Anisotropic Banach Spaces and Thermodynamic Formalism for Dispersing Billiard Maps
Lecture 3: Geometric Potentials and Pressure, Part I

Mark Demers
Fairfield University
Research supported in part by NSF grant DMS 1800321

Spring School on Transfer Operators
Research Semester: Dynamics, Transfer Operators and Spectra
Centre Interfacultaire Bernoulli, EPFL
March 22 - 26, 2021

Lecture 3: Geometric Potentials and Pressure, Part I

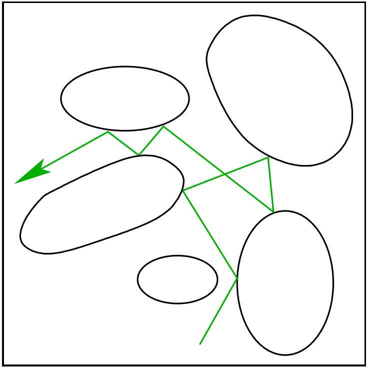
Goal for today: Introduce geometric potentials and formulate definition of associated topological pressure for finite horizon Lorentz gas. Present initial results regarding growth lemmas, topological and metric pressures.

Lecture 4: Introduce Banach spaces adapted to the geometric potentials on which we prove a spectral gap for the transfer operator. Use this to prove existence and uniqueness of equilibrium states and analyticity of pressure function.

Reference: V. Baladi and M. Demers, *Thermodynamic formalism for dispersing billiards*, preprint 2020.

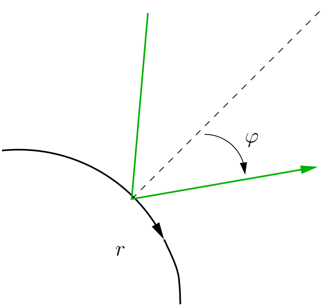
Periodic Lorentz gas (Sinai Billiard) [Sinai '68]

- Billiard table $\mathcal{Q} = \mathbb{T}^2 \setminus \cup_i B_i$; scatterers B_i.
- Boundaries of scatterers are $\mathcal{C}^3$ and have strictly positive curvature.
- Billiard flow is given by a point particle moving at unit speed with elastic collisions at the boundary

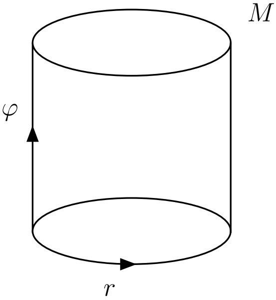


Assume **Finite Horizon** condition: there is an upper bound on the free flight time between collisions.

The Associated Billiard Map



- r = position coordinate oriented clockwise on boundary of scatterer ∂B_i
- φ = angle outgoing trajectory makes with normal to scatterer



$M = \left(\cup_i \partial B_i \right) \times [-\frac{\pi}{2}, \frac{\pi}{2}]$, the natural "collision" cross-section for the billiard flow.

$T : (r, \varphi) \to (r', \varphi')$ is the first return map: the **billiard map**.

- a hyperbolic map with singularities

Pressure and Equilibrium States

Given a function ϕ, define the **pressure** of ϕ by,

$$P(\phi) := \sup \left\{ h_\nu(T) + \int \phi \, d\nu : \nu \text{ invariant prob. measure} \right\}$$

If μ is an invariant probability for T satisfying $h_\mu(T) + \int \phi \, d\mu = P(\phi)$, then μ is an **equilibrium state** for ϕ.

For Hölder continuous ϕ, the existence and uniqueness of equilibrium states has been established for many systems.

- uniformly hyperbolic systems (Anosov and Axiom A) [Sinai '72], [Bowen '74], [Ruelle '78]
- nonuniformly hyperbolic maps and flows
  - Markov partitions [Sarig '11], [Lima, Matheus '18], [Buzzi, Crovisier, Sarig '19]
  - Young towers [Pesin, Senti, Zhang '16]
  - non-uniform specification [Climenhaga, Thompson '13], [Burns, Climenhaga, Fisher, Thompson '18]

Geometric Potentials

Important family of potentials: geometric potentials,

$$t\phi = -t \log J^u T, \quad t \in \mathbb{R} .$$

- $t = 1$ gives the smooth invariant measure $\mu_{\mathsf{SRB}} = \cos\varphi dr d\varphi$. This is an equilibrium state for ϕ and uniqueness is proved in a class of measures whose support decays sufficiently near singularities [Katok, Strelcyn '86].
- $t = 0$ yields the measure of maximal entropy [Baladi, D. '20]. This is Bernoulli (and hence mixing) and globally unique, but its rate of mixing is not known.
- $t < 0$ implies $P(t) = \infty$ since $J^u T$ is unbounded near tangential collisions. Today restrict to $t > 0$.
- [Chen, Wang, Zhang '20] proves existence (but not uniqueness) of equilibrium state for t near 1 using Young towers.

Associated Transfer Operator

The main tool we will use is the transfer operator associated to the potential $t\phi = -t \log J^u T$.

For a smooth hyperbolic system, the transfer operator with spectral radius $e^{P(t\phi)}$ is

$$\widetilde{\mathcal{L}}_t f = \frac{f \circ T^{-1}}{((J^u T)^t J^s T) \circ T^{-1}}$$

For a billiard, setting $E(x) = \sin(\angle(E^s(x), E^u(x)))$,

$$\frac{\cos \varphi(x)}{\cos \varphi(Tx)} = J_{\mathsf{Leb}} T(x) = J^s T(x) J^u T(x) \frac{E(Tx)}{E(x)} ,$$

$$\implies (J^u T)^t J^s T = \left(\frac{E \cos \varphi}{(E \cos \varphi) \circ T} \right)^t (J^s T)^{1-t}$$

So $\widetilde{\mathcal{L}}_t$ has the same spectrum as

$$\mathcal{L}_t f = \frac{f \circ T^{-1}}{(J^s T)^{1-t} \circ T^{-1}}$$

Associated Transfer Operator

$$\mathcal{L}_t f = \frac{f \circ T^{-1}}{(J^sT)^{1-t} \circ T^{-1}}$$

For $t = 1$, this corresponds to using μ_{SRB} as the conformal measure. We will identify a function f with the measure $d\mu = f d\mu_{\mathsf{SRB}}$. Then acting on distributions,

$$\mathcal{L}_t \mu(\psi) = \mu\left(\frac{\psi \circ T}{(J^sT)^{1-t}}\right), \quad \text{test function } \psi$$

Construct equilibrium state μ_t out of left and right eigenvectors of $\mathcal{L}_t$ corresponding to the eigenvalue of maximum modulus.

Sources of difficulty:

- T has discontinuities so a topological definition of pressure must overcome the effect of this cutting.
- The potential is not Hölder continuous
  - $J^sT \approx \cos\varphi$ so the potential is unbounded
  - J^sT is not continuous on any open set

Weight Function for Topological Pressure

To control the evolution of $\mathcal{L}_t^n f$, must control integrals of the type,

$$\int_W \mathcal{L}_t^n f \, \psi \, dm_W = \int_{T^{-n}W} f \, \psi \circ T^n \, |J^s T^n|^t \, dm_{T^{-n}W} \, .$$

- $W \in \mathcal{W}^s_H$, the set of (weakly) homogeneous local stable manifolds with uniformly bounded curvature
- m_W is arclength measure on W
- $\psi \in C^\alpha(W)$ is a Hölder continuous test function
- f is an element of our Banach space (closure of $C^1(M)$ in some norm)

$T^{-n}W = \cup_i W_i$, $W_i \in \mathcal{G}_n(W)$, homogeneous components.

We need to estimate precisely how $\sum_{W_i} |J^s T^n|^t_{C^0(W_i)}$ grows as a function of n and W. This resembles the expression from our growth lemma in Lecture 2.

Homogeneity Strips and Modified One-step Expansion

$$H_{\pm k} = \{(r, \varphi) \in M : (k+1)^{-q} \le |\varphi \mp \frac{\pi}{2}| \le k^{-q}\}, \quad k \ge k_0$$

For $V \in \widehat{\mathcal{W}}^s$, let V_i denote the homogeneous connected components of $T^{-1}V$.

Lemma (Modified One-step Expansion)

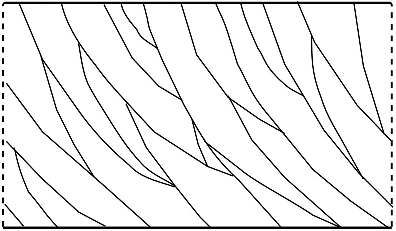
Fix $t_0 > 0$ and $q \ge 2/t_0$. There exists $\theta(t_0) < 1$, $k_0(t_0), \delta_0(t_0) > 0$ such that for all $V \in \widehat{\mathcal{W}}^s$,

$$\sup_{|V| \le \delta_0} \sum_{V_i} |J_{V_i} T|_*^t < \theta^t, \quad \textit{for all } t \ge t_0.$$

The proof is similar to the standard estimate: near a tangential collision, $\sum_{k \ge k_0} |J_{V_i} T|_*^t \sim \sum_{k \ge k_0} k^{-qt} \le C k_0^{-1}$. Then k_0 can be chosen large enough (and δ_0 small enough) to make θ^t arbitrarily close to Λ^{-t}, where $\Lambda = 1 + 2\mathcal{K}_{\min}\tau_{\min}$.

A Definition of Topological Pressure

- Define $\mathcal{S}_n = \cup_{i=0}^n T^{-i}\mathcal{S}_0$,
  $\mathcal{S}_n^{\mathbb{H}} = \cup_{i=0}^n T^{-i}\mathcal{S}_0^{\mathbb{H}}$
- Let $\mathcal{M}_0^n$ = connected components of $M \setminus \mathcal{S}_n$,
- $\mathcal{M}_0^{n,\mathbb{H}}$ = connected components of $M \setminus (\mathcal{S}_{n-1}^{\mathbb{H}} \cup T^{-n}\mathcal{S}_0)$



$M \setminus \mathcal{S}_n$

Define for $t > 0$,

- $$Q_n(t) := \sum_{A \in \mathcal{M}_0^{n,\mathbb{H}}} \sup_{x \in A \cap M'} |J^sT^n(x)|^t, \qquad M' = M \setminus (\cup_{n\in\mathbb{Z}}\mathcal{S}_n)$$
- $$P_*(t) := \lim_{n\to\infty} \frac{1}{n} \log Q_n(t)$$
- The limit exists since the sequence $\log Q_n(t)$ is subadditive: $Q_{n+k}(t) \le Q_n(t)Q_k(t)$. It follows, $Q_n(t) \ge e^{nP_*(t)}$.

Properties of $P_*(t)$ and Variational Inequality

Theorem

For a finite horizon Sinai billiard:

a) $P_*(t)$ is a convex, continuous, decreasing function for $t > 0$;

b) $P_*(t)$ satisfies a variational inequality,

$$P_*(t) \geq P(t) = \sup\left\{h_\mu(T) - t\int \log J^uT\,d\mu : \mu\ T\text{-inv. prob.}\right\}$$

Proof. (a) follows from $Q_n(\alpha t + (1-\alpha)s) \leq Q_n(t)^\alpha Q_n(s)^{1-\alpha}$.
(b) relies on the continuation of singularities property. This implies that setting $\mathcal{P} = \mathcal{M}_0^1$, then the elements of $\mathcal{P}_{-n}^n = \bigvee_{i=-n}^n T^{-i}\mathcal{P}$ are simply connected. This plus the uniform hyperbolicity of T implies $\mathcal{P}$ is a generating partition. Then using that $\int_M \log J^sT\,d\mu = -\int_M \log J^uT\,d\mu$ for an invariant measure μ, a standard estimate (e.g. [Walters '82]) implies $h_\mu(T) - t\int_M \log J^uT\,d\mu \leq P_*(t)$. □

Definition of $t_* > 1$

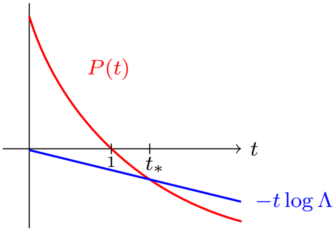
Want to prove that $P_*(t) = P(t)$ for $t \in (0, t_*)$ for some $t_* > 1$.

To do this, need to prove exact exponential growth of $Q_n(t)$:

$$\exists C_2 > 0 \;\; s.t. \;\; e^{nP_*(t)} \le Q_n(t) \le C_2 e^{nP_*(t)} \,,$$

and uniform growth along stable curves,

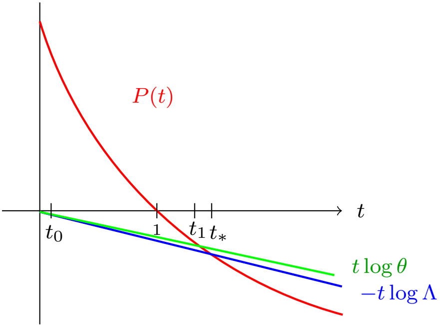
$$\exists c_0 > 0 \, s.t. \, \forall W \in \widehat{\mathcal{W}}^s, |W| \ge \delta_1, \sum_{W_i \in \mathcal{G}_n(W)} |J_{W_i} T^n|^t_{C^0(W_i)} \ge c_0 Q_n(t) \,.$$



- $t_* := \sup\{t > 0 : -t \log \Lambda < P(t)\}$
  Pressure Gap: $\Lambda^{-t} < e^{P(t)}$ for $t < t_*$
- Two cases: $t \in (0, 1]$ and $t \in (1, t_*)$.
- Fix $t_0 > 0$ and prove above estimates for $t \in [t_0, 1]$.
- Fix $t_1 < t_*$ and prove above estimates for $t \in [1, t_1]$.

Restricting to $t \in [t_0, t_1]$

Fixing $t_0 > 0$ and $t_1 < t_*$, we can choose $\theta < 1$ such that the intersection point between $t \log \theta$ and $P(t)$ is to the right of t_1.



Then we can choose q, k_0 and δ_0 so that the one-step expansion holds for the chosen θ uniformly for all $t \in [t_0, t_1]$.

Note: For $t < t_1$, $\theta^t < e^{P(t)} \leq e^{P_*(t)}$.

Growth Lemmas for $t \in [t_0, 1]$

For $\delta_1 < \delta_0$, let $\mathcal{G}_n^{\delta_1}(W)$ denote the analogous collection as $\mathcal{G}_n(W)$, but with respect to the length scale δ_1 rather than δ_0.

Lemma ('Long' elements of $\mathcal{G}_n(W)$ carry most weight)

$\forall \varepsilon > 0\ \exists \delta_1, n_1 > 0$ *s.t* $\forall W \in \widehat{\mathcal{W}}^s$ *with* $|W| \geq \delta_1/3$ *and all* $n \geq n_1$,

$$\sum_{\substack{W_i \in \mathcal{G}_n^{\delta_1}(W_i) \\ |W_i| < \delta_1/3}} |J_{W_i} T^n|_{C^0(W_i)}^t \leq \varepsilon \sum_{W_i \in \mathcal{G}_n^{\delta_1}(W)} |J_{W_i} T^n|_{C^0(W_i)}^t$$

Proof uses one-step expansion and grouping according to most recent long ancestor, together with the following lower bound:

$$\sum_{W_i \in \mathcal{G}_k^{\delta_1}(V)} |J_{W_i} T^k|_{C^0(W_i)}^t = \sum_{W_i \in \mathcal{G}_k^{\delta_1}(V)} |J_{W_i} T^k|_{C^0(W_i)} |J_{W_i} T^k|_{C^0(W_i)}^{t-1}$$

$$\geq C_1 \Lambda^{k(1-t)} \sum_{W_i \in \mathcal{G}_k^{\delta_1}(V)} \frac{|T^k W_i|}{|W_i|} \geq C_1 \Lambda^{k(1-t)} |V| \delta_1^{-1} .$$

Prevalence of 'long' partition elements

Define $\mathcal{A}_n(\delta) = \{A \in \mathcal{M}_0^{n,\mathbb{H}} : \text{diam}^u(T^n A) \geq \delta/3\}$.

Lemma ('Long' elements of $\mathcal{M}_{-n}^{0,\mathbb{H}}$ carry most weight)

There exist $\delta_2 > 0$ and $c_0 > 0$ such that

$$\sum_{A \in \mathcal{A}_n(\delta_2)} \sup_{x \in A} |J^s T^n(x)|^t \geq c_0 \, Q_n(t) \, , \ \forall n \in \mathbb{N} \, , \ \forall t \in [t_0, 1] \, .$$

Uses version of one-step expansion for elements of $\mathcal{M}_{-n}^{0,\mathbb{H}}$, as well as the following distortion bound:

$\exists C > 0$ s.t. for all $n \geq 1$, if $W_1, W_2 \in \widehat{\mathcal{W}}_{\mathbb{H}}^s$ are such that $W_1, W_2 \subset A \in \mathcal{M}_0^{n,\mathbb{H}}$, and all $x \in W_1$, $y \in W_2$,

$$\left| \log \frac{J_{W_1} T^n(x)}{J_{W_2} T^n(y)} \right| \leq C \, .$$

Uses that the stable cones are globally defined, even though E^s is only measurable.

Uniform Growth for $W \in \widehat{\mathcal{W}}^s$ and Supermultiplicativity

Proposition

a) There exists $c_1 > 0$ s.t. for any $W \in \widehat{\mathcal{W}}^s$ with $|W| \ge \delta_1/3$,

$$\sum_{W_i \in \mathcal{G}_n(W)} |J_{W_i} T^n|^t_{C^0(W_i)} \ge c_1 Q_n(t)\,,\ \forall n \ge 1\,,\ \forall t \in [t_0, 1]\,.$$

b) There exists $c_2 > 0$ s.t. for all $k, n \ge 1$,

$$Q_{n+k}(t) \ge c_2 Q_n(t) Q_k(t)\,.$$

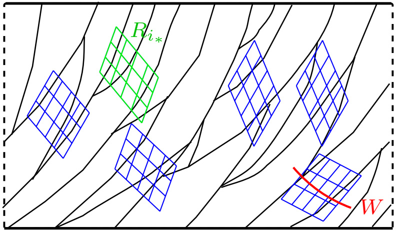
(b) follows from (a) and first growth lemma, since

$$\sum_{W_i \in \mathcal{G}^{\delta_1}_{n+k}(W)} |J_{W_i} T^{n+k}|^t_{C^0} \ge C \sum_{V_j \in L^{\delta_1}_n(W)} |J_{V_j} T^n|^t_{C^0} \sum_{W_i \in \mathcal{G}^{\delta_1}_k(V_j)} |J_{W_i} T^k|^t_{C^0}$$

Immediate corollary of (b) is **exact exponential growth of $Q_n(t)$**:

$$e^{nP_*(t)} \le Q_n(t) \le 2c_2^{-1} e^{nP_*(t)} \quad \forall n \ge 1\,,\ \forall t \in [t_0, 1]\,.$$

Justification for (a) Lower Bound on Growth



$M \setminus \mathcal{S}_{-n}$

- 'Cover' M with $k(\delta_2)$ Cantor rectangles R_i s.t. any stable/ unstable curve of length $\delta_2/3$ properly crosses at least one R_i
- $\mathcal{A}_n^i := \{A \in \mathcal{A}_n(\delta_2) \subset \mathcal{M}_0^{n,\mathbb{H}} : T^n A \text{ properly crosses } R_i\}$
- $\exists i_*$ s.t. $\sum_{A \in \mathcal{A}_n^{i_*}} \sup_A |J^s T^n|^t \geq \frac{c_0}{k} Q_n(t)$
- Take $W \in \widehat{\mathcal{W}}^s$ with $|W| \geq \delta_1/3 \geq \delta_2/3$. Crosses one R_j.
- Use mixing of SRB measure to ensure that $V \subset T^{-N}W$ crosses R_{i_*}, N depends only on δ_2.
- Then $\sum_{W_i \in \mathcal{G}_n(V)} |J_{W_i} T^n|_{C^0}^t$ will be comparable to $\sum_{A \in \mathcal{A}_n^{i_*}} \sup_A |J^s T^n|^t$ using our generalized distortion bound.

Growth Lemmas for $t > 1$

The main issue for $t > 1$ is getting a lower bound on the sum over $\mathcal{G}_n(t)$. We do this by interpolation.

Holder inequality: $t > 1$, choose $s < 1$, $\alpha \in (0,1)$ s.t. $1 = \alpha t + (1-\alpha)s$.

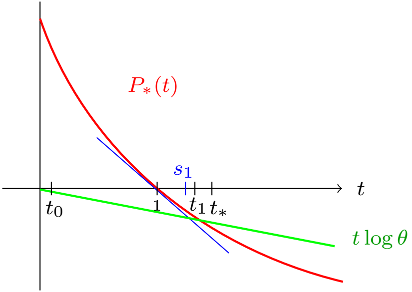
$$\sum_i a_i = \sum_i a_i^{\alpha t + (1-\alpha)s} \leq \left(\sum_i a_i^t \right)^{\alpha} \left(\sum_i a_i^s \right)^{1-\alpha}$$

This implies,

$$\sum_{W_i \in \mathcal{G}_n(W)} |J_{W_i} T^n|_{C^0}^t \geq \frac{\left(\sum_i |J_{W_i} T^n|_{C^0}\right)^{1/\alpha}}{\left(\sum_i |J_{W_i} T^n|_{C^0}^s\right)^{(1-\alpha)/\alpha}} \geq C e^{n P_*(s)(\alpha-1)/\alpha}$$

$\alpha = \frac{1-s}{t-s} \implies \frac{\alpha-1}{\alpha} = \frac{1-t}{1-s}$. So we can make the lower bound arbitrarily close to $e^{-n(t-1)\chi}$ where $\chi = \lim_{s \to 1^-} \frac{P_*(s)}{1-s}$.

Interpolating from $t = 1$ to $t = t_1$



- Set $s_1 =$ intersection point of sub-tangent to $P_*(t)$ at $t = 1$ with line $t \log \theta$.

- Our interpolation gives a lower bound up to s_1:
  $\sum_{W_i \in \mathcal{G}_n(W)} |J_{W_i} T^n|_{C^0}^t \geq C' e^{-n(t-1)\chi}$ for all $t \in [1, s_1]$
  Use this to prove growth lemmas since $\theta^t < e^{\chi(t-1)}$ if $t < s_1$.
- Next interpolate from s_1 to s_2, where s_2 is the intersection point of the subtangent to $P_*(t)$ at $t = s_1$ with the line $t \log \theta$. Continuing inductively, this process accumulates on t_* and passes t_1 in finitely many steps. So we establish the growth lemmas and exact exponential growth of $Q_n(t)$ with constants depending only on t_0 and t_1.