Anisotropic Banach Spaces and Thermodynamic Formalism for Dispersing Billiard Maps
Lecture 5: Measure of Maximal Entropy, Part I

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Lecture 5: Measure of Maximal Entropy, Part I

Goal for today: Describe changes necessary to study the case $t = 0$ corresponding to the measure of maximal entropy; formulate definition of topological entropy for billiard. Present initial results regarding growth lemmas, topological and metric entropies.

Lecture 6: Introduce Banach spaces adapted to the potential in the case $t = 0$. We are not able to prove a spectral gap for the transfer operator, yet we have enough control to prove existence and uniqueness of the measure of maximal entropy.

Reference: V. Baladi and M. Demers, *On the measure of maximal entropy for finite horizon Sinai billiard maps*, J. Amer. Math. Soc. **33** (2020), 381–449.

Map and Transfer Operator

- Billiard Map $T(r, \varphi) = (r_1, \varphi_1)$ is the collision map associated to a finite horizon Lorentz gas
  - Billiard table $Q = \mathbb{T}^2 \setminus \cup_i B_i$; scatterers B_i.
  - Boundaries of scatterers are $\mathcal{C}^3$ and have strictly positive curvature.
- Assume **Finite Horizon** condition: there is no trajectory making only tangential collisions $\implies$ an upper bound on the free flight time between collisions.
- Transfer operator for geometric potential with $t = 0$,

$$\mathcal{L}_0 f = \frac{f \circ T^{-1}}{J^s T \circ T^{-1}}$$

  - $J^s T \approx \cos \varphi$ so the potential is unbounded
  - $J^s T$ is not continuous on any open set

Previous Results on Topological Entropy

A continuous map on a noncompact set $M' \subset M$.

- The set $\mathcal{S}_0 = \{(r, \varphi) \in M : \varphi = \pm\frac{\pi}{2}\}$ corresponds to tangential collisions.
- For $n \in \mathbb{Z}$, $\mathcal{S}_n = \cup_{i=0}^{n} T^{-i}\mathcal{S}_0$ is the singularity set for T^n.
- Define $M' = M \setminus (\cup_{n=-\infty}^{\infty}\mathcal{S}_n)$. $T : M' \circlearrowleft$ is a continuous map.

[Chernov '91] studied the topological entropy of the billiard map T on an invariant subset $M_1 \subset M'$ using a countable Markov partition η_1. He showed that

$$h_{\mathsf{top}}(T, M') \geq h_{\mathsf{top}}(T, M_1) = h_{\mathsf{top}}(\sigma, \Sigma_1),$$

where (σ, Σ_1) is the TMC derived from the Markov partition η_1.

Also, $P_n(T) \geq e^{n h_{\mathsf{top}}(T, M_1)}$, $\quad P_n = \#$ periodic pts of period n.

Weight Function for Topological Entropy

To control the evolution of $\mathcal{L}_0^n f$, must control integrals of the type,

$$\int_W \mathcal{L}_0^n f \, \psi \, dm_W = \int_{T^{-n}W} f \, \psi \circ T^n \, dm_{T^{-n}W} \, .$$

- $W \in \mathcal{W}^s$, the set of local stable manifolds with uniformly bounded curvature
- m_W is arclength measure on W
- $\psi \in C^\alpha(W)$ is a Hölder continuous test function
- f is an element of our Banach space (closure of $C^1(M)$ in some norm)

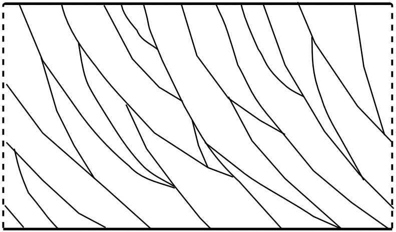
$T^{-n}W = \cup_i W_i$, smooth, connected components.

We need to estimate precisely how $\sum_{W_i} 1$ grows as a function of n and W. **Without a Jacobian, the growth lemmas will look different; we cannot use homogeneity strips.**

Step 1: A Definition of Topological Entropy

- Let $\mathcal{M}^n_{-k}$ = connected components of $M \setminus (\mathcal{S}_{-k} \cup \mathcal{S}_n)$.
- Define

$$h_* = \lim_{n \to \infty} \frac{1}{n} \log \# \mathcal{M}_0^n$$



$M \setminus \mathcal{S}_n$

- The limit exists since the sequence $\log \# \mathcal{M}_0^n$ is subadditive: $\# \mathcal{M}_0^{n+m} \le \# \mathcal{M}_0^n \cdot \# \mathcal{M}_0^m$.
- h_* is the exponential rate of growth of the number of pieces created by the discontinuities of T. It does not depend on a choice of metric.
- $T^n \mathcal{S}_n = \mathcal{S}_{-n} \implies \# \mathcal{M}_0^n = \# \mathcal{M}^0_{-n}$. So $h_*(T) = h_*(T^{-1})$.

Connection to Bowen Definitions

For $n \geq 1$, define the dynamical distance:

$$d_n(x, y) = \max_{0 \leq i \leq n} d(T^i x, T^i y),$$

where $d(x, y) =$ Euclidean distance on each $M_i = \partial B_i \times [-\frac{\pi}{2}, \frac{\pi}{2}]$, and $d(x, y) = 10 \max_i \text{diam}(M_i)$ when x, y are in different M_i.

- E is (n, ε)**-separated set** if for all $x \neq y \in E$, $d_n(x, y) > \varepsilon$.
  $r_n(\varepsilon) :=$ maximum cardinality of any (n, ε)-separated set
- F is (n, ε)**-spanning set** if $\forall\, x \in M$, $\exists\, y \in F$ s.t. $d_n(x, y) \leq \varepsilon$.
  $s_n(\varepsilon) :=$ minimum cardinality of any (n, ε)-spanning set

Define

$$h_{\text{sep}} = \lim_{\varepsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log r_n(\varepsilon)$$

$$h_{\text{span}} = \lim_{\varepsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log s_n(\varepsilon)$$

Connection to Bowen Definitions

The finite horizon condition implies the following lemma.

Lemma

There exists ε_0, depending only on the table Q, such that for all $n \geq 1$, if x, y lie in different elements of $\mathcal{M}_0^n$, then $d_n(x, y) \geq \varepsilon_0$.

Lemma + uniform hyperbolicity of $T \implies$

Proposition ([Baladi, D. '20])

For a finite horizon Lorentz gas,

$$h_* = h_{\mathsf{sep}} = h_{\mathsf{span}}.$$

Moreover, for any $k \geq 1$,

$$h_* = \lim_{n \to \infty} \frac{1}{n} \log \# \mathcal{M}_{-k}^n,$$

where $\mathcal{M}_{-k}^n$ are the connected components of $M \setminus (\mathcal{S}_{-k} \cup \mathcal{S}_n)$.

Dynamical Refinements of Partitions

In order to connect h_* to measure theoretic entropy, it is convenient to express it in terms of dynamical refinements of a partition. Define

$\mathcal{P} :=$ maximal connected sets on which T and T^{-1} are continuous

For each $k, n \in \mathbb{N}$,

- Define $\mathcal{P}_{-k}^{n} = \bigvee_{i=-k}^{n} T^{-i}\mathcal{P}$, a partition of M
- $\mathring{\mathcal{P}}_{-k}^{n} =$ interiors of elements of $\mathcal{P}_{-k}^{n}$,
  a partition of $M \setminus (\mathcal{S}_{-k} \cup \mathcal{S}_n)$
- $\mathcal{P}_{-k}^{n} \approx \mathring{\mathcal{P}}_{-k}^{n} +$ isolated points due to multiple tangencies
- $\mathring{\mathcal{P}}_{-k}^{n} = \mathcal{M}_{-k-1}^{n+1}$
  This relies on the continuation of singularities property

Characterization of h_* and Variational Inequality

Theorem ([Baladi, D. '20])

For a finite horizon Sinai billiard,

- $h_* = h_{\mathsf{sep}} = h_{\mathsf{span}}$
- For any $k \geq 0$,

$$h_* = \lim_{n\to\infty} \frac{1}{n} \log \# \mathcal{M}_{-k}^n = \lim_{n\to\infty} \frac{1}{n} \log \# \mathcal{P}_{-k}^n$$

- h_* satisfies a variational inequality,

$$h_* \geq \sup\{h_\mu(T) : \mu \text{ is a } T\text{-invariant Borel prob. measure}\}$$

The variational inequality is a straightforward consequence of the fact that $\mathcal{P}$ is a generator for T, and the entropy of a partition is bounded by its cardinality.

$\mathcal{G}_n(W)$ and Linear Complexity Bound

To obtain a precise estimate on the spectral radius of $\mathcal{L}_0$, we will need precise estimates on the growth rates of $\#\mathcal{M}_0^n$ and $\#\mathcal{G}_n(W)$.

- Define $\mathcal{G}_n(W)$ without homogeneity strips
- For $W \in \widehat{\mathcal{W}}^s$, define $\mathcal{G}_1(W)$ to be the maximal, connected components of $T^{-1}W$ subdivided to length at most δ_0 (t.b.d.)
- Define $\mathcal{G}_n(W) = \{\mathcal{G}_1(W_i) : W_i \in \mathcal{G}_{n-1}(W)\}$.

Recall the linear complexity bound.

For $x \in M$, let $N(\mathcal{S}_n, x)$ denote the number of singularity curves in $\mathcal{S}_n$ that meet at x. Define $N(\mathcal{S}_n) = \sup_{x \in M} N(\mathcal{S}_n, x)$.

Lemma (Bunimovich, Chernov, Sinai '90)

Assume finite horizon. There exists $K > 0$ depending only on the configuration of scatterers such that $N(\mathcal{S}_n) \le Kn$ for all $n \ge 1$.

Choose n_0 s.t. $n_0^{-1} \log(Kn_0 + 1) < h_*$. Choose δ_0 s.t. any stable curve of length $\le \delta_0$ is cut into at most $Kn_0 + 1$ pieces by $\mathcal{S}_{-n_0}$.

Fragmentation Lemma (Growth Lemma)

Let $L_n^\delta(W) = \{W_i \in \mathcal{G}_n^\delta(W) : |W| \geq \delta/3\}$.
$Sh_n^\delta(W) = \mathcal{G}_n^\delta(W) \setminus L_n^\delta(W)$.

Lemma ([Baladi, D. '20])

For all $\varepsilon > 0$ there exists $n_1, \delta > 0$ s.t. for all $n \geq n_1$,

$$\#Sh_n^\delta(W) \leq \varepsilon \#\mathcal{G}_n^\delta(W) \quad \text{for all } W \in \widehat{\mathcal{W}}^s \text{ with } |W| \geq \delta/3.$$

Idea of Proof: Choose $\varepsilon > 0$ and n_1 s.t. $3C_0^{-1}(Kn_1 + 1)\Lambda^{-n_1} < \varepsilon$. Choose $\delta > 0$ s.t. if $|W| < \delta$ then $T^{-n_1}W$ comprises at most $Kn_1 + 1$ connected components of length at most δ_0.

Then $Sh_{n_1}^\delta(W)$ contains at most $Kn_1 + 1$ elements while $|T^{-n_1}W| \geq C_0\Lambda^{n_1}\delta/3$, where $\Lambda = 1 + 2\mathcal{K}_{\min}\tau_{\min}$.

Thus $\#\mathcal{G}_{n_1}^\delta(W) \geq C_0\Lambda^{n_1}/3$ and so $\frac{\#Sh_{n_1}^\delta(W)}{\#\mathcal{G}_{n_1}^\delta(W)} \leq \varepsilon$ by choice of n_1.

Argument can be iterated, grouping by most recent long ancestor.

Fragmentation of $\mathcal{M}_0^n$

This lemma can also be formulated for elements of $\mathcal{M}_0^n$ and $\mathcal{M}_{-n}^0$.

Let δ_1, $n_1 \ge n_0$ correspond to $\varepsilon = 1/4$ in fragmentation lemma: For all $W \in \widehat{\mathcal{W}}^s$ with $|W| \ge \delta_1/3$,

$$\# L_n^{\delta_1}(W) \ge \tfrac{3}{4} \# \mathcal{G}_n^{\delta_1}(W) \, , \quad \forall n \ge n_1 \, .$$

$$\text{Define} \;\; L_s(\mathcal{M}_0^n) := \{ A \in \mathcal{M}_0^n : \text{diam}^s(A) \ge \delta_1/3 \}$$
$$L_u(\mathcal{M}_{-n}^0) := \{ B \in \mathcal{M}_{-n}^0 : \text{diam}^u(B) \ge \delta_1/3 \}$$

Lemma

There exists $c_0 > 0$ s.t. for all $n \ge 1$,

$$\# L_s(\mathcal{M}_0^n) \ge c_0 \delta_1 \# \mathcal{M}_0^n \quad \textit{and} \quad \# L_u(\mathcal{M}_{-n}^0) \ge c_0 \delta_1 \# \mathcal{M}_{-n}^0 \, .$$

Fragmentation Lemmas $\implies$ Uniform Bounds on Growth

Proposition

a) $\exists\, c_1 > 0$ such that for any $W \in \widehat{\mathcal{W}}^s$ with $|W| \geq \delta_1/3$,

$$\#\mathcal{G}_n(W) \geq c_1 \#\mathcal{M}_0^n \qquad \forall n \geq 1\,.$$

b) There exists $c_2 > 0$ such that for all $k, n \geq 1$,

$$\#\mathcal{M}_0^{n+k} \geq c_2 \#\mathcal{M}_0^n \cdot \#\mathcal{M}_0^k\,.$$

(b) implies **exact exponential growth** of $\#\mathcal{M}_0^n$,

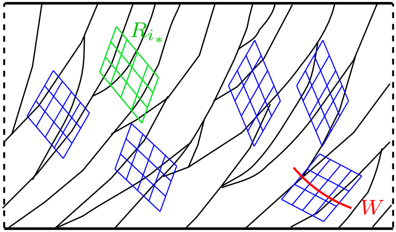
$$e^{nh_*} \leq \#\mathcal{M}_0^n \leq 2c_2^{-1} e^{nh_*} \quad \text{for all } n \geq 1.$$

(a) + fragmentation lemma $\implies$ (b) since

$$\#\mathcal{G}_{n+k}(W) \geq \sum_{V_j \in L_n^{\delta_1}(W)} \#\mathcal{G}_k(V_j) \geq \#L_n^{\delta_1}(W) c_1 \#\mathcal{M}_0^k$$

$$\geq \tfrac{3c_1}{4} \#\mathcal{G}_n^{\delta_1}(W) \#\mathcal{M}_0^k \geq \frac{3c_1^2}{4} \#\mathcal{M}_0^n \#\mathcal{M}_0^k$$

Justification for (a) Lower Bound on Growth of $\#\mathcal{G}_n(W)$



$M \setminus \mathcal{S}_{-n}$

- 'Cover' M with $k(\delta_2)$ Cantor rectangles R_i s.t. any stable/ unstable curve of length $\delta_1/3$ properly crosses at least one R_i
- $\exists i_*$ s.t. $\#\{L_u(\mathcal{M}^0_{-n})$ properly crossing $R_{i_*}\} \geq \frac{c_0 \delta_1}{k} \#\mathcal{M}^0_{-n}$
- Take $W \in \widehat{\mathcal{W}}^s$ with $|W| \geq \delta_1/3$. Crosses one R_j.
- Use mixing of SRB measure to ensure that $T^{-N}W$ crosses R_{i_*}, N depends only on δ_1.
- Then $\#\mathcal{G}_{n+N}(W) \geq \frac{c_0 \delta_1}{k} \#\mathcal{M}^n_0$

Uniform Growth of Stable Curves

A corollary of our uniform bounds is the uniform growth rate of $|T^{-n}W|$ in terms of h_*.

Corollary

There exists $C > 0$ s.t. for all $W \in \widehat{\mathcal{W}}^s$ with $|W| \ge \delta_1/3$, for all $n \ge n_1$,

$$Ce^{nh_*} \le |T^{-n}W| \le C^{-1}e^{nh_*} .$$

Proof: Our uniform bounds give,

$$Ce^{nh_*} \le \#\mathcal{G}_n(W) \le C^{-1}e^{nh_*}$$

But also, $|T^{-n}W| \le \delta_0 \#\mathcal{G}_n(W)$, giving the upper bound. And

$$|T^{-n}W| = \sum_{W_i \in \mathcal{G}_n^{\delta_1}} |W_i| \ge \frac{\delta_1}{3} \# L_n^{\delta_1}(W) \ge \frac{\delta_1}{4} \# \mathcal{G}_n^{\delta_1}(W) ,$$

giving the lower bound. □