Anisotropic Banach Spaces and Thermodynamic Formalism for Dispersing Billiard Maps
Lecture 4: Geometric Potentials and Pressure, Part II

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Lecture 4: Geometric Potentials and Pressure, Part II

Goal for today: Introduce Banach spaces adapted to the geometric potentials on which we prove a spectral gap for the transfer operator. Use this to prove existence and uniqueness of equilibrium states and analyticity of pressure function.

Recall from Lecture 3,

$$\mathcal{L}_t f(x) = \frac{f(T^{-1}x)}{|J^s T(T^{-1}x)|^{1-t}},$$

is the transfer operator corresponding to the potential $-t \log J^u T$, $t > 0$. Want to construct equilibrium state from the product of left and right maximal eigenvectors of $\mathcal{L}_t$.

Restrict to $t \in (0, t_*)$, where $t_* > 1$ is defined by

$$-t_* \log \Lambda = P(t_*).$$

Reference: V. Baladi and M. Demers, *Thermodynamic formalism for dispersing billiards*, preprint 2020.

Stable Curves and Definition of Pressure

Fix $0 < t_0 < t_1 < t_*$. Consider $t \in [t_0, t_1]$.

- Choose $\theta > \Lambda^{-1}$ s.t. $\theta^t < e^{P(t)}$ for all $t \in [t_0, t_1]$.
- Fix $q \geq 2/t_0$ and choose $k_0, \delta_0 > 0$ s.t. one-step expansion holds for all $|W| \leq \delta_0$.
- $\widehat{\mathcal{W}}^s_H$ set of homogeneous cone-stable curves with bounded curvature and length $\leq \delta_0$.
  $\mathcal{W}^s_H \subset \widehat{\mathcal{W}}^s_H$ set of (weakly) homogeneous stable manifolds.
- Define $\mathcal{S}_n = \cup_{i=0}^n T^{-i}\mathcal{S}_0$, $\qquad \mathcal{S}_n^{\mathbb{H}} = \cup_{i=0}^n T^{-i}\mathcal{S}_0^{\mathbb{H}}$
- $\mathcal{M}_0^{n,\mathbb{H}}$ = connected components of $M \setminus (\mathcal{S}_{n-1}^{\mathbb{H}} \cup T^{-n}\mathcal{S}_0)$
- $Q_n(t) := \displaystyle\sum_{A \in \mathcal{M}_0^{n,\mathbb{H}}} \sup_{x \in A \cap M'} |J^s T^n(x)|^t, \qquad M' = M \setminus (\cup_{n \in \mathbb{Z}} \mathcal{S}_n)$
- $P_*(t) := \displaystyle\lim_{n \to \infty} \frac{1}{n} \log Q_n(t)$

Definition of Norms: Weak Norm

Fix $0 < \alpha \le 1/(q+1)$.

For $f \in C^1(M)$, define the **weak norm** of f by

$$|f|_w = \sup_{W \in \mathcal{W}^s_H} \sup_{\substack{\psi \in \mathcal{C}^\alpha(W) \\ |\psi|_{\mathcal{C}^\alpha(W)} \le 1}} \int_W f \, \psi \, dm_W \, .$$

Define $\mathcal{B}_w$ to be the completion of $C^1(M)$ in the $|\cdot|_w$ norm.

Remark: Norms defined on stable manifolds $\mathcal{W}^s$ rather than cone-stable curves $\widehat{\mathcal{W}}^s$. We make this choice because J^sT varies Hölder continuously along $W \in \mathcal{W}^s$, but only measurably transverse to stable direction. These norms are not well suited to study perturbations of the dynamics.

For $t = 1$, J^sT disappears and one can use $\widehat{\mathcal{W}}^s$ instead. Such norms are robust under perturbations [D., Zhang '13].

Definition of Norms: Strong Norm

Choose $p > q + 1$, $\beta \in (1/p, \alpha)$ and $\gamma < \min\{1/p, \alpha - \beta\}$.

Define the **strong stable norm** of f by

$$\|f\|_s = \sup_{W \in \mathcal{W}^s_H} \sup_{\substack{\psi \in \mathcal{C}^\beta(W) \\ |\psi|_{\mathcal{C}^\beta(W)} \le |W|^{-1/p}}} \int_W f \, \psi \, dm_W$$

Define the **strong unstable norm** of f by

$$\|f\|_u = \sup_{\varepsilon \le \varepsilon_0} \sup_{\substack{W_1, W_2 \in \mathcal{W}^s_H \\ d(W_1, W_2) \le \varepsilon}} \sup_{\substack{|\psi_i|_{\mathcal{C}^\alpha(W_i)} \le 1 \\ d_0(\psi_1, \psi_2) = 0}} \varepsilon^{-\gamma} \left| \int_{W_1} f \psi_1 - \int_{W_2} f \psi_2 \right|$$

The **strong norm** of f is defined to be $\|f\|_{\mathcal{B}} = \|f\|_s + c_u \|f\|_u$,

Define $\mathcal{B}$ to be the completion of $C^1(M)$ in the $\| \cdot \|_{\mathcal{B}}$ norm.

Distances Between Curves and Test Functions

- View $W \in \widehat{\mathcal{W}}^s$ as the graph of a function of the r-coordinate over an interval I_W,

$$W = \{G_W(r) : r \in I_W\} = \{(r, \varphi_W(r)) : r \in I_W\}\,.$$

- Given $W_1, W_2 \in \widehat{\mathcal{W}}^s$ with functions φ_{W_1}, φ_{W_2}, define

$$d(W_1, W_2) = |I_{W_1} \triangle I_{W_2}| + |\varphi_{W_1} - \varphi_{W_2}|_{C^1(I_{W_1} \cap I_{W_2})}\,,$$

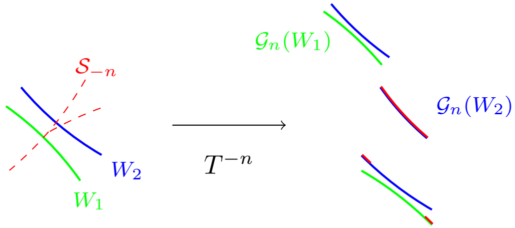
  if W_1 and W_2 lie in the same homogeneity strip, and $d(W_1, W_2) = \infty$ otherwise.

- If $d(W_1, W_2) < \infty$, then for $\psi_1 \in C^0(W_1)$, $\psi_2 \in C^0(W_2)$, define

$$d_0(\psi_1, \psi_2) = |\psi_1 \circ G_{W_1} - \psi_2 \circ G_{W_2}|_{C^0(I_{W_1} \cap I_{W_2})}\,.$$

Lasota-Yorke: Unmatched Pieces

For strong unstable norm, estimate $\left|\int_{W_1} \mathcal{L}_t^n f\, \psi_1 - \int_{W_2} \mathcal{L}_t^n f\, \psi_2\right|$



- Unmatched pieces have length at most $\Lambda^{-j}\varepsilon$ is they are cut by a singularity curve at time $-j$.
- Use the strong stable norm to estimate,

$$\int_{W_i} \mathcal{L}_t^n f\, \psi = \int_{V_j} \mathcal{L}_t^{n-j} f\, \psi \circ T^j |J_{V_j} T^j|^t \le \Lambda^{-j/p} \varepsilon^{1/p} \|\mathcal{L}_t^{n-j} f\|_s |J_{V_j} T^j|_{C^0}^t$$

- $\|\cdot\|_s$ acts as 'weak norm' for $\|\cdot\|_u$ to control unmatched pieces.

Regularity of J^sT

Since $\mathcal{B}$ is defined as the completion of $C^1(M)$ in $\| \cdot \|_{\mathcal{B}}$, a priori, it is not clear that $\mathcal{L}_t$ acts continuously on $\mathcal{B}$.

Lemma ([Chernov, Markarian '06])

For $W \in \mathcal{W}^s_H$ and $\eta > 0$, let $W_u(\eta) = \{$ points in W whose unstable manifold extends a length at least η on both sides of $W\}$. Then $m_W(W \setminus W_u(\eta)) \le C\eta$ for some $C > 0$ indep. of W and η.

Lemma ([Baladi, D. '20])

$\exists\, C_1, C_2 > 0$ such that for any homogeneous unstable curve U and any $\rho > 0$, there exists $U' \subset U$ with $m_U(U \setminus U') \le C_1 \rho$ such that

$$\left| \frac{J^sT(x)}{J^sT(y)} - 1 \right| \le C_2 \left(\rho^{-\frac{q}{q+1}} d(x,y) + d(x,y)^{1/(q+1)} \right) .$$

These two lemmas allow us to approximate $\mathcal{L}_t f$ by C^1 functions in the $\| \cdot \|_{\mathcal{B}}$ norm.

Banach Spaces

Theorem ([Baladi, D. '20])

- We have a sequence of continuous inclusions,

$$\mathcal{C}^1(M) \subset \mathcal{B} \subset \mathcal{B}_w \subset (\mathcal{C}^\alpha(M))^*.$$

- The embedding of the unit ball of $\mathcal{B}$ in $\mathcal{B}_w$ is compact.
- There exist $C, C_n > 0$ such that for all $f \in \mathcal{B}$, $n \geq 0$,

$$|\mathcal{L}_t^n f|_w \leq C Q_n(t) |f|_w \,,$$
$$\|\mathcal{L}_t^n f\|_s \leq C\big(\Lambda^{-(\beta-1/p)n} Q_n(t) + \theta^{(t-1/p)n}\big)\|f\|_s + C_n|f|_w)$$
$$\|\mathcal{L}_t^n f\|_u \leq C Q_n(t)\big(n^\gamma \Lambda^{-\gamma n}\|f\|_u + C_n\|f\|_s\big)\,.$$

Implies the spectral radius of $\mathcal{L}_t$ on $\mathcal{B}$ is at most $e^{P_*(t)}$ and its essential spectral radius $< e^{P_*(t)}$ **if** $\theta^t < e^{P_*(t)}$ (**pressure gap**).

To prove $\mathcal{L}_t$ is quasi-compact, we need a **lower bound** on the spectral radius.

Lower Bound on Spectral Radius

The lower bound follows from our uniform growth result from Lecture 3:

There exists $c_1 > 0$ s.t. for any $W \in \widehat{\mathcal{W}}^s$ with $|W| \geq \delta_1/3$,

$$\sum_{W_i \in \mathcal{G}_n(W)} |J_{W_i} T^n|^t_{C^0(W_i)} \geq c_1 Q_n(t) \, , \, \forall n \geq 1 \, , \, \forall t \in [t_0, 1] \, .$$

Let $W \in \mathcal{W}^s_H$ with $|W| \geq \delta_1/3$, choose $\psi \equiv 1$. For any $n \geq 1$,

$$\int_W \mathcal{L}_t^n 1 = \sum_{W_i \in \mathcal{G}_n(W)} \int_{W_i} |J_{W_i} T^n|^t \geq e^{-C_d} \sum_{W_i \in \mathcal{G}_n(W)} |J_{W_i} T^n|^t_{C^0(W_i)}$$
$$\geq e^{-C_d} c_1 Q_n(t) \geq e^{-C_d} c_1 e^{n P_*(t)}$$

Thus $\|\mathcal{L}^n 1\|_s \geq C e^{n P_*(t)}$ and so the spectral radius of $\mathcal{L}$ is $e^{P_*(t)}$.

Spectral Decomposition of $\mathcal{L}_t$

Our exact exponential growth from Lecture 3 implies:

$$\|\mathcal{L}_t^n\|_{\mathcal{B}} \le C Q_n(t) \le C' e^{nP_*(t)} ,$$

so that the peripheral spectrum of $\mathcal{L}_t$ has no Jordan blocks.

There exist a finite set $\{\theta_j\}_{j=0}^N$, $\theta_0 = 0$, linear operators $\Pi_j, R : \mathcal{B} \circlearrowleft$ satisfying $\Pi_i \Pi_j = \Pi_j R = R \Pi_j = 0$ with spectral radius of $R < 1$, such that

$$e^{-P_*(t)} \mathcal{L}_t = \sum_{j=1}^{N} e^{2\pi\theta_j} \Pi_j + R$$

Proof of spectral gap follows similar lines as for Baker's map: Define $\nu_t = \Pi_0 1$. Show all eigenvectors corresponding to the peripheral spectrum are measures absolutely continuous wrt ν_t, and θ_j must be rational. Use mixing to show 1 is simple for $\mathcal{L}_t^k$ for $k \ge 1$. (Lack of smoothness complicates argument.)

A Spectral Gap for $\mathcal{L}_t$

Theorem ([Baladi, D. '20])

For each $t_0 > 0$ and $t_1 < t_*$, there exists a Banach space $\mathcal{B} = \mathcal{B}(t_0, t_1)$ such that $\mathcal{L}_t$ has a spectral gap:

- $e^{P_*(t)}$ is the eigenvalue of maximum modulus, it is simple, and the remainder of the spectrum of $\mathcal{L}_t$ is contained in a disk of radius $\bar\sigma e^{P_*(t)}$, where $\bar\sigma < 1$ is uniform for $t \in [t_0, t_1]$.

Letting ν_t and $\tilde\nu_t$ denote the maximal right and and left eigenvectors for $\mathcal{L}_t$, define

$$\mu_t(\psi) = \frac{\langle \nu_t, \psi \tilde\nu_t \rangle}{\langle \nu_t, \tilde\nu_t \rangle}, \quad \psi \in C^\alpha(M).$$

Then μ_t is an invariant probability measure for T, and enjoys exponential decay of correlations against Hölder observables.

μ_t has no atoms, gives 0 weight to any C^1 curve and is positive on open sets. Moreover, $\int |\log d(x, \mathcal{S}_{\pm 1})| \, d\mu_t < \infty$.

Entropy of μ_t and a Variational Principle

Define $B(x,n,\varepsilon) = \{y \in M : d(T^{-i}x, T^{-i}y) \le \varepsilon, \forall i \in [0,n]\}$.

Proposition (Measure of Bowen Balls)

There exists $C > 0$ s.t. for all $x \in M$, $n \ge 1$, and $y \in B(x,n,\varepsilon)$,

$$\mu_t(B(x,n,\varepsilon)) \le C e^{-nP_*(t) + t \log J^sT^n(T^{-n}y)}.$$

- [Brin, Katok '81] $\implies$ for μ_t-a.e. $x \in M$,

$$\lim_{\varepsilon \to 0} \limsup_{n \to \infty} -\frac{1}{n} \log \mu_t(B(x,n,\varepsilon)) = h_{\mu_t}(T).$$

- This plus the Proposition implies

$$h_{\mu_t}(T) \ge P_*(t) - t \int \log J^sT \, d\mu_t = P_*(t) + t \int \log J^uT \, d\mu_t$$

- But $P_*(t) \ge h_{\mu_t}(T) - t \int \log J^uT \, d\mu_t$ since $P_*(t) \ge P(t)$.
- Conclude: $P_*(t) = h_{\mu_t}(T) - t \int \log J^uT \, d\mu_t = P(t)$.

Uniqueness of Equilibrium State

We prove uniqueness using the concept of **tangent measure**.

We say μ is a C^1-tangent measure at t if

$$P(-t \log J^u T + \phi) \geq P(t) + \int \phi \, d\mu \,, \quad \text{for all } \phi \in C^1(M)$$

If μ is an equilibrium state for $-t \log J^u T$, then μ is a tangent measure [Walters '82].

We show there can be only one tangent measure for each t by showing that for each $\phi \in C^1(M)$, the perturbed transfer operator defined by

$$\mathcal{L}_{t,z\phi} f = \frac{f \circ T^{-1}}{(J^s T)^{1-t} \circ T^{-1}} e^{z\phi \circ T^{-1}} \,, \quad z \in \mathbb{C} \,,$$

is an analytic perturbation of $\mathcal{L}_t$.

Generalized Variational Principle

Theorem ([Baladi, D. '20])

Let $t \in [t_0, t_1]$ and $\phi \in C^1(M)$. For $|z|$ sufficiently small,

- $\mathcal{L}_{t,z\phi}$ has a spectral gap on $\mathcal{B}$;
- the spectral radius of $\mathcal{L}_{t,z\phi}$ is $e^{P(-t\log J^uT + z\phi)}$;
- restricting to $z \in \mathbb{R}$,

$$\frac{d}{dz} e^{P(-t\log J^uT+z\phi)}\Big|_{z=0} = e^{P(t)} \int \phi \, d\mu_t \, ;$$

- finally, $P_*(t\log J^sT + z\phi) = P(-t\log J^uT + z\phi)$ and there exists a unique equilibrium measure attaining the supremum.

The derivative formula for $e^{P(-t\log J^uT+z\phi)}$ implies that any tangent measure μ must satisfy $\int \phi d\mu = \int \phi d\mu_t$ for all C^1 functions ϕ. Thus $\mu = \mu_t$, so μ_t is unique.

Analyticity of $P(t)$

Since J^sT is not piecewise Hölder, a separate set of arguments is needed to show that $\mathcal{L}_t$ is analytic as a function of t, $t \in [t_0, t_1]$.

Theorem

The function $t \mapsto P(t)$ is analytic on $(0, t_*)$, with

$$P'(t) = \int \log J^sT \, d\mu_t = -\int \log J^uT \, d\mu_t < 0 \, ,$$

$$P''(t) = \sum_{k \geq 0} \left[\int (\log J^sT \circ T^k) \log J^sT \, d\mu_t - (P'(t))^2 \right] \geq 0 \, .$$

Moreover, $P''(t) = 0$ if and only if $\log J^sT = f - f \circ T + P'(t)$ for some $f \in L^2(\mu_t)$.

If there exists $t_1 \neq t_2$ in $(0, t_*)$ such that $\mu_{t_1} = \mu_{t_2}$, then $P(t)$ is affine on $(0, t_*)$ and $\log J^sT$ is μ_t-a.e. cohomologous to a constant for all $t \in (0, t_*)$.