Anisotropic Banach Spaces and Thermodynamic Formalism for Dispersing Billiard Maps
Lecture 6: Measure of Maximal Entropy, Part II

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Goal for today: Introduce Banach spaces adapted to the potential for the measure of maximal entropy. Due to these modifications, we are not able to prove a spectral gap for the transfer operator, yet we will gain enough control to prove existence and uniqueness of the measure of maximal entropy.

Recall from Lecture 5,

$$\mathcal{L}_0 f(x) = \frac{f(T^{-1}x)}{J^s T(T^{-1}x)} \, ,$$

is the transfer operator corresponding to the potential for the measure of maximal entropy (MME). Want to construct the MME from the product of left and right maximal eigenvectors of $\mathcal{L}_0$.

Reference: V. Baladi and M. Demers, *On the measure of maximal entropy for finite horizon Sinai billiard maps*, J. Amer. Math. Soc. **33** (2020), 381–449.

Map and Transfer Operator

- Billiard Map $T(r, \varphi) = (r_1, \varphi_1)$ is the collision map associated to a finite horizon Lorentz gas
  - Billiard table $Q = \mathbb{T}^2 \setminus \cup_i B_i$; scatterers B_i.
  - Boundaries of scatterers are $\mathcal{C}^3$ and have strictly positive curvature.
- Assume **Finite Horizon** condition: there is no trajectory making only tangential collisions $\implies$ an upper bound on the free flight time between collisions.
- Transfer operator for geometric potential with $t = 0$,

$$\mathcal{L}_0 f = \frac{f \circ T^{-1}}{J^s T \circ T^{-1}}$$

  - $J^s T \approx \cos \varphi$ so the potential is unbounded
  - $J^s T$ is not continuous on any open set

Stable Curves and Definition of Topological Entropy

- $\widehat{\mathcal{W}}^s$ set of cone-stable curves with bounded curvature and length $\le \delta_0$. $\mathcal{W}^s \subset \widehat{\mathcal{W}}^s$ set of local stable manifolds.
- For $n \in \mathbb{Z}$, define $\mathcal{S}_n = \cup_{i=0}^{-n} T^i \mathcal{S}_0$,
- $\mathcal{M}_0^n =$ connected components of $M \setminus \mathcal{S}_n$ for $n \ge 1$
- $h_* := \lim_{n\to\infty} \frac{1}{n} \log \#\mathcal{M}_0^n$

In Lecture 5, we established:

$\exists \delta_1, c_1 > 0$ such that for any $W \in \widehat{\mathcal{W}}^s$ with $|W| \ge \delta_1/3$,

$$\#\mathcal{G}_n(W) \ge c_1 \#\mathcal{M}_0^n \,, \qquad \text{for all } n \ge 1.$$

and exact exponential growth of $\#\mathcal{M}_0^n$: $\exists C_2 \ge 1$ s.t.

$$e^{nh_*} \le \#\mathcal{M}_0^n \le C_2 e^{nh_*} \,, \qquad \text{for all } n \ge 1.$$

New Assumption: 'Sparse Recurrence' to Singularities

All results today will use following additional assumption on T.

- Fix $n_0 \in \mathbb{N}$ and an angle φ_0 close to $\pi/2$.
- Let $s_0 \in (0,1)$ be the smallest number such that any orbit of length n_0 has at most $s_0 n_0$ collisions with $|\varphi| \geq \varphi_0$.

Finite horizon guarantees that we can always choose n_0 and φ_0 so that $s_0 < 1$. (Indeed, no triple tangencies implies that $s_0 \leq \frac{2}{3}$.)

Assumption: $h_* > s_0 \log 2$

Fact: If W is a local stable manifold, then $|T^{-1}W| \leq C|W|^{1/2}$.

Our assumption ensures that the growth due to tangential collisions does not exceed the exponential rate of growth given by h_*.

Toy Calculation in Previous Norms

Recall that the strong stable norm for $t > 0$ was

$$\|f\|_s = \sup_{W \in \mathcal{W}^s} \sup_{|\psi|_{C^\alpha(W)} \le |W|^{-1/p}} \int_W f\psi \, dm_W \, ,$$

and the weight $|W|^{-1/p}$ was needed to control the contribution from unmatched pieces in the strong unstable norm estimate.

But now we have no Jacobian or homogeneity strips. So suppose $W \in \mathcal{W}^s$ s.t. $T^{-1}W$ has a single component with $|T^{-1}W| \approx |W|^{1/2}$. Then if $\psi = |W|^{-1/p}$,

$$\int_W \mathcal{L}_0 f \, \psi = |W|^{-1/p} \int_{T^{-1}W} f \le \|f\|_s \frac{|T^{-1}W|^{1/p}}{|W|^{1/p}} \approx \|f\|_s |W|^{-1/2p}$$

and taking sup over $W \in \mathcal{W}^s$ yields ∞. The spectral radius of $\mathcal{L}_0 = \infty$ on such a space, for any $p > 0$.

To avoid this, we use a logarithmic weight instead.

Definition of Norms: Weak Norm

Choose $\alpha, \beta, \varsigma > 0$ and $\gamma > 1$ such that

$$\beta < \alpha \le 1/3, \quad 2^{s_0 \gamma} < e^{h_*}, \quad \varsigma < \gamma \, .$$

Choose n_0 so that

$$\frac{1}{n_0} \log(K n_0 + 1) < h_* - \gamma s_0 \log 2 \, ,$$

where K is from the linear bound on complexity.

Fix the length scale $\delta_0 > 0$ so that any $W \in \mathcal{W}^s$ (with $|W| \le \delta_0$) is cut into at most $K n_0 + 1$ pieces by $\mathcal{S}_{-n_0}$.

For $f \in C^1(M)$, define the **weak norm** of f by

$$|f|_w = \sup_{W \in \mathcal{W}^s} \sup_{\substack{\psi \in \mathcal{C}^\alpha(W) \\ |\psi|_{\mathcal{C}^\alpha(W)} \le 1}} \int_W f \, \psi \, dm_W \, .$$

Define $\mathcal{B}_w$ to be the completion of $C^1(M)$ in the $| \cdot |_w$ norm.

Definition of Norms: Strong Norm

Define the **strong stable norm** of f by

$$\|f\|_s = \sup_{W \in \mathcal{W}^s} \sup_{\substack{\psi \in \mathcal{C}^\beta(W) \\ |\psi|_{\mathcal{C}^\beta(W)} \le |\log |W||^\gamma}} \int_W f \, \psi \, dm_W$$

Define the **strong unstable norm** of f by

$$\|f\|_u = \sup_{\varepsilon \le \varepsilon_0} \sup_{\substack{W_1, W_2 \in \mathcal{W}^s \\ d(W_1, W_2) \le \varepsilon}} \sup_{\substack{|\psi_i|_{\mathcal{C}^\alpha(W_i)} \le 1 \\ d_0(\psi_1, \psi_2) = 0}} |\log \varepsilon|^\varsigma \left| \int_{W_1} f \psi_1 - \int_{W_2} f \psi_2 \right|$$

The **strong norm** of f is defined to be $\|f\|_{\mathcal{B}} = \|f\|_s + \|f\|_u$,

Define $\mathcal{B}$ to be the completion of $C^1(M)$ in the $\| \cdot \|_{\mathcal{B}}$ norm.

Recall Distances Between Curves and Test Functions

- View $W \in \widehat{\mathcal{W}}^s$ as the graph of a function of the r-coordinate over an interval I_W,

$$W = \{G_W(r) : r \in I_W\} = \{(r, \varphi_W(r)) : r \in I_W\} .$$

- Given $W_1, W_2 \in \widehat{\mathcal{W}}^s$ with functions φ_{W_1}, φ_{W_2}, define

$$d(W_1, W_2) = |I_{W_1} \triangle I_{W_2}| + |\varphi_{W_1} - \varphi_{W_2}|_{C^1(I_{W_1} \cap I_{W_2})} ,$$

if $I_{W_1} \cap I_{W_2} \neq \emptyset$, and $d(W_1, W_2) = \infty$ otherwise.

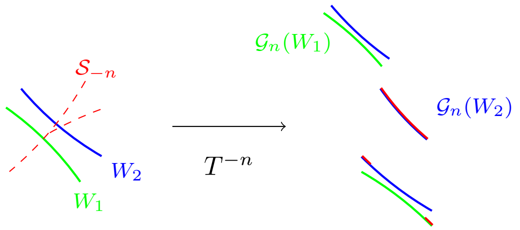
- If $d(W_1, W_2) < \infty$, then for $\psi_1 \in C^0(W_1)$, $\psi_2 \in C^0(W_2)$, define

$$d_0(\psi_1, \psi_2) = |\psi_1 \circ G_{W_1} - \psi_2 \circ G_{W_2}|_{C^0(I_{W_1} \cap I_{W_2})} ,$$

No contraction of $\|\cdot\|_u$

The logarithmic modulus of continuity in the strong unstable norm prevents contraction of $\|\cdot\|_u$.

For strong unstable norm, estimate $\left|\int_{W_1} \mathcal{L}_0^n f\,\psi_1 - \int_{W_2} \mathcal{L}_0^n f\,\psi_2\right|$



- If $d(W^1, W^2) \le \varepsilon$, and if $W_i^1 \in \mathcal{G}_n(W^1)$, $W_i^2 \in \mathcal{G}_n(W_i^2)$ are matched, then $d(W_i^1, W_i^2) \le C\Lambda^{-n}\varepsilon$.
- But the contraction is $\frac{|\log C\Lambda^{-n}\varepsilon|^\varsigma}{|\log \varepsilon|^\varsigma}$, and taking the supremum over $\varepsilon > 0$ yields 1.

Banach Spaces and Inequalities

Theorem ([Baldi, D. '20])

- We have a sequence of inclusions,

$$\mathcal{C}^1(M) \subset \mathcal{B} \subset \mathcal{B}_w \subset (\mathcal{C}^\alpha(M))^*.$$

- The embedding of the unit ball of $\mathcal{B}$ in $\mathcal{B}_w$ is compact.
- Assume $h_* > s_0 \log 2$. There exists $C > 0$ such that for all $f \in \mathcal{B}$, $n \geq 0$,

$$\begin{aligned} |\mathcal{L}^n f|_w &\leq C|f|_w \# \mathcal{M}_0^n \\ \|\mathcal{L}^n f\|_s &\leq C(\sigma^n \|f\|_s + |f|_w) \# \mathcal{M}_0^n \,, \quad \text{for some } \sigma < 1 \\ \|\mathcal{L}^n f\|_u &\leq C(\|f\|_u + \|f\|_s) \# \mathcal{M}_0^n \end{aligned}$$

The inequalities above are not true Lasota-Yorke inequalities due to lack of contraction in the strong unstable norm.

Bounds on the Spectral Radius of $\mathcal{L}_0$

Although we do not prove quasi-compactness of $\mathcal{L}_0$ on $\mathcal{B}$, we do have good control of $\|\mathcal{L}_0^n\|_{\mathcal{B}}$.

- Our upper bound $\#\mathcal{M}_0^n \le C_2 e^{nh_*}$ plus our 'Lasota-Yorke' inequalities imply that $\|\mathcal{L}_0^n\|_{\mathcal{B}} \le Ce^{nh_*}$, for all $n \ge 1$.

- Our lower bound on $\#\mathcal{G}_n(W)$ implies that

$$\|\mathcal{L}_0^n 1\|_s \ge |\mathcal{L}_0^n 1|_w \ge \int_W \mathcal{L}_0^n 1 = \sum_{W_i \in \mathcal{G}_n^{\delta_1}(W)} |W_i|$$

$$\ge \frac{\delta_1}{3}\frac{3}{4}\#\mathcal{G}_n^{\delta_1}(W) \ge Ce^{nh_*} .$$

This implies that the sequence $e^{-nh_*}\mathcal{L}_0^n 1$ is uniformly bounded away from 0 and ∞ in the strong norm. We use this fact to construct an eigenmeasure for $\mathcal{L}_0$ with eigenvalue e^{h_*}.

Construction of μ_*

- The sequence

$$\nu_n = \frac{1}{n}\sum_{k=0}^{n-1} e^{-kh_*}\mathcal{L}_0^k 1, \text{ is uniformly bounded in } \mathcal{B}.$$

  By compactness, a subsequence converges in $\mathcal{B}_w$.
  Let $\nu \in \mathcal{B}_w$ be a limit point of ν_n. ν is a measure.

- Similarly, let $\tilde{\nu} \in (B_w)^*$ be a limit point of the sequence

$$\frac{1}{n}\sum_{k=0}^{n-1} e^{-kh_*}(\mathcal{L}_0^*)^k(d\mu_{\mathsf{SRB}}).$$

- Define $\quad \mu_*(\psi) = \dfrac{\tilde{\nu}(\psi\nu)}{\tilde{\nu}(\nu)}, \quad$ for $\psi \in C^1(M)$.

Since $\mathcal{L}_0\nu = e^{h_*}\nu$ and $\mathcal{L}_0^*\tilde{\nu} = e^{h_*}\tilde{\nu}$, we have $\mu_*(\psi \circ T) = \mu_*(\psi)$, i.e. μ_* is an invariant measure for T.

Hyperbolicity of μ_*

Key Fact: Although $\nu \in \mathcal{B}_w$, it follows from the convergence of ν_n to ν in the $|\cdot|_w$ norm that $\|\nu\|_{\mathcal{B}} < \infty$.

This implies estimates of the form:

- For any $k \in \mathbb{Z}$, $\exists C_k > 0$ s.t.

  $$\nu(\mathcal{N}_\varepsilon(\mathcal{S}_k)) \le C_k(-\log \varepsilon)^{-\gamma}, \qquad \mu_*(\mathcal{N}_\varepsilon(\mathcal{S}_k)) \le C_k(-\log \varepsilon)^{-\gamma}.$$

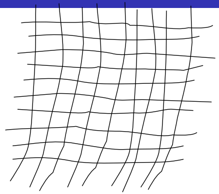
  $\mathcal{N}_\varepsilon(\mathcal{S}_k) = \varepsilon$-neighborhood of $\mathcal{S}_k$ in M, $\gamma > 1$.

- $\displaystyle\int_M -\log d(x, \mathcal{S}_{\pm 1})\, d\mu_*(x) < \infty$ (μ_* is T-adapted).

- μ_*-a.e. $x \in M$ has a stable and unstable manifold of positive length. The same is true with respect to ν.

Ergodicity of μ_*

Since μ_* is hyperbolic, we cover a full measure set of M with Cantor rectangles, and study the properties of μ_* on each rectangle.



A Cantor Rectangle R

Lemma (Absolute continuity of holonomy)

On each Cantor rectangle R, the holonomy map sliding along unstable manifolds in R is absolutely continuous with respect to the conditional measures of μ_* on stable manifolds.

That $\|\nu\|_{\mathcal{B}} < \infty$ is crucial to the proof of the lemma.

Consequences:

- Each Cantor rectangle R belongs to one ergodic component.
- Since T is topologically mixing, we can force images of rectangles to overlap $\implies (T^n, \mu_*)$ is ergodic for all n.

Mixing and Bernoulli Property of μ_*

- The local product structure of the Cantor rectangles, together with a global argument showing that a full measure set of points on each component of M can be connected by a network of stable/unstable manifolds, enables us to prove that (T, μ_*) is K-mixing, following techniques of [Pesin '77, '92].

- K-mixing + hyperbolicity + absolute continuity of μ_* + bounds on $\mu_*(\mathcal{N}_\varepsilon(\mathcal{S}_{\pm 1}))$
  $\implies$ the partition $\mathcal{M}_{-1}^{1}$ is **very weakly Bernoulli**, following the technique of [Chernov, Haskell '96].

  Since $\bigvee_{n=-\infty}^{\infty} T^{-n}(\mathcal{M}_{-1}^{1})$ generates the full σ-algebra for T, this implies by [Ornstein, Weiss '73] that (T, μ_*) is Bernoulli.

Entropy of μ_*

Define $B(x, n, \varepsilon) = \{y \in M : d(T^{-i}x, T^{-i}y) \le \varepsilon, \forall i \in [0, n]\}$.

Proposition (Measure of Bowen Balls)

There exists $C > 0$ s.t. for all $x \in M$ and $n \ge 1$,

$$\mu_*(B(x, n, \varepsilon)) \le Ce^{-nh_*}.$$

- [Brin, Katok '81] $\implies$ for μ_*-a.e. $x \in M$,

$$\lim_{\varepsilon \to 0} \limsup_{n \to \infty} -\frac{1}{n} \log \mu_*(B(x, n, \varepsilon)) = h_{\mu_*}(T^{-1}) = h_{\mu_*}(T).$$

- This plus the Proposition implies $h_{\mu_*}(T) \ge h_*$
- But $h_* \ge h_{\mu_*}(T)$ by Theorem 1.
- Conclude: $h_* = h_{\mu_*}(T)$.

Uniqueness of μ_*

The Bowen argument for uniqueness uses

$$\forall \varepsilon > 0, \exists C > 0 \text{ s.t. for } \mu_*\text{-a.e. } x \in M, \;\; \mu_*(B(x,n,\varepsilon)) \ge Ce^{-nh_*}.$$

This **fails for billiards** due to rate of approach to singularity set.
Rather: $\forall \eta > 0$ and μ_*-a.e. $x \in M$,

$$\exists C = C(\eta, x) > 0 \text{ s.t. } \mu_*(B(x,n,\varepsilon)) \ge Ce^{-n(h_*+\eta)}.$$

This is not sufficient for the Bowen argument.

However: we prove a version of this estimate that 'most' $x \in M$ belong to an element of $\mathcal{M}_0^j$ satisfying good lower bounds 'often.'

Choose $\bar{n} \in \mathbb{N}$ s.t. $(K\bar{n}+1)^{1/\bar{n}} < e^{h_*/4}$

Choose $\delta_2 > 0$ s.t. if $A \in \mathcal{M}_{-k}^n$ satisfies

$$\max\{\text{diam}^u(A), \text{diam}^s(A)\} \le \delta_2 \,,$$

then $A \setminus \mathcal{S}_{\pm\bar{n}}$ consist of at most $K\bar{n}+1$ connected components.

Nonuniform Lower Bounds

$$Sh_0^{2n} := \{A \in \mathcal{M}_0^{2n} : \forall j, 0 \le j \le n/2, \quad T^j A \subset E \in \mathcal{M}_0^{2n-j} \text{ s.t. } \text{diam}^s(E) < \delta_2\}$$

Similar definition for Sh^0_{-2n} with $\text{diam}^u(E)$ replacing $\text{diam}^s(E)$.

Lemma

Let $B_{2n} = \{A \in \mathcal{M}_0^{2n} :$ *either* $A \in Sh_0^{2n}$ *or* $T^{2n} A \in Sh^0_{-2n}\}$
There exists $C > 0$ *s.t. for all* $n \ge 1$, $\# B_{2n} \le C e^{7nh_*/4}$.

Lemma

$\forall k \ge 1$, *if* $E \in \mathcal{M}_0^k$ *with* $\text{diam}^s(E) \ge \delta_2$ *and* $\text{diam}^u(T^k E) \ge \delta_2$

then $\quad \mu_*(E) \ge C_{\delta_2} e^{-kh_*}, \quad$ *for some* $C_{\delta_2} > 0$.

If $A \in G_{2n} = \mathcal{M}_0^{2n} \setminus B_{2n}$, then $\exists j \le n$ s.t. $T^j A \subset E \in \mathcal{M}_0^{2n-j}$ and E satisfies second lemma.

Together with a time shift to group elements of $\mathcal{M}_0^{2n}$ according to $\mathcal{M}_0^{2n-j}$, this is sufficient to adapt the Bowen argument.

Variational Principle and Measure of Maximal Entropy

Theorem ([Baladi, D. '20])

Let T be the billiard map corresponding to a finite horizon periodic Lorentz gas. Assume $h_* > s_0 \log 2$. Then,

$$h_* = \lim_{n\to\infty} \frac{1}{n} \log \#\mathcal{M}_0^n = \sup_\mu h_\mu(T).$$

Moreover, there exists a unique T-invariant measure μ_* such that

- $h_{\mu_*}(T) = h_*$
- $h_* = P(0) = \lim_{t\downarrow 0} P(t) = \lim_{t\downarrow 0} P_*(t)$
- $h_* = h_{\mathsf{top}}(T, M')$
- (T, μ_*) is Bernoulli and positive on open sets
- $\int -\log d(x, \mathcal{S}_{\pm 1})\, d\mu_*(x) < \infty$

Last item implies that μ_* is T-adapted. By [Lima, Matheus '18], Buzzi '20], $\exists C > 0$ such that $P_n(T) \geq Ce^{nh_*}$, for n large.

Open Questions

- Can one establish a rate of mixing for μ_*?
- Other limit theorems and properties of μ_*? e.g. Central Limit Theorem
- Can one find a finite horizon Sinai billiard table with $h_* \leq s_0 \log 2$?
- If so, does an MME exist and is it T-adapted?
- For the geometric potentials $-t \log J^u T$, what happens for $t > t_*$? Is $P(t)$ analytic for all $t > 0$ or is there a phase transition at some $t_\star > 1$? If so, how does $t_\star$ depend on the table?