where we have used $(\mathcal{L}_t^*)^j \tilde{\nu}_t = e^{P_*(t)j} \tilde{\nu}_t$. Also, due to the invariance of μ_t ,

$$\langle (f_\nu)^k \nu_t, \psi \tilde{\nu}_t \rangle = e^{-2\pi i\varpi k j} \langle (f_\nu)^k \circ T^{-j} \nu_t, \psi \tilde{\nu}_t \rangle = e^{-2\pi i\varpi k j} \langle \nu_t, \tilde{\nu}_t \rangle \mu_t((f_\nu)^k \psi \circ T^j).$$

Putting these two expressions together, we estimate,

$$\begin{aligned} & \left| \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=0}^{n-1} e^{-P_*(t)j - 2\pi i \varpi k j} \langle \mathcal{L}_t^j(g\nu_t), \psi \tilde{\nu}_t \rangle - \langle (f_\nu)^k \nu_t, \psi \tilde{\nu}_t \rangle \right| \\ & \leq \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=0}^{n-1} \langle \nu_t, \tilde{\nu}_t \rangle \mu_t(|g - (f_\nu)^k|) |\psi|_\infty \leq \varepsilon \langle \nu_t, \tilde{\nu}_t \rangle |\psi|_\infty. \end{aligned}$$

Since $\langle (f_\nu)^k \nu_t, \tilde{\nu}_t \rangle \neq 0$ and $\varepsilon > 0$ was arbitrary, this estimate shows that (i) $\Pi_{\varpi k}(g\nu_t) \neq 0$, so that $\mathbb{V}_{\varpi k}$ is not empty, and (ii) $\nu_{k,t} = (f_\nu)^k \nu_t$ can be approximated by elements of $\mathbb{V}_{\varpi k}$, and so must belong to $\mathbb{V}_{\varpi k}$, as claimed.

Step 2: \mathcal{L}_t has a spectral gap. It suffices to show that the ergodicity of (T, μ_{SRB}) implies that the positive eigenvalue $e^{P_*(t)}$ is simple. For then applying Step 1, suppose $\nu \in \mathbb{V}_\varpi$ for $\varpi = a/b$. Then both $\mathcal{L}_t^b \nu = e^{P_*(t)b} \nu$ and $\mathcal{L}_t^b \nu_t = e^{P_*(t)b} \nu_t$, so that \mathcal{L}_t^b has eigenvalue $e^{P_*(t)b}$ of multiplicity 2, and this is also its spectral radius, contradicting the fact that (T^b, μ_{SRB}) is also ergodic.

Now, suppose $\nu \in \mathbb{V}_0$. By Lemma 4.13(b), there exists $f_\nu \in L^\infty(\nu_t)$ such that $d\nu = f_\nu d\nu_t$. We will show that f_ν is ν_t -a.e. a constant.

By (4.52) $f_\nu \circ T = f_\nu$, ν_t -a.e. so that setting

$$S_n f_\nu = \sum_{j=0}^{n-1} f_\nu \circ T^j,$$

we see that $\frac{1}{n} S_n f_\nu = f_\nu$ for all $n \geq 1$. Thus f_ν is constant on stable manifolds. Next, since the factor measure $\hat{\nu}_t$ is equivalent to $\hat{\mu}_{\text{SRB}}$ on the index set Ξ by Lemma 4.17, it follows that $f_\nu = f_\nu \circ T$ on $\hat{\mu}_{\text{SRB}}$ -a.e. $W_\xi \in \mathcal{F}$. So $f_\nu = f_\nu \circ T$, μ_{SRB} -a.e. Since μ_{SRB} is ergodic, f_ν is constant μ_{SRB} -a.e. But since f_ν is constant on each stable manifold $W_\xi \in \mathcal{F}$, it follows that there exists $c > 0$ such that $f_\nu = c$ for $\hat{\mu}_{\text{SRB}}$ -a.e. $\xi \in \Xi$, and once again using the equivalence of $\hat{\mu}_{\text{SRB}}$ and $\hat{\nu}_t$, we conclude that f_ν is constant ν_t -a.e. \square

Proof of Theorem 4.1. All claims except the last sentence of the theorem follow from Propositions 4.12 and 4.16, and Lemma 4.18. Exponential decay of correlations for C^α functions with rate v satisfying (4.4) follows from the classical spectral decomposition

$$\mathcal{L}_t^k f = e^{kP_*(t)} [c_t(f) \cdot \nu_t + \mathcal{R}_t^k(f)], \text{ where } \exists C < \infty \text{ s. t. } \|\mathcal{R}_t^k f\| < C v^k \|f\|, \forall k \geq 0, \forall f \in \mathcal{B},$$

and $c_t(f) = \frac{\langle f, \tilde{\nu}_t \rangle}{\langle \nu_t, \tilde{\nu}_t \rangle}$. Indeed, by [DZ2, Lemma 5.3] for $\psi \in C^\alpha(M)$,

$$(4.53) \quad \psi \circ T^{-j} f \in \mathcal{B} \text{ and } \|\psi \circ T^{-j} f\|_{\mathcal{B}} \leq C_j |\psi|_{C^\alpha} \|f\|_{\mathcal{B}} \text{ for all } j \geq 1,$$

we find for $f_1, f_2 \in C^\alpha(M)$ (using (4.53) with $j = k$),

$$\begin{aligned} \int (f_1 \circ T^k) f_2 d\mu_t &= \frac{\langle (f_1 \cdot f_2 \circ T^{-k} \nu_t), \tilde{\nu}_t \rangle}{\langle \nu_t, \tilde{\nu}_t \rangle} = e^{-kP_*(t)} \frac{\langle f_1 \mathcal{L}_t^k(f_2 \nu_t), \tilde{\nu}_t \rangle}{\langle \nu_t, \tilde{\nu}_t \rangle} \\ &= c_t(f_2 \nu_t) \frac{\langle f_1 \nu_t, \tilde{\nu}_t \rangle}{\langle \nu_t, \tilde{\nu}_t \rangle} + \frac{\langle f_1 \mathcal{R}_t^k(f_2 \nu_t), \tilde{\nu}_t \rangle}{\langle \nu_t, \tilde{\nu}_t \rangle} = \int f_1 d\mu_t \int f_2 d\mu_t + \frac{\langle f_1 \mathcal{R}_t^k(f_2 \nu_t), \tilde{\nu}_t \rangle}{\langle \nu_t, \tilde{\nu}_t \rangle}, \end{aligned}$$

and we have, using again (4.53) (with $j = 0$),

$$\left| \langle f_1 \mathcal{R}_t^k(f_2 \nu_t), \tilde{\nu}_t \rangle \right| \leq |f_1|_{C^\alpha} \|\mathcal{R}_t^k(f_2 \nu_t)\| \leq C |f_1|_{C^\alpha} v^k \|f_2 \nu_t\| \leq C |f_1|_{C^\alpha} |f_2|_{C^\alpha} v^k.$$

Exponential mixing for Hölder functions of exponent smaller than α then follows from mollification (a lower exponent may worsen the rate of mixing). Finally, mixing is obtained by a standard argument: Since μ is a Borel probability measure and M is a compact metric space (and thus a normal topological space), any $f \in L^2(\mu)$ can be approximated by a sequence of continuous

functions in the $L^2(\mu)$ norm, using Urysohn functions. So, by Cauchy–Schwartz, we may reduce to proving mixing for continuous test functions. Clearly, Lipschitz functions form a subalgebra of the Banach algebra of continuous functions, the constant function $\equiv 1$ is Lipschitz, and for any $x \neq y$ in M there exists a Lipschitz function \tilde{f} with $\tilde{f}(x) \neq \tilde{f}(y)$. Since M is a compact metric space the Stone–Weierstrass theorem implies that any continuous function on M can be approached in the supremum norm by a sequence of Lipschitz functions on M . Since we have proved mixing for Lipschitz functions, the proof is concluded. \square

5. FINAL PROPERTIES OF μ_t (PROOF OF THEOREM 1.1 AND THEOREM 2.5)

In this section we show Proposition 5.1, Corollary 5.3, Proposition 5.5, Lemma 5.6, and Proposition 5.7, which, together with Theorem 4.1, give Theorem 1.1.

5.1. Measuring Neighbourhoods of Singularity Sets – μ_t is T -adapted. In this section, we show Proposition 5.1, which gives in particular that μ_t is T -adapted. For any $\varepsilon > 0$ and any $A \subset M$, we set $\mathcal{N}_\varepsilon(A) = \{x \in M \mid d(x, A) < \varepsilon\}$. The proof will be based on controlling the measure of small neighbourhoods of singularity sets.

Proposition 5.1. *Let μ_t be given by Theorem 4.1 for $t \in [t_0, t_1]$, with $p > 2$ the norm parameter.*

- a) *For any C^1 curve S uniformly transverse to the stable cone, there exists $C > 0$ such that $\mu_t(\mathcal{N}_\varepsilon(S)) \leq C\varepsilon^{1/p}$ for all $\varepsilon > 0$.*
- b) *The measure μ_t has no atoms. We have $\mu_t(\mathcal{S}_n) = 0$ for any $n \in \mathbb{Z}$, and $\mu_t(W) = 0$ for any local stable or unstable manifold W .*
- c) *The measure μ_t is adapted, i.e., $\int |\log d(x, \mathcal{S}_{\pm 1})| d\mu_t < \infty$.*
- d) *For any $p' > 2p$, μ_t -almost every x and each $n \in \mathbb{Z}$, there exists $C > 0$ such that*

$$(5.1) \quad d(T^j x, \mathcal{S}_n) \geq Cj^{-p'}, \forall j \geq 0.$$

- e) *μ_t -almost every $x \in M$ has stable and unstable manifolds of positive lengths.*

Proof. We proceed as in [BD, Corollary 7.4]. The key fact is that for any $n \in \mathbb{N}$ there exists $C_n < \infty$ such that for all $\varepsilon > 0$

$$(5.2) \quad \mu_t(\mathcal{N}_\varepsilon(\mathcal{S}_{-n})) < C_n \varepsilon^{1/p}, \quad \mu_t(\mathcal{N}_\varepsilon(\mathcal{S}_n)) < C_n \varepsilon^{1/(2p)}.$$

Denoting by $1_{n,\varepsilon}$ the indicator function of the set $\mathcal{N}_\varepsilon(\mathcal{S}_{-n})$, Proposition 4.16(a) implies

$$\mu_t(\mathcal{N}_\varepsilon(\mathcal{S}_{-n})) = \langle 1_{n,\varepsilon} \nu_t, \tilde{\nu}_t \rangle \leq C |1_{n,\varepsilon} \nu_t|_w,$$

for $n \geq 0$. The bound $|1_{n,\varepsilon} f|_w \leq A_n \|f\|_s |\varepsilon|^{1/p}$ for all $f \in \mathcal{B}$ follows exactly as the proof of [BD, Lemma 7.3], replacing the logarithmic modulus of continuity $|\log \varepsilon|^{-\gamma}$ in the strong stable norm there by our Hölder modulus of continuity $\varepsilon^{1/p}$, and using the fact that \mathcal{S}_{-n} is uniformly transverse to the stable cone. This proves the first inequality in (5.2). The second follows from the invariance of μ_t , together with the fact that $T(\mathcal{N}_\varepsilon(\mathcal{S}_n)) \subset \mathcal{N}_{C\varepsilon^{1/2}}(\mathcal{S}_{-n})$.

Claim a) of the proposition follows from the proof of (5.2), since the only property required of \mathcal{S}_{-n} is that it comprises finitely many smooth curves uniformly transverse to the stable cone. The bound (5.2) applied to arbitrary stable curves immediately implies that μ_t has no atoms, and that $\mu_t(\mathcal{S}_n) = 0$ for any $n \in \mathbb{Z}$. Next, if we had $\mu_t(W) > 0$ for a local stable manifold, then $\mu_t(T^n W) > 0$ for all $n > 0$. Since μ_t is a probability measure and T^n is continuous on stable manifolds, $\cup_{n \geq 0} T^n W$ must be the union of finitely many smooth curves (indeed, if $\cup_{n \geq 0} T^n W$ comprised infinitely many smooth curves, then $\mu_t(M) = \infty$ by the invariance of μ_t). Since $|T^n W| \rightarrow 0$, there is a subsequence (n_j) such that $\cap_{j \geq 0} T^{n_j} W = \{x\}$. Thus $\mu_t(\{x\}) > 0$, a contradiction. For an unstable manifold W , use the fact that T^{-n} is continuous on W . So we have established b).

To show c), choose $p' > 2p$. Then by (5.2)

$$\begin{aligned} \int_{M \setminus \mathcal{N}_1(\mathcal{S}_1)} |\log d(x, \mathcal{S}_1)| d\mu_t &= \sum_{j \geq 1} \int_{\mathcal{N}_{j-p'}(\mathcal{S}_1) \setminus \mathcal{N}_{(j+1)-p'}(\mathcal{S}_1)} |\log d(x, \mathcal{S}_1)| d\mu_t \\ &\leq p' \sum_{j \geq 1} \log(j+1) \cdot \mu_t(\mathcal{N}_{j-p'}(\mathcal{S}_1)) \leq p' C_1 \sum_{j \geq 1} \log(j+1) \cdot j^{-p'/(2p)} < \infty. \end{aligned}$$

A similar estimate holds for $\int |\log d(x, \mathcal{S}_{-1})| d\mu_t$.

Next, fix $\eta > 0$, $p' > 2p$ and $n \in \mathbb{Z}_+$. Since both sums

$$(5.3) \quad \sum_{j \geq 1} \mu_t(\mathcal{N}_{\eta j-p'}(\mathcal{S}_{-n})) \leq \tilde{C} C_{-n} \eta^{\frac{1}{p}} \sum_{j \geq 1} j^{-\frac{p'}{p}}, \quad \sum_{j \geq 1} \mu_t(\mathcal{N}_{\eta j-p'}(\mathcal{S}_n)) \leq \tilde{C} C_n \eta^{\frac{1}{2p}} \sum_{j \geq 1} j^{-\frac{p'}{2p}},$$

are finite, the Borel–Cantelli Lemma implies that μ_t -almost every $x \in M$ visits $\mathcal{N}_{\eta j-p'}(\mathcal{S}_n)$ only finitely many times. This gives (5.1) and thus claim d). Finally, the existence of nontrivial stable and unstable manifolds claimed in e) follows from the Borel–Cantelli estimate (5.3) by a standard argument, choosing $p' > 2p$ and $\eta \geq 1$ such that $\Lambda^j > \eta^{-1} j^{p'}$ for all j (see [CM, Sect. 4.12]). \square

5.2. μ_t is an Equilibrium State. Variational Principle for $P_*(t)$. For $\epsilon > 0$, $x \in M$, and $n \geq 1$ denote by $B_n(x, \epsilon)$ the dynamical (Bowen) ball for T^{-1} :

$$(5.4) \quad B_n(x, \epsilon) = \{y \in M \mid d(T^{-j}(y), T^{-j}(x)) \leq \epsilon, \forall 0 \leq j \leq n\}.$$

Proposition 5.2 (Upper Bounds on the Measure of Dynamical Balls). *Let $t_0 \in (0, 1)$ and $t_1 \in (1, t_*)$. There exists $A < \infty$ such that for all small enough $\epsilon > 0$, all $x \in M$, and all $n \geq 1$, the measure μ_t constructed in Theorem 4.1 for $t \in [t_0, t_1]$ satisfies*

$$(5.5) \quad \mu_t(\overline{B_n(x, \epsilon)}) \leq A e^{-nP_*(t) + t \sum_{k=1}^n \log J^s T(T^{-k}(x))}.$$

Corollary 5.3 (Equilibrium State for $-t \log J^u$. Variational principle for $P_*(t)$). *The measure μ_t constructed in Theorem 4.1 for $t \in (0, t_*)$ satisfies $P_{\mu_t}(-t \log J^u T) = P_*(t) = P(t)$.*

Proof of Corollary 5.3. By definition we have $P_{\mu_t}(-t \log J^u T) \leq P(t)$, and Proposition 2.4 gives $P(t) \leq P_*(t)$, so it is enough to show $P_{\mu_t}(-t \log J^u T) \geq P_*(t)$. We follow [BD, Cor. 7.17]. Since $\int |\log d(x, \mathcal{S}_{\pm 1})| d\mu_t < \infty$ by Proposition 5.1, and μ_t is ergodic, we may apply [DWY, Prop. 3.1] (a slight generalization of the Brin–Katok local theorem [BK], using [M, Lemma 2], continuity of the map is not used) to T^{-1} . This gives that for μ_t -almost every $x \in M$,

$$\lim_{\epsilon \rightarrow 0} \liminf_{n \rightarrow \infty} -\frac{1}{n} \log \mu_t(B_n(x, \epsilon)) = \lim_{\epsilon \rightarrow 0} \limsup_{n \rightarrow \infty} -\frac{1}{n} \log \mu_t(B_n(x, \epsilon)) = h_{\mu_t}(T^{-1}) = h_{\mu_t}(T).$$

Using (5.5) it follows that for any ϵ sufficiently small,

$$\limsup_{n \rightarrow \infty} -\frac{1}{n} \log \mu_t(B_n(x, \epsilon)) \geq P_*(t) - \lim_{n \rightarrow \infty} \frac{t}{n} \sum_{k=1}^n \log J^s T(T^{-k}(x)) \geq P_*(t) - t \int_M \log J^s T d\mu_t,$$

for all μ_t -typical x . Thus applying (2.9), we get $P_{\mu_t}(-t \log J^u T) \geq P_*(t)$. \square

Proof of Proposition 5.2. For $x \in M$ and $n \geq 0$, let $1_{B_{n,\epsilon}^B}$ denote the indicator function of $B_n(x, \epsilon)$. Since ν_t is attained as the (averaged) limit of $e^{-nP_*(t)} \mathcal{L}_t^n 1$ in the weak (and strong) norm and since we have $\int_W (\mathcal{L}_t^n 1) \psi dm_W \geq 0$ whenever $\psi \geq 0$, it follows that, viewing ν_t as a leafwise distribution,

$$(5.6) \quad \int_W \psi \nu_t \geq 0, \quad \text{for all } \psi \geq 0.$$

Then the inequality $|\int_W \psi \nu_t| \leq \int_W |\psi| \nu_t$ implies that the supremum in the weak norm can be obtained by restricting to $\psi \geq 0$. In addition, for each $n \geq 0$,

$$(5.7) \quad \begin{aligned} \int_W \psi \mathcal{L}_t^n \nu_t &= \lim_k e^{-kP_*(t)} \int_W \psi \mathcal{L}_t^n (\mathcal{L}_t^k 1) dm_W \\ &= \lim_k e^{-kP_*(t)} \int_{T^{-n}W} \psi \circ T^n \mathcal{L}_t^k 1 |J^s T^n|^t dm_{T^{-n}W} = \int_{T^{-n}W} \psi \circ T^n |J^s T^n|^t \nu_t, \end{aligned}$$

for each $W \in \mathcal{W}^s$ and $\psi \in C^\beta(W)$.

Let $W \in \mathcal{W}^s$ be a curve intersecting $B_n(x, \epsilon)$, and let $\psi \in C^\alpha(W)$ satisfy $\psi \geq 0$ and $|\psi|_{C^\alpha(W)} \leq 1$. Then, since $\mathcal{L}_t \nu_t = e^{P_*(t)} \nu_t$, we have

$$(5.8) \quad \int_W \psi 1_{n,\epsilon}^B \nu_t = \int_W \psi 1_{n,\epsilon}^B e^{-nP_*(t)} \mathcal{L}_t^n \nu_t = e^{-nP_*(t)} \sum_{W_i \in \mathcal{G}_n(W)} \int_{W_i} (\psi \circ T^n) (1_{n,\epsilon}^B \circ T^n) |J^s T^n|^t \nu_t.$$

In the proof of [BD, Prop. 7.12] we showed that $1_{n,\epsilon}^B f \in \mathcal{B}_w$ (and \mathcal{B}) for each $f \in \mathcal{B}$ and $n \geq 0$. In the proof of [BD, Lemma 3.4], we found (using our strong notion of finite horizon) $\tilde{\epsilon} > 0$ such that there if x, y lie in different elements of \mathcal{M}_0^n , then $\max_{0 \leq i \leq n} d(T^i x, T^i y) \geq \tilde{\epsilon}$. Since $B_n(x, \epsilon)$ is defined with respect to T^{-1} , we will use the time reversal counterpart of this property: If $\epsilon < \tilde{\epsilon}$, we conclude that $B_n(x, \epsilon)$ is contained in a single component of \mathcal{M}_{-n}^0 , i.e., $B_n(x, \epsilon) \cap \mathcal{S}_{-n} = \emptyset$, so that T^{-n} is a diffeomorphism of $B_n(x, \epsilon)$ onto its image. Note that $T^{-n}(B_n(x, \epsilon))$ is contained in a single component of \mathcal{M}_0^n , denoted $A_{n,\epsilon}$. Thus, $W_i \cap A_{n,\epsilon} = W_i$ for each $W_i \in \mathcal{G}_n(W)$. By (5.6),

$$\int_{W_i} (\psi \circ T^n) 1_{T^{-n}(B_n(x,\epsilon))} |J^s T^n|^t \nu_t \leq \int_{W_i} (\psi \circ T^n) |J^s T^n|^t \nu_t.$$

In the proof of [BD, Prop. 7.12] we observed that there are at most two $W_i \in \mathcal{G}_n(W)$ having nonempty intersection with $T^{-n}(B_n(x, \epsilon))$. Using these facts together with (4.11) and (4.12) (which implies $\| |J^s T^n|^t \|_{C^\alpha(W_i)} \leq C \| |J^s T^n|^t \|_{C^0(W_i)}$), we sum over $W'_i \in \mathcal{G}_n(W)$ such that $W'_i \cap T^{-n}(B_n(x, \epsilon)) \neq \emptyset$, to obtain

$$\int_W \psi 1_{n,\epsilon}^B \nu_t \leq e^{-nP_*(t)} \sum_i \int_{W'_i} (\psi \circ T^n) |J^s T^n|^t \nu_t \leq 2C e^{-nP_*(t)+t \sum_{k=0}^{n-1} \log J^s T(T^{k-n}(x))} |\nu_t|_w,$$

where we also used the distortion bounds from Lemma 2.2 to switch to $J^s T^n(T^{-n}x)$ since $T^{-n}x$ may not belong to W'_i . This yields $|1_{n,\epsilon}^B \nu_t|_w \leq 2C e^{-nP_*(t)+t \log J^s T^n(T^{-n}x)} |\nu_t|_w$. Applying Proposition 4.16(a) gives (5.5). \square

5.3. Definition of h_* . Sparse Recurrence. Proof that $\lim_{t \downarrow 0} P(t) = h_*$. In [BD, Lemma 3.3] we showed that the limit below exists

$$h_* := \lim_{n \rightarrow \infty} \frac{1}{n} \log \# \mathcal{M}_0^n.$$

The number h_* generalises topological entropy, in particular, $P(0) \leq h_*$ [BD, Theorem 2.3].

Using h_* , we can state the sparse recurrence condition:

Definition 5.4 (Sparse Recurrence to Singularities). *For $\varphi < \pi/2$ and $n \in \mathbb{N}$, define $s_0(\varphi, n) \in (0, 1]$ to be the smallest number such that any orbit of length n has at most $s_0 n$ collisions whose angles with the normal are larger than φ in absolute value. We say that T satisfies the sparse recurrence condition if there exist $\varphi_0 < \pi/2$ and $n_0 \in \mathbb{N}$ such that $h_* > s_0(\varphi_0, n_0) \log 2$.*

We refer to [BD, §2.4] for a discussion of the sparse recurrence condition. We proved in [BD] that sparse recurrence implies $P(0) = h_*$. The following proposition connects h_* to $P_*(t)$ for $t > 0$, despite the use of different partitions, \mathcal{M}_0^n and $\mathcal{M}_0^{n,\mathbb{H}}$.

Proposition 5.5. *If T satisfies sparse recurrence then $\lim_{t \downarrow 0} P_*(t) = \lim_{t \downarrow 0} P(t) = h_*$, and $\lim_{t \downarrow 0} h_{\mu_t} = h_*$.*

Assuming the sparse recurrence condition [BD, Theorem 2.4] we have $P(0) = h_*$. So in this case the function $P(t)$ is continuous on $[0, t_*)$. In the general case, we cannot exclude $P(0) < h_*$ even if we can show $\lim_{t \downarrow 0} P(t) = P(0)$.

Proof. Recall that $P(t) = P_*(t)$ for $t \in (0, t_*)$ (using Proposition 2.4 and Corollary 5.3).

Showing³⁹ $\lim_{t \downarrow 0} P(t) \leq h_*$ does not require the sparse recurrence condition: Any invariant probability measure μ satisfies $\int_M \log J^u T d\mu \geq \log \Lambda$ due to (1.2). Also, $h_\mu(T) \leq h_*$ by [BD, Theorem 2.3]. Thus for $t > 0$, we have $P(t) \leq h_* - t \log \Lambda$, so that, $\lim_{t \downarrow 0} P(t) \leq h_*$.

To prove the lower bound, assume the sparse recurrence condition, and let μ_0 denote the measure of maximal entropy for T constructed in [BD, Theorem 2.4] (called μ_* in that paper). Since μ_0 is T-adapted [BD, Theorem 2.6], the Jacobian $J^u T$ is defined μ_0 -almost everywhere and $\int \log J^u T d\mu_0 = \chi_{\mu_0}^+ < \infty$. Thus for $t > 0$,

$$P(t) \geq P_{\mu_0}(-t \log J^u T) = h_{\mu_0} - t \int_M \log J^u T d\mu_0 = h_* - t \chi_{\mu_0}^+,$$

and $\lim_{t \downarrow 0} P(t) \geq h_*$.

Finally, since $P(t) \leq h_{\mu_t} \leq h_*$, we must have $\lim_{t \downarrow 0} h_{\mu_t} = h_*$ as well. \square

5.4. Full Support of μ_t . It follows from Lemma 4.17 that the measure ν_t is fully supported on M . In this section, we will prove the analogous property for μ_t combining mixing of the SRB measure and a direct use of Cantor rectangles, bypassing the absolute continuity argument which was used in [BD, Section 7.3] to show full support of the measure of maximal entropy there. Recall the definition of maximal Cantor rectangle $R = R(D)$ comprising the intersection of all homogeneous stable and unstable manifolds completely crossing a solid rectangle D as described in the proof of Proposition 3.14. The boundary of the solid rectangle D comprises two stable and unstable manifolds which also belong to R . Let $\Xi_R \subset \mathcal{W}^s$ denote the family of stable manifolds corresponding to R (i.e. the set of homogeneous stable manifolds that completely cross D).

Lemma 5.6. *For any maximal Cantor rectangle R , if $\mu_{\text{SRB}}(\cup_{W \in \Xi_R} W) > 0$ then we also have $\mu_t(\cup_{W \in \Xi_R} W) > 0$. Consequently, for any nonempty open set $O \subset M$, we have $\mu_t(O) > 0$.*

Proof. Let $\psi \in C^1(M)$ such that $\psi \geq 0$ and $\psi \equiv 1$ on $\cup_{W \in \Xi_R} W$. Due to the spectral decomposition of \mathcal{L}_t^* , setting $c = \langle \nu_t, \tilde{\nu}_t \rangle^{-1}$, we have

$$(5.9) \quad \mu_t(\psi) = c \lim_{n \rightarrow \infty} e^{-nP_*(t)} \langle \psi \nu_t, (\mathcal{L}_t^*)^n d\mu_{\text{SRB}} \rangle = c \lim_{n \rightarrow \infty} e^{-nP_*(t)} \langle \mathcal{L}_t^n(\psi \nu_t), d\mu_{\text{SRB}} \rangle.$$

Then, using the disintegration of μ_{SRB} , introduced before Lemma 4.17, into conditional measures on a fixed foliation $\mathcal{F} = \{W_\xi\}_{\xi \in \Xi}$ of stable manifolds, and a transverse measure $\hat{\mu}_{\text{SRB}}$ on the index set Ξ , and recalling (5.7), we estimate for $n \geq 0$,

$$\begin{aligned} \langle \mathcal{L}_t^n(\psi \nu_t), d\mu_{\text{SRB}} \rangle &= \int_{\Xi} |W_\xi|^{-1} d\hat{\mu}_{\text{SRB}}(\xi) \int_{W_\xi} \mathcal{L}_t^n(\psi \nu_t) \rho_\xi \\ &= \int_{\Xi} |W_\xi|^{-1} d\hat{\mu}_{\text{SRB}}(\xi) \sum_{W_i \in \mathcal{G}_n(W_\xi)} \int_{W_i} \psi \nu_t |J^s T^n|^t \rho_\xi \circ T^n \\ &\geq C \int_{\Xi} |W_\xi|^{-1} d\hat{\mu}_{\text{SRB}}(\xi) \sum_{W_i \in \mathcal{G}_n(W_\xi)} |J^s T^n|_{C^0(W_i)}^t \int_{W_i} \psi \nu_t, \end{aligned}$$

where in the last line we have used (4.47), bounded distortion for $J^s T$ and the positivity of ν_t . Next, note that if $W_i \in \mathcal{G}_n(W_\xi)$ properly crosses⁴⁰ R , then using again the positivity of ν_t , we have

$$(5.10) \quad \int_{W_i} \psi \nu_t \geq \zeta(\ell_R),$$

³⁹Note that the limit exists since $P(t) = P_*(t)$ is monotonic.

⁴⁰See the proof of Proposition 3.14 for the definition of proper crossing.

where ℓ_R is the minimum length of a stable manifold in Ξ_R and ζ is a function depending only on t (uniform in $[t_0, t_1]$) and $\delta_1(t_0, t_1)$ from (3.47) via (4.50). Thus letting $\mathcal{G}_n^R(W_\xi)$ denote those elements of $\mathcal{G}_n(W)$ that properly cross R , we have

$$(5.11) \quad \langle \mathcal{L}_t^n(\psi\nu_t), d\mu_{\text{SRB}} \rangle \geq C'\zeta(\ell_R) \int_{\Xi} |W_\xi|^{-1} d\hat{\mu}_{\text{SRB}}(\xi) \sum_{W_i \in \mathcal{G}_n^R(W_\xi)} |J^s T^n|_{C^0(W_i)}^t.$$

As in the proof of Proposition 3.14, by [CM, Lemma 7.87], we choose a finite number of locally maximal homogeneous Cantor rectangles $\mathcal{R}(\delta_1) = \{R_1, \dots, R_k\}$ such that there exists $n_* = n_*(\delta_1, R)$ such that $T^{-n_*}(D(R_i))$ contains a homogeneous connected component that properly crosses R for all $i = 1, \dots, k$. Therefore, if $V \in \mathcal{W}^s$ has $|V| \geq \delta_1/3$, then at least one element of $\mathcal{G}_{n_*}(V)$ properly crosses R . Thus, if $|W_\xi| \geq \delta_1/3$ and $n - n_* \geq n_1$, then using (3.31), and letting δ'_1 denote the minimum length of a stable manifold belonging to any of the R_i ,

$$(5.12) \quad \begin{aligned} \sum_{W_i \in \mathcal{G}_n^R(W_\xi)} |J^s T^n|_{C^0(W_i)}^t &\geq e^{-tC_d} \sum_{W_j \in L_{n-n_*}^{\delta'_1}(W_\xi)} |J^s T^{n-n_*}|_{C^0(W_j)}^t |J^s T^{n_*}|_{C^0(W_i)}^t \\ &\geq \frac{3}{4} e^{-tC_d} C(\delta'_1)^{t \left(\frac{2q+1}{q+1}\right)^{n_*}} \sum_{W_j \in \mathcal{G}_{n-n_*}^{\delta'_1}(W_\xi)} |J^s T^{n-n_*}|_{C^0(W_j)}^t \\ &\geq \frac{3}{4} e^{-tC_d} C(\delta'_1)^{t \left(\frac{2q+1}{q+1}\right)^{n_*}} c_1 e^{(n-n_*)P_*(t)}, \end{aligned}$$

where in the second line we have estimated $J^s T^{n_*}$ from below on W_i as in (3.39) using the fact that $|W_i| \geq \delta'_1$, and in the third line we have applied Propositions 3.14 and 3.15.

Substituting (5.12) into (5.11) and letting Ξ^{δ_1} denote those elements $W_\xi \in \mathcal{F}$ with $|W_\xi| \geq \delta_1/3$,

$$e^{-nP_*(t)} \langle \mathcal{L}_t^n(\psi\nu_t), d\mu_{\text{SRB}} \rangle \geq C''\zeta(\ell_R) \delta_1^{-1} (\delta'_1)^{t \left(\frac{2q+1}{q+1}\right)^{n_*}} e^{-n_*P_*(t)} \hat{\mu}_{\text{SRB}}(\Xi^{\delta_1}).$$

Since this lower bound is independent of n , by (5.9) we have $\mu_t(\psi) > 0$, and since this holds for all $\psi \in C^1(M)$ with $\psi \equiv 1$ on $\cup_{W \in \Xi_R} W$, the first statement of the lemma is proved. Then the second statement of the lemma follows from the fact that any nonempty open set $O \subset M$ has a locally maximal Cantor set R such that $D(R) \subset O$ and $\mu_{\text{SRB}}(R) > 0$. \square

5.5. Uniqueness of Equilibrium State. (Strong) Variational Principle for $P_*(t, g)$. In this section, we prove the following uniqueness result:

Proposition 5.7. *For any $0 < t < t_*$, the measure μ_t from Theorem 4.1 is the unique equilibrium state for $-t \log J^u T$.*

The proof of the proposition will give a more general statement (shown at the end of this section):

Theorem 5.8 (Strong Variational Principle for $P_*(t, g)$). *For any $[t_0, t_1] \subset (0, t_*)$ there exists $v_0 > 0$ such that for any C^1 function $g : M \rightarrow \mathbb{R}$ with $|g|_{C^1} \leq v_0$ we have*

$$P_*(t, g) = P(t, g) = \max \left\{ h_\mu + \int (-t \log J^u T + g) d\mu : \mu \text{ a } T\text{-invariant probability measure} \right\},$$

and the equilibrium state for $-t \log J^u + g$ is unique.

(We restrict to C^1 functions g for simplicity. The result also holds Hölder g of suitable exponent.)

Fix $0 < t_0 < t_1 < t_*$. For $\phi \in C^1(M)$, $t \in [t_0, t_1]$, and $v \in \mathbb{R}$, define the transfer operator $\mathcal{L}_{t,v} = \mathcal{L}_{t,v,\phi}$ by

$$\mathcal{L}_{t,v} f = \frac{f \circ T^{-1}}{|J^s T|^{1-t} \circ T^{-1}} e^{v\phi \circ T^{-1}}, \quad \text{for all } f \in C^1(M).$$

Since $\mathcal{L}_{t,v} f = e^{v\phi \circ T^{-1}} \mathcal{L}_t f$ and the discontinuities of $\phi \circ T^{-1}$ are uniformly transverse to the stable cone, [DZ2, Lemma 5.3] implies that $\mathcal{L}_{t,v} f \in \mathcal{B}$ (with $\mathcal{B} = \mathcal{B}(t_0, t_1)$ the space for \mathcal{L}_t)

and $\|\mathcal{L}_{t,v}f\|_{\mathcal{B}} \leq C\|f\|_{\mathcal{B}}|e^{v\phi}|_{C^1}$, so that $\mathcal{L}_{t,v}$ defines a bounded linear operator on \mathcal{B} . By [DZ1, Lemma 6.1] the map $v \mapsto \mathcal{L}_{t,v}$ is analytic. Thus since $\mathcal{L}_t = \mathcal{L}_{t,0}$ has a spectral gap, so does $\mathcal{L}_{t,v}$ for $|v|$ sufficiently small, and the leading eigenvalue $\lambda_{t,v}$ varies analytically in v [Ka, VII, Thm 1.8, II.1.8]; moreover, $\lambda_{t,0} = e^{P(t)}$ and, with μ_t from Theorem 4.1, we have [Ka, II.2.1, (2.1), (2.33)]

$$(5.13) \quad \frac{d}{dv}\lambda_{t,v}\Big|_{v=0} = e^{P(t)} \int \phi d\mu_t, \quad \forall t \in [t_0, t_1].$$

Recalling the definition $P(t, v\phi)$ in (1.7), the following result will give Proposition 5.7:

Proposition 5.9. *Fix $0 < t_0 < t_1 < t_*$. For $\phi \in C^1(M)$, $t \in [t_0, t_1]$, and $v \in \mathbb{R}$, with $|v|$ sufficiently small, the spectral radius of $\mathcal{L}_{t,v}$ on $\mathcal{B}(t_0, t_1)$ is $\lambda_{t,v\phi} = e^{P(t,v\phi)}$.*

Proof of Proposition 5.7. We use tangent measures, inspired by the proof of [Br, Theorem 16]: If μ is an equilibrium state for $-t \log J^u T$ then μ is a C^1 -tangent measure at t (see e.g.⁴¹ [W, Theorem 9.14]) in the sense that,

$$(5.14) \quad P(t, \phi) \geq P(t, 0) + \int \phi d\mu \quad \text{for all } \phi \in C^1(M).$$

Thus, Proposition 5.9 together with (5.13) imply that

$$\begin{aligned} \int \phi d\mu_t &= \lim_{v \downarrow 0} \frac{P(t, v\phi) - P(t, 0)}{v} \geq \int \phi d\mu \quad \text{and} \\ \int \phi d\mu_t &= \lim_{v \uparrow 0} \frac{P(t, v\phi) - P(t, 0)}{v} \leq \int \phi d\mu. \end{aligned}$$

Thus $\int \phi d\mu_t = \int \phi d\mu$ for all $\phi \in C^1(M)$. Since M is a compact metric space, $C^1(M)$ is dense in $C^0(M)$ and so $\mu = \mu_t$ showing the uniqueness claim in the proposition. \square

Proof of Proposition 5.9. Let $|v|$ be small enough such that $g := v\phi$ satisfies (3.6), (3.30), (3.46) and their analogues for the number of interpolations needed to reach $t_1 < t_*$ in Section 3.6. The constants $n_1, n_2, \delta_1, \delta_2, C_2, c_0, c_1, c_2$, and C_κ from Section 3 then hold for all $g = v'\phi$ with $|v'| < |v|$ and all $t \in [t_0, t_1]$. In particular the constants $c_1(v') > 0$ from Proposition 3.14 and Proposition 3.18(a) and $c_2(v') > 0$ in Proposition 3.15 and Proposition 3.18(b) are uniform in $|v'| < |v|$ and $t \in [t_0, t_1]$.

Step 1. The Spectral Radius $\lambda_{t,v}$ of $\mathcal{L}_{t,v}$ on \mathcal{B} is $e^{P_(t,v\phi)}$.* Possibly reducing $|v|$ further, $\mathcal{L}_{t,v}$ has a spectral gap on \mathcal{B} , as observed above. The upper bound on $\lambda_{t,v} \leq e^{P_*(t,v\phi)}$ can thus be proved as in Proposition 4.12, once we know that the spectral radius of $\mathcal{L}_{t,v}$ on \mathcal{B}_w is at most $e^{P_*(t,v\phi)}$. For this, by the upper bound in Proposition 3.18(b), it suffices to find $C < \infty$ such that

$$(5.15) \quad |\mathcal{L}_{t,v}^n f|_w \leq C Q_n(t, v\phi) |f|_w, \quad \forall f \in C^1.$$

To prove (5.15), note that due to (2.10), we have for $W \in \mathcal{W}^s$ and $W_i \in \mathcal{G}_n(W)$,

$$(5.16) \quad |e^{vS_n\phi}|_{C^\alpha(W_i)} \leq (1 + C_* |\nabla\phi|_{C^0} \cdot \delta_0^{1-\alpha}) |e^{vS_n\phi}|_{C^0(W_i)},$$

then, for $W \in \mathcal{W}^s$ and $\psi \in C^\alpha(W)$ with $|\psi|_{C^\alpha(W)} \leq 1$, we follow (4.10) and apply (4.11), (4.12), Lemma 3.4, and (5.16) to write,

$$\begin{aligned} \int_W \mathcal{L}_{t,v}^n f \psi dm_W &\leq \sum_{W_i \in \mathcal{G}_n(W)} |f|_w |\psi \circ T^n|_{C^\alpha(W)} \|J^s T^n|^t e^{vS_n\phi}|_{C^\alpha(W_i)} \\ &\leq |f|_w C_1^{-1} (1 + 2^t C_d) (1 + C_* |\nabla\phi|_{C^0}) \sum_{W_i \in \mathcal{G}_n(W)} |J^s T^n|^t |e^{vS_n\phi}|_{C^0(W_i)} \leq C |f|_w Q_n(t, v\phi). \end{aligned}$$

The lower bound $\lambda_{t,v} \geq e^{P_*(t,v\phi)}$ on the spectral radius follows as in the proof of Proposition 4.12.

⁴¹The standard definitions use C^0 rather than C^1 in (5.14) For our purposes, C^1 will suffice.

Step 2. $P_*(t, v\phi) = P(-t \log J^u T + v\phi)$. Denoting by $\nu_{t,v}$ the eigenmeasure associated to $e^{P_*(t,v\phi)}$ and by $\tilde{\nu}_{t,v}$ the eigenmeasure of the dual operator $\mathcal{L}_{t,v}^*$, defined as in Lemma 4.13 and (4.46), we construct an invariant probability measure $\mu_{t,v}$ as in Proposition 4.16.

We claim the following analogue of Proposition 5.2: There exists $A < \infty$ such that for all sufficiently small $|v|$, all $\epsilon > 0$ sufficiently small, all $x \in M$ and $n \geq 1$,

$$(5.17) \quad \mu_{t,v}(B_n(x, \epsilon)) \leq A e^{-nP_*(t,v\phi) + t \log J^s T^n(T^{-n}x) + vS_n\phi(T^{-n}x)},$$

where $B_n(x, \epsilon)$ is the Bowen ball defined in (5.4). Using (5.17), the proof of Corollary 5.3 yields that $P_{\mu_{t,v}}(-t \log J^u T + v\phi) \geq P_*(t, v\phi)$, and this, together with Proposition 2.4 yields $P(-t \log J^u T + v\phi) = P_*(t, v\phi)$. By Step 1, this ends the proof of Proposition 5.9. (In addition, we have established that $\mu_{t,v}$ is an equilibrium state for $-t \log J^u T + v\phi$.)

Finally, (5.17) follows easily from the proof of Proposition 5.2. The property in (5.6) extends to $\nu_{t,v}$ due to its definition as a limit of $e^{-nP_*(t,v\phi)} \mathcal{L}_{t,v}^n 1$. The analogue of (5.7) holds for the same reason, so that the modification of (5.8) yields,

$$\int_W \psi 1_{n\epsilon}^B \nu_{t,v} = e^{-nP_*(t,v\phi)} \sum_{W_i \in \mathcal{G}_n(W)} \int_{W_i} (\psi \circ T^n)(1_{n\epsilon}^B \circ T^n) |J^s T^n|^t e^{vS_n\phi} \nu_{t,v},$$

where $1_{n\epsilon}^B$ denotes the indicator function of $B_n(x, \epsilon)$. The subsequent estimates in the proof of Proposition 5.2 go through with the obvious changes, so that

$$|1_{n\epsilon}^B \nu_{t,v}|_w \leq C' e^{-nP_*(t,v\phi) + t \log J^s T^n(T^{-n}x) + vS_n\phi(T^{-n}x)},$$

where the only additional factor needed is the distortion constant $C_* |\nabla\phi|_{C^0}$ from (2.10). Applying the analogue of Proposition 4.16(a) completes the proof of (5.17). \square

Proof of Theorem 5.8. The upper bound $P(t, g) \leq P_*(t, g)$ is the content of Proposition 2.4. Taking $v\phi = g$, the equilibrium state for $-t \log J^u + g$ is $\mu_{t,v}$ constructed in Step 2 of the proof of Proposition 5.9. The proof of uniqueness can be obtained by a straightforward adaptation of the argument proving uniqueness of the equilibrium state for $-t \log J^u$, up to taking small enough $|g|_{C^1}$. \square

6. ANALYTICITY, DERIVATIVES OF $P(t)$, AND STRICT CONVEXITY (PROOF OF THEOREM 1.2)

This section contains the proof of Theorem 1.2 and Corollaries 1.4, 1.5, and 1.6.

The maximal eigenvalue of \mathcal{L}_t^n is $\exp(nP(t))$. Showing that $nP(t)$ is analytic for some integer $n \geq 1$ is equivalent to showing that $P(t)$ is analytic. Recall the one-step expansion factor $\theta^{-1} > 1$ from Lemma 3.1. In the remainder of this section⁴²:

$$\text{Fix } n_0 \geq 1 \text{ such that } |J^s T^{n_0}| < C_0 \theta^{n_0} \leq \frac{1}{2}, \text{ and set } \mathcal{T} := T^{n_0}.$$

By standard results on analytic perturbations of simple isolated eigenvalues [Ka], analyticity of $P(t) = P_*(t)$ will be an immediate consequence of the following result:

Proposition 6.1 (Analyticity of $t \mapsto \mathcal{L}_t^{n_0}$). *Fix $0 < t_0 < t_1 < t_*$. Then the map $t \mapsto \mathcal{L}_t^{n_0}$ is analytic from (t_0, t_1) to the space of bounded operators from \mathcal{B} to \mathcal{B} , with*

$$(6.1) \quad \partial_t^j \mathcal{L}_t^{n_0}(f)|_{t=w} = \mathcal{L}_w^{n_0}((\log J^s \mathcal{T})^j f), \quad \forall j \geq 1, \quad \forall w \in (t_0, t_1), \quad \forall f \in \mathcal{B}.$$

Proof. We claim that it suffices to prove that, for any $0 < t_0 < t_1 < t_*$, we have

$$(6.2) \quad \text{there exists } C < \infty \text{ such that } \|\mathcal{L}_w^{n_0}((\log J^s \mathcal{T})^j f)\|_{\mathcal{B}} \leq j(Cj)^j \|f\|_{\mathcal{B}}, \quad \forall f \in \mathcal{B},$$

for all $w \in (t_0, t_1)$ and all $j \geq 0$. (The bound (6.2) is the content of Proposition 6.3.)

⁴²The value 1/2 below is for convenience, giving the number $-\log 2$ in Lemma 6.2; what is important is $C_0 \theta^{n_0} < 1$.

Indeed, by the Stirling formula, (6.2) implies

$$(6.3) \quad \frac{\|\mathcal{L}_w^{n_0}(f(\log J^s \mathcal{T})^j)\|_{\mathcal{B}}}{j!} \leq j C^j C_{Stirling}^j \|f\|_{\mathcal{B}}.$$

Now, for $w \in (t_0, t_1)$ and $t \in \mathbb{C}$, first write

$$\mathcal{L}_t^{n_0} f = \frac{f \circ \mathcal{T}^{-1}}{|J^s \mathcal{T}|^{1-w} \circ \mathcal{T}^{-1}} e^{(w-t) \log J^s \mathcal{T} \circ \mathcal{T}^{-1}} = \frac{f \circ \mathcal{T}^{-1}}{|J^s \mathcal{T}|^{1-w} \circ \mathcal{T}^{-1}} \sum_{j=0}^{\infty} \frac{(w-t)^j}{j!} (\log J^s \mathcal{T})^j \circ \mathcal{T}^{-1}$$

where (6.3), with $w = 1$ and $f \equiv 1$, gives that the series converges in norm for $|w-t| < (CC_{Stirling})^{-1}$. Then note that

$$(6.4) \quad \frac{f \circ \mathcal{T}^{-1}}{|J^s \mathcal{T}|^{1-w} \circ \mathcal{T}^{-1}} \sum_{j=0}^{\infty} \frac{(w-t)^j}{j!} (\log J^s \mathcal{T})^j \circ \mathcal{T}^{-1} = \sum_{j=0}^{\infty} \frac{(w-t)^j}{j!} \mathcal{L}_w^{n_0}(f(\log J^s \mathcal{T})^j),$$

where the sum commutes with $\mathcal{L}_w^{n_0}$ due to (6.3), with w and f , so that this series also converges in norm for $|w-t| < (CC_{Stirling})^{-1}$. The radius of convergence is independent of $w \in (t_0, t_1)$, giving the claimed analyticity there. The power series representation (6.4) immediately implies (6.1). \square

The key to the analyticity result in this section is the following elementary lemma which extends the distortion estimate Lemma 2.1 to expressions of the type $(\log |J^s \mathcal{T}|)^j |J^s \mathcal{T}|^t$:

Lemma 6.2 (Distortion for $\exp(\Psi)(\Psi)^j$). *Fix $I \subset \mathbb{R}$ a compact interval and let $\Psi : I \rightarrow \mathbb{R}_-$. Then, for any $v > 0$, there exists $C_v < \infty$ such that*

$$(6.5) \quad |\exp(v\Psi)|\Psi|^j|_{C^0(I)} \leq (C_v j)^j, \quad \forall j \geq 1.$$

In addition, if $|\sup \Psi| = \inf |\Psi| \geq \log 2$ and there exist $\alpha \in (0, 1)$ and $C_{\Psi} < \infty$ such that⁴³

$$(6.6) \quad |\Psi(x) - \Psi(y)| \leq C_{\Psi} |x - y|^{\alpha}, \quad \forall x, y \in I,$$

then

$$(6.7) \quad |\log |\Psi(x)| - \log |\Psi(y)|| \leq 4C_{\Psi} |x - y|^{\alpha}, \quad \forall x, y \in I,$$

and, for any $t > 0$,

$$(6.8) \quad |\exp(t\Psi)|\Psi|^j|_{C^{\alpha}(I)} \leq (1 + eC_{\Psi}(4j + t)) |\exp(t\Psi)|\Psi|^j|_{C^0(I)}, \quad \forall j \geq 0.$$

(The lemma will be applied to $\Psi = \log |J^s \mathcal{T}|$, with $\alpha \leq 1/(q+1)$, and I an interval giving an arc length parametrisation of a weakly homogeneous stable manifold.)

Proof. The proof of (6.5) is a straightforward exercise in calculus (with $C_v = (e \cdot v)^{-1}$): It suffices to show that $\sup_{X \in [0,1]} |\log X|^j X^v \leq (\frac{j}{e \cdot v})^j$.

Next, for any $x, y \in I$, the Mean Value Theorem applied to the logarithm yields, for some Z between $|\Psi(x)|$ and $|\Psi(y)|$,

$$(6.9) \quad |\log |\Psi(x)| - \log |\Psi(y)|| \leq \frac{1}{Z} |\Psi(x) - \Psi(y)| \leq \frac{C_{\Psi} |x - y|^{\alpha}}{\log 2} \leq 4C_{\Psi} |x - y|^{\alpha}.$$

From (6.9) we get (6.7) and also, for any $x, y \in I$,

$$\begin{aligned} \left| \log \frac{\exp(t\Psi(x))|\Psi(x)|^j}{\exp(t\Psi(y))|\Psi(y)|^j} \right| &\leq j |\log |\Psi(x)| - \log |\Psi(y)|| + t |\Psi(x) - \Psi(y)| \\ &\leq (j4C_{\Psi} + tC_{\Psi}) |x - y|^{\alpha}. \end{aligned}$$

⁴³The bound (6.6) is equivalent to $\exp(-C_{\Psi}|x-y|^{\alpha}) \leq \exp(\Psi(x))/\exp(\Psi(y)) \leq \exp(C_{\Psi}|x-y|^{\alpha})$ or, for small enough $|x-y|$, to $|1 - \exp(\Psi(x))/\exp(\Psi(y))| \leq C_{d,\Psi}|x-y|^{\alpha}$.

This implies

$$(6.10) \quad \exp(-C_\Psi(4j+t)|x-y|^\alpha) \leq \frac{\exp(t\Psi(x))|\Psi(x)|^j}{\exp(t\Psi(y))|\Psi(y)|^j} \leq \exp(C_\Psi(4j+t)|x-y|^\alpha).$$

For $|x-y|^\alpha < (4jC_\Psi + tC_\Psi)^{-1}$ (other pairs (x, y) are trivial to handle), (6.10) implies

$$\left| 1 - \frac{\exp(t\Psi(x))|\Psi(x)|^j}{\exp(t\Psi(y))|\Psi(y)|^j} \right| \leq eC_\Psi(4j+t)|x-y|^\alpha.$$

Multiplying both sides above by $\exp(t\Psi(y))|\Psi(y)|^j \leq |\exp(t\Psi)|\Psi|^j|_{C^0(I)}$, proves (6.8). \square

Recalling that $\mathcal{T} = T^{n_0}$ for fixed n_0 , we define, for all integers $j \geq 0$,

$$(6.11) \quad \mathcal{M}_t^{(j)} f := \mathcal{L}_t^{n_0}((\log |J^s \mathcal{T}|)^j f),$$

acting on measurable functions. We first prove (6.2):

Proposition 6.3. *For any $0 < t_0 < t_1 < t_*$, there exists $C < \infty$ such that*

$$(6.12) \quad \|\mathcal{M}_t^{(j)} f\|_{\mathcal{B}} \leq C^j j^{j+1} \|f\|_{\mathcal{B}}, \quad \forall j \geq 1, \forall f \in \mathcal{B}, \forall t \in [t_0, t_1].$$

Remark 6.4. *A modification of the proof of Lemma 4.3 shows that for any $f \in C^1(M)$, $\mathcal{M}_t^{(j)} f$ can be approximated by $C^1(M)$ functions in the \mathcal{B} norm, using the fact that Lemma 4.10 holds for the function $(\log |J^s \mathcal{T}|)^j |J^s \mathcal{T}|^t$ by Lemma 6.2. By density of $C^1(M)$ in \mathcal{B} , this, together with Proposition 6.3, implies $\mathcal{M}_t^{(j)} f \in \mathcal{B}$ for all $f \in \mathcal{B}$ and $j \geq 0$.*

Proof of Proposition 6.3. It is enough to consider $f \in C^1(M)$. We first bound the stable norm. Fix $W \in \mathcal{W}^s$, and $\psi \in C^\beta(W)$ such that $|\psi|_{C^\beta(W)} \leq |W|^{-1/p}$. For $t > 0$, we have

$$(6.13) \quad \begin{aligned} \int_W \mathcal{M}_t^{(j)} f \psi \, dm_W &= \sum_{W_i \in \mathcal{G}_{n_0}(W)} \int_{W_i} f(\psi \circ \mathcal{T}) |J^s \mathcal{T}|^t (\log |J^s \mathcal{T}|)^j \, dm_{W_i} \\ &\leq \sum_{W_i \in \mathcal{G}_{n_0}(W)} \|f\|_s |\psi \circ \mathcal{T}|_{C^\beta(W)} \frac{|W_i|^{1/p}}{|W|^{1/p}} \| |J^s \mathcal{T}|^t (\log |J^s \mathcal{T}|)^j \|_{C^\beta(W)}. \end{aligned}$$

On the one hand, we have seen in §4.3.1 that $|\psi \circ \mathcal{T}|_{C^\beta(W_i)} \leq \tilde{C} |\psi|_{C^\beta(W)}$. On the other hand, recalling that $\sup_{W_i \in \mathcal{G}_n} |J^s \mathcal{T}|_{C^0(W_i)} < 1$, and using (6.5) from Lemma 6.2, for any $v > 0$, there exists C_v such that for any $W_i \in \mathcal{G}_n(W)$, all $t \in [t_0, t_1]$, and all $j \geq 1$,

$$(6.14) \quad \sup_{W_i} (|\log |J^s \mathcal{T}|^j| |J^s \mathcal{T}|^t) \leq (jC_v)^j \sup_{W_i} |J^s \mathcal{T}|^{t-v}.$$

Therefore, since $\beta < \alpha$, choosing⁴⁴ $v < t_0/2 - 1/p$ and applying Lemma 6.2, we deduce from (6.13) and (6.14) that for all $j \geq 1$ and $f \in C^1$, taking $C'_d = 1 + eC_\Psi(4+t)$ from (6.8),

$$\begin{aligned} \int_W \mathcal{M}_t^{(j)} f \psi \, dm_W &\leq C_v^j j^j j C'_d \sum_{W_i \in \mathcal{G}_{n_0}(W)} \|f\|_s \frac{|W_i|^{1/p}}{|W|^{1/p}} |J^s \mathcal{T}|_{C^0(W_i)}^{t-v} \\ &\leq C'_d C_2[0] j (jC_v)^j \|f\|_s Q_{n_0}(t - v - 1/p), \end{aligned}$$

where $C_\Psi = C_d$ by (2.3), and we used Lemma 3.4 with $\varsigma = 1/p$. Taking the suprema over $\psi \in C^\beta(W)$ with $|\psi|_{C^\beta(W)} \leq |W|^{-1/p}$ and $W \in \mathcal{W}^s$ yields $C_s < \infty$ such that

$$(6.15) \quad \|\mathcal{M}_t^{(j)}(f)\|_s \leq j \frac{(jC_s)^j}{2} \|f\|_s, \quad \forall j \geq 1, \forall f \in C^1, \forall t \in [t_0, t_1].$$

⁴⁴This is always possible since $p > q + 1$ and $t_0 q \geq 4$ from Definition 3.2.

For the unstable norm, let $\varepsilon < \varepsilon_0$ and let $W^1, W^2 \in \mathcal{W}^s$ with $d_{\mathcal{W}^s}(W^1, W^2) \leq \varepsilon$. For $\ell = 1, 2$, we partition $\mathcal{T}^{-1}W^\ell$ into matched pieces U_k^ℓ and unmatched pieces V_i^ℓ as in §4.3.3, and we find, for any $\psi_\ell \in C^\alpha(W^\ell)$ with $|\psi_\ell|_{C^\alpha(W^\ell)} \leq 1$ and $d(\psi_1, \psi_2) = 0$,

$$(6.16) \quad \left| \int_{W^1} \mathcal{M}_t^{(j)}(f) \psi_1 - \int_{W^2} \mathcal{M}_t^{(j)}(f) \psi_2 \right| \\ \leq \sum_k \left| \int_{U_k^1} f(\psi_1 \circ \mathcal{T}) |J^s \mathcal{T}|^t (\log |J^s \mathcal{T}|)^j - \int_{U_k^2} f(\psi_2 \circ \mathcal{T}) |J^s \mathcal{T}|^t (\log |J^s \mathcal{T}|)^j \right| \\ + \sum_{\ell, i} \left| \int_{V_i^\ell} f(\psi_\ell \circ \mathcal{T}) |J^s \mathcal{T}|^t (\log |J^s \mathcal{T}|)^j \right|.$$

For the unmatched pieces, adapting (4.19), by using Lemma 6.2 combined with Lemma 2.1 and (6.14), we find, for $\ell = 1, 2$ (choosing again $v < t_0/2 - 1/p$ so that $t - v - 1/p > t_0/2$),

$$(6.17) \quad \sum_{\ell, i} \left| \int_{V_i^\ell} f(\psi_\ell \circ \mathcal{T}) |J^s \mathcal{T}|^t (\log |J^s \mathcal{T}|)^j \right| \\ \leq \|f\|_s C_1^{-1} j C'_d(j C_v)^j \sum_{\ell, i} |\mathcal{T} V_i^\ell|^{1/p} \| |J^s \mathcal{T}|^{t-v-1/p} \|_{C^0(V_i^\ell)} \\ \leq \|f\|_s 4C_2[0] C_1^{-1} j C'_d(j C_v)^j \varepsilon^{1/p} Q_{n_0}(t - v - 1/p), \quad \forall t \in [t_0, t_1], \quad \forall j \geq 1,$$

using Lemma 3.4 with $\varsigma = 0$. Next, we consider matched pieces. Recalling (4.17), we define

$$(\widetilde{\log J^s \mathcal{T}})^j(x) := (\log J^s \mathcal{T})^j \circ G_{U_k^2} \circ G_{U_k^1}^{-1}(x), \quad \forall x \in U_k^1, \quad \forall j \geq 1.$$

Now, using $\widetilde{\psi}_2$ and $\widetilde{J^s \mathcal{T}} = \widetilde{J^s} T^{n_0}$ as defined above (4.22), and injecting Lemma 6.2 in the proof of Sublemma 4.8(b) gives, for v as above,

$$(6.18) \quad |(\psi_1 \circ \mathcal{T})(\log J^s \mathcal{T})^j |J^s \mathcal{T}|^t - \widetilde{\psi}_2(\widetilde{\log J^s \mathcal{T}})^j |\widetilde{J^s \mathcal{T}}|^t|_{C^\beta(U_k^1)} \\ \leq C(j C_v)^j j C'_d 2^t |J^s T^{n_0}|_{C^0(U_k^1)}^{t-v} \varepsilon^{\alpha-\beta}, \quad \forall k, \quad \forall j, \quad \forall t \in [t_0, t_1].$$

Then we split

$$(6.19) \quad \left| \int_{U_k^1} f(\psi_1 \circ \mathcal{T}) (\log J^s \mathcal{T})^j |J^s \mathcal{T}|^t - \int_{U_k^2} f(\psi_2 \circ \mathcal{T}) (\log J^s \mathcal{T})^j |J^s \mathcal{T}|^t \right| \\ \leq \left| \int_{U_k^1} f \left((\psi_1 \circ \mathcal{T}) (\log J^s \mathcal{T})^j |J^s \mathcal{T}|^t - \widetilde{\psi}_2(\widetilde{\log J^s \mathcal{T}})^j |\widetilde{J^s \mathcal{T}}|^t \right) \right| \\ (6.20) \quad + \left| \int_{U_k^1} f \widetilde{\psi}_2(\widetilde{\log J^s \mathcal{T}})^j |\widetilde{J^s \mathcal{T}}|^t - \int_{U_k^2} f(\psi_2 \circ \mathcal{T}) (\log J^s \mathcal{T})^j |J^s \mathcal{T}|^t \right|.$$

We estimate (6.19) for all $t \in [t_0, t_1]$ and $j \geq 1$ using (6.18),

$$\left| \int_{U_k^1} f \left((\psi_1 \circ \mathcal{T}) (\log J^s \mathcal{T})^j |J^s \mathcal{T}|^t - \widetilde{\psi}_2(\widetilde{\log J^s \mathcal{T}})^j |\widetilde{J^s \mathcal{T}}|^t \right) \right| \leq \|f\|_s \delta_0^{1/p} j C'_d(C_v j)^j 2^t |J^s \mathcal{T}|_{C^0(I_k)}^{t-v} \varepsilon^{\alpha-\beta}.$$

Then, noting that $d(\psi_2 \circ \mathcal{T}) (\log J^s \mathcal{T})^j |J^s \mathcal{T}|^t, \widetilde{\psi}_2(\widetilde{\log J^s \mathcal{T}})^j |\widetilde{J^s \mathcal{T}}|^t = 0$ by definition, and that the C^α norms of both test functions are bounded by $C(j C_v)^j j C'_d |J^s \mathcal{T}|_{C^0(I_k)}^{t-v}$, using Lemma 6.2, we estimate (6.20) for all $f \in C^1$ and $t \in [t_0, t_1]$ as follows

$$\left| \int_{U_k^1} f \widetilde{\psi}_2(\widetilde{\log J^s \mathcal{T}})^j |\widetilde{J^s \mathcal{T}}|^t - \int_{U_k^2} f(\psi_2 \circ \mathcal{T}) (\log J^s \mathcal{T})^j |J^s \mathcal{T}|^t \right| \leq \|f\|_u d_{\mathcal{W}^s}(U_k^1, U_k^2)^\gamma j C'_d(j C_v)^j C |J^s \mathcal{T}|_{C^0(I_k)}^{t-v} \\ \leq C'(j C_v)^j j C'_d \|f\|_u n_0^\gamma \Lambda^{-n_0 \gamma} \varepsilon^\gamma |J^s \mathcal{T}|_{C^0(I_k)}^{t-v},$$

where we used Lemma 4.8(a) in the second inequality. Putting these estimates into (6.19), combining with (6.17) in (6.16), and summing over k gives, for all $t \in [t_0, t_1]$ and $j \geq 1$,

$$\begin{aligned} & \left| \int_{W_1} \mathcal{M}_t^{(j)} f \psi_1 - \int_{W_2} \mathcal{M}_t^{(j)} f \psi_2 \right| \\ & \leq j(j\bar{C})^j (\|f\|_u n_0^\gamma \Lambda^{-n_0 \gamma} \varepsilon^\gamma Q_{n_0}(t-v) + \|f\|_s (\varepsilon^{1/p} Q_{n_0}(t-1/p-v) + \varepsilon^{\alpha-\beta} Q_{n_0}(t-v))). \end{aligned}$$

Finally, since $\alpha - \beta \leq \gamma$ and $1/p \leq \gamma$, while n_0 is fixed, we have found $C_u < \infty$ such that

$$\|\mathcal{M}_t^{(j)} f\|_u \leq j \frac{(jC_u)^j}{2} (\|f\|_s + \|f\|_u), \quad \forall f \in C^1, \quad \forall t \in [t_0, t_1], \quad \forall j \geq 1.$$

With (6.15), taking $C = \max\{C_s, C_u\}$, this concludes the proof of Proposition 6.3. \square

Proof of Theorem 1.2. Since $\exp(n_0 P(t)) > 0$ is a simple isolated eigenvalue of $\mathcal{L}_t^{n_0}$, analyticity of $\exp(n_0 P(t))$ is an immediate consequence of Proposition 6.1 and [Ka, VII, Theorem 1.8, II.1.8]. Since $\inf_{[t_0, t_1]} \exp(n_0 P(t)) > 0$, the function $P(t)$ is also analytic. The formulas

$$n_0 P'(t) \exp(n_0 P(t)), \quad n_0 P''(t) \exp(n_0 P(t)) + n_0^2 P'(t)^2 \exp(n_0 P(t))$$

can be read off [Ka, II.2.2, (2.1), (2.33) p.79] (taking $m = 1$ there). It is then easy to extract the claimed formula (1.11) for $P'(t)$. In order to⁴⁵ establish (1.12) for $P''(t)$, use (1.11), and note that, recalling $\chi_t = P'(t) = \int \log J^s T d\mu_t$,

$$\begin{aligned} & \sum_{k \geq 0} \left[\int (\log |J^s T| \circ T^k) \log |J^s T| d\mu_t - \chi_t^2 \right] \\ & = \langle \log |J^s T| (1 - e^{-P_*(t)} \mathcal{L}_t)^{-1} \left((\log |J^s T| - \chi_t) \nu_t \right), \tilde{\nu}_t \rangle. \end{aligned}$$

If there exists $f \in L^2(\mu_t)$ such that $\log |J^s T| - \chi_t = f - f \circ T$ then it is easy to see that $P''(t) = 0$. For the converse statement, we will use a martingale CLT result à la Gordin (see e.g. Viana [BDV, Theorem E.11]) as in [DRZ]: Let \mathcal{A}_0 be the sigma-algebra generated by the $(\mu_t \text{-mod } 0)$ partition of M into maximal connected, strongly homogeneous local stable manifolds for T (this partition is measurable since it has a countable generator, see e.g. [CM, §5.1]). Then $\mathcal{A}_n = T^{-n} \mathcal{A}_0$, for $n \in \mathbb{Z}$, is a decreasing sequence of sigma algebras. Therefore, if $P''(t) = 0$, to obtain $f \in L^2(\mu_t)$ such that $\log |J^s T| = \chi_t + f - f \circ T$ from Gordin's Theorem ([BDV, Theorem E.11] or [DRZ, Theorem 5.1]), we only need to check the following two conditions:

$$(6.21) \quad \sum_{n=0}^{\infty} \| \log |J^s T| - E((\log |J^s T| - \chi_t) | \mathcal{A}_{-n}) \|_{L^2(\mu_t)} < \infty,$$

$$(6.22) \quad \sum_{n=0}^{\infty} \| E((\log |J^s T| - \chi_t) | \mathcal{A}_n) \|_{L^2(\mu_t)} < \infty.$$

We first discuss (6.21). If $n \geq 0$, then the elements of \mathcal{A}_{-n} are of the form $T^n(V^s(x))$ where $V^s(x)$ is the maximal connected, strongly homogeneous stable manifold of (almost every) x . From Lemma 6.2, the function $\log |J^s T|$ is (Hölder) continuous on $T^n(V^s(x))$ for any $n \geq 1$, so, letting $\mathcal{A}_{-n}(x)$ be the element of \mathcal{A}_{-n} containing x , we have

$$E(\log |J^s T| | \mathcal{A}_{-n})(x) = \log |J^s T|(y),$$

for some $y \in \mathcal{A}_{-n}(x)$. Thus (see the proof of [DRZ, (5.3)]), (6.7) with $\alpha = 1/(q+1)$ gives

$$\begin{aligned} & \| \log |J^s T| - E((\log |J^s T| - \chi_t) | \mathcal{A}_{-n}) \|_{L^2(\mu_t)} \\ & \leq \| \log |J^s T| - E((\log |J^s T| - \chi_t) | \mathcal{A}_{-n}) \|_{L^\infty(\mu_t)} \leq C \Lambda^{-n/(q+1)}, \quad \forall n \geq 1, \quad \forall t \in [t_0, t_1]. \end{aligned}$$

⁴⁵Formulas (1.11)–(1.12) are classical in smooth hyperbolic dynamics, see [Ru, Chap. 5, ex. 5b] for $P''(t)$.

(The length of any element $T^n(V^s(x))$ in \mathcal{A}_{-n} is bounded by $C_0\Lambda^{-n}$.) This proves (6.21).

To establish (6.22), we also adapt the argument in [DRZ], starting from

$$\begin{aligned} & \sum_{n=0}^{\infty} \|E((\log |J^s T| - \chi_t)|\mathcal{A}_n)\|_{L^2(\mu_t)} \\ &= \sum_{n=0}^{\infty} \sup \left\{ \int (\log |J^s T| - \chi_t) \cdot (\psi \circ T^n) d\mu_t \mid \psi \in L^2(\mathcal{A}_0, \mu_t) \text{ with } \|\psi\|_{L^2(\mu_t)=1} \right\}. \end{aligned}$$

The key new ingredient is the fact that,⁴⁶ since $\nu_t = e^{-n_0 P(t)} \mathcal{L}_t^{n_0}(\nu_t)$, with $\nu_t \in \mathcal{B}$, we get from Proposition 6.3 and Remark 6.4 that

$$(6.23) \quad (\log |J^s T^{n_0}| \circ T^{-n_0})\nu_t = e^{-n_0 P(t)} \mathcal{M}_t^{(1)}(\nu_t) \in \mathcal{B} \subset \mathcal{B}_w.$$

By definition, any \mathcal{A}_0 -measurable function ψ is constant on each curve in \mathcal{A}_0 . If in addition ψ is bounded, then for any $k \geq 0$, $\psi \circ T^k \in C^\alpha(\mathcal{W}_{\mathbb{H}}^s)$ and $|\psi \circ T^k|_{C^\alpha(\mathcal{W}_{\mathbb{H}}^s)} = |\psi|_{C^0(\mathcal{W}_{\mathbb{H}}^s)} =: |\psi|_\infty$. Thus by Lemma 4.14 and (4.48),

$$(6.24) \quad \psi \circ T^k \tilde{\nu}_t \in \mathcal{B}_w^* \text{ and } |\langle f, \psi \circ T^k \tilde{\nu}_t \rangle| \leq C' |f|_w |\psi|_\infty, \forall f \in \mathcal{B}_w.$$

Then, recalling that $\mu_t(f) = \langle f \nu_t, \tilde{\nu}_t \rangle / \langle \nu_t, \tilde{\nu}_t \rangle$ for suitable f , following [DRZ], we write for $n \geq n_0$, and any bounded \mathcal{A}_0 -measurable function ψ ,

$$\begin{aligned} \int (\log |J^s T| - \chi_t) \cdot (\psi \circ T^n) d\mu_t &= \frac{1}{n_0} \int (\log |J^s T^{n_0}| \circ T^{-n_0} - n_0 \chi_t) \cdot (\psi \circ T^{n-n_0}) d\mu_t \\ &= \frac{1}{n_0} \langle (\log |J^s T^{n_0}| \circ T^{-n_0} - n_0 \chi_t) \nu_t, (\psi \circ T^{n-n_0}) \tilde{\nu}_t \rangle / \langle \nu_t, \tilde{\nu}_t \rangle \\ (6.25) \quad &= \frac{1}{n_0} \langle e^{(n-n_0)P(t)} \mathcal{L}_t^{n-n_0} ((\log |J^s T^{n_0}| \circ T^{-n_0} - n_0 \chi_t) \nu_t), \psi \tilde{\nu}_t \rangle / \langle \nu_t, \tilde{\nu}_t \rangle. \end{aligned}$$

(The expressions in the first line are well defined and coincide because $(\log |J^s T| - \chi_t) \in L^1(d\mu_t)$ and ψ is bounded. The expression in the second line is well defined by (6.23) and (6.24). Therefore, the second equality holds due to the definition of μ_t in Proposition 4.16(b). The last equality is clear.) Clearly $\tilde{\nu}_t(\nu_t(\log |J^s T^{n_0}| \circ T^{-n_0})) = n_0 \chi_t$, so that Corollary 1.5 and (6.23) give constants $\rho < 1$ and $C'_1, C'_2 < \infty$ such that for all $n \geq n_0$ and all $t \in [t_0, t_1]$

$$\begin{aligned} & |e^{(n-n_0)P(t)} \mathcal{L}_t^{n-n_0} (\nu_t(\log |J^s T^{n_0}| \circ T^{-n_0} - n_0 \chi_t))|_w \\ & \leq \|e^{(n-n_0)P(t)} \mathcal{L}_t^{n-n_0} (\nu_t(\log |J^s T^{n_0}| \circ T^{-n_0} - n_0 \chi_t))\|_{\mathcal{B}} \\ (6.26) \quad & \leq C'_1 \rho^{n-n_0} \|\nu_t(\log |J^s T^{n_0}| \circ T^{-n_0})\|_{\mathcal{B}} \leq C'_2 \rho^n. \end{aligned}$$

Next, (6.25) together with the bounds (6.24) and (6.26), gives $C < \infty$ such that, for any bounded function ψ which is \mathcal{A}_0 -measurable,

$$(6.27) \quad \left| \int (\log |J^s T| - \chi_t) \cdot (\psi \circ T^n) d\mu_t \right| \leq C \rho^n |\psi|_{L^\infty(M)}, \quad \forall n \geq 1, \quad \forall t \in [t_0, t_1].$$

Then the proof of Lemma 5.1(c) (the T -adapted property of μ_t) not only implies that $\log |J^s T|(x) \leq C \log(d(x, \mathcal{S}_1))$ is in $L^1(d\mu_t)$ but also in $L^\ell(d\mu_t)$ for all $\ell \geq 1$ (use that $\sum_{j \geq 1} (\log(j+1))^\ell j^{-p'/(2p)} < \infty$ for all ℓ if $p' > 2p$). It follows that [DRZ, Lemma 5.2] holds for $\bar{s} = (\log |J^s T|) - \chi_t$, bootstrapping the L^∞ bound (6.27) to the required L^2 control (6.22). The only change required in the proof (since the observable \bar{s} in [DRZ, Lemma 5.2] is bounded while ours is not), is to replace the second term on the right-hand side of [DRZ, eq. (5.7)] by the Hölder bound $(\int |\bar{s}|^3 d\mu_t)^{1/3} (\int |\psi - \psi_L|^{3/2} d\mu_t)^{2/3}$ (where $\psi_L(x) = \psi(x)$ if $|\psi(x)| \leq L$ and $\psi_L(x) = 0$ otherwise). Then using the fact that $\psi \in L^2(\mu_t)$,

$$\int |\psi - \psi_L|^{3/2} d\mu_t = \int \mathbf{1}_{|\psi| > L} \cdot |\psi|^{3/2} d\mu_t \leq L^{-1/2} |\psi|_{L^2}^2,$$

⁴⁶Since $f \equiv 1 \in \mathcal{B}$, Remark 6.4 also implies $\log |J^s T^{n_0}| \circ T^{-n_0} = \log |J^s T| \circ T^{-1} = \mathcal{M}_1^{(1)}(1) \in \mathcal{B}$.

using the Markov bound $\mu(|\psi| > L) \leq L^{-2}|\psi|_{L^2}^2$. Setting $L = \rho^{-3n/4}$ instead of $L = \rho^{-n/2}$ in [DRZ, eq. (5.8)] completes the proof of [DRZ, Lemma 5.2] with modified rate $\rho^{n/4}$ for our observable \bar{s} . This verifies (6.22) and concludes the proof that $\log |J^s T|$ is cohomologous in $L^2(\mu_t)$ to the constant $\chi_t < 0$ if $P''(t) = 0$.

Finally, $P''(t) \geq 0$ implies that $P'(t)$ is increasing so that $\int \log J^u T d\mu_t = -P'(t)$ is decreasing, while $h_{\mu_t} = P(t) - tP'(t)$ is decreasing since $P(t)$ and $-tP'(t)$ are decreasing. \square

Proof of Corollary 1.4. The bounds on $\mathcal{M}_t^{(j)}$ from Proposition 6.3 apply also to the dual operator $(\mathcal{M}_t^{(j)})^*$ acting on \mathcal{B}^* . Thus, by Proposition 6.1, $(\mathcal{L}_t^{n_0})^*$ has the analogous representation as a power series and is analytic. It follows that both ν_t and $\tilde{\nu}_t$ are analytic for $t \in (0, t_*)$.

By [DZ2, Lemma 5.3], if $\psi \in C^\alpha(M)$, then $\psi\nu_t \in \mathcal{B}$ and $\|\psi\nu_t\|_{\mathcal{B}} \leq \bar{C}\|\nu_t\|_{\mathcal{B}}|\psi|_{C^\alpha(M)}$, for some $\bar{C} > 0$ independent of ψ and ν_t . So for each $\psi \in C^\alpha(M)$, $\mu_t(\psi)$ is analytic on $(0, t_*)$.

To prove continuity of μ_t on C^0 functions, we shall use the following bound. Fix $[t_0, t_1] \subset (0, t_*)$, and define

$$C_\nu = \sup_{t \in [t_0, t_1]} \max\{\|\nu_t\|_{\mathcal{B}}, \|\tilde{\nu}_t\|_{\mathcal{B}^*}\}, \quad C'_\nu = \sup_{t \in [t_0, t_1]} \max\{\|\nu'_t\|_{\mathcal{B}}, \|\tilde{\nu}'_t\|_{\mathcal{B}^*}\},$$

where ν'_t and $\tilde{\nu}'_t$ denote the derivatives of ν_t and $\tilde{\nu}_t$ as operators on \mathcal{B} and \mathcal{B}^* , respectively. Then for $\psi \in C^\alpha(M)$,

$$\begin{aligned} \mu_t(\psi) - \mu_s(\psi) &= \langle \psi\nu_t, \tilde{\nu}_t - \tilde{\nu}_s \rangle + \langle \psi(\nu_t - \nu_s), \tilde{\nu}_s \rangle \\ (6.28) \quad &\leq \bar{C}|\psi|_{C^\alpha(M)}\|\nu_t\|_{\mathcal{B}}\|\tilde{\nu}_t - \tilde{\nu}_s\|_{\mathcal{B}^*} + \bar{C}|\psi|_{C^\alpha(M)}\|\nu_t - \nu_s\|_{\mathcal{B}}\|\tilde{\nu}_s\|_{\mathcal{B}^*} \\ &\leq C_\star|t - s||\psi|_{C^\alpha(M)}, \end{aligned}$$

where $C_\star = 2\bar{C}C_\nu C'_\nu$.

Next, let $\rho : \mathbb{R}^2 \rightarrow [0, \infty)$ denote a C^∞ bump function, supported on the unit disk and with $\int \rho dm = 1$. For $\varepsilon > 0$, let $\rho_\varepsilon(z) = \varepsilon^{-2}\rho(z/\varepsilon)$, and define M^ε to be the extension of the phase space M by a strip of width ε along each component of S_0 .

For $\psi \in C^0(M)$, extend ψ to M^ε by making ψ constant on vertical lines outside M . Then $\psi \in C^0(M^\varepsilon)$. For $x \in M$, define the convolution

$$\psi_\varepsilon(x) = \int \rho_\varepsilon(x - y)\psi(y)dm(y).$$

Note that $|\psi_\varepsilon|_{C^\alpha} \leq C\varepsilon^{-\alpha}$. Since M^ε is compact, ψ is uniformly continuous, so there exists a decreasing function $\eta : \mathbb{R}^+ \rightarrow \mathbb{R}^+$, $\lim_{\varepsilon \rightarrow 0} \eta(\varepsilon) = 0$, such that for all $\varepsilon > 0$, $|\psi_\varepsilon - \psi|_{C^0(M)} \leq \eta(\varepsilon)$.

Now fix $t \in (t_0, t_1)$. For $\delta > 0$, choose $\varepsilon > 0$ such that $\eta(\varepsilon) < \delta/3$ and choose $s_0 \in (t_0, t_1)$ such that $C_\star|s_0 - t|C\varepsilon^{-\alpha} \leq \delta/3$. Then using (6.28), if $|s - t| < |s_0 - t|$,

$$|\mu_t(\psi) - \mu_s(\psi)| \leq |\mu_t(\psi - \psi_\varepsilon)| + |\mu_t(\psi_\varepsilon) - \mu_s(\psi_\varepsilon)| + |\mu_s(\psi_\varepsilon - \psi)| \leq 2\eta(\varepsilon) + C_\star|s - t||\psi_\varepsilon|_{C^\alpha} < \delta.$$

Thus $\mu_s(\psi) \rightarrow \mu_t(\psi)$ as $s \rightarrow t$. \square

Proof of Corollary 1.5. For a compact subinterval I of $(0, t_*)$ the bound $\sigma(t)$ in Proposition 4.12 satisfies $\sigma_I := \sup_{t \in I} \sigma(t) < 1$. If each \mathcal{L}_t , for $t \in I$, has its spectrum on \mathcal{B} contained in $e^{P_\star(t)} \cup \{|z| < \sigma_I \cdot e^{P_\star(t)}\}$, the corollary follows. Otherwise, use Proposition 6.1 and continuity [Ka, §IV.3.5] of any (finite) set of eigenvalues of finite multiplicities of bounded operators. \square

Proof of Corollary 1.6. The corollary follows from (6.21) and (6.22), using Gordin's Theorem ([BDV, Theorem E.11] or [DRZ, Theorem 5.1]). \square

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