Anisotropic Banach Spaces and Thermodynamic Formalism for Dispersing Billiard Maps
Lecture 2: Geometry of Dispersing Billiards

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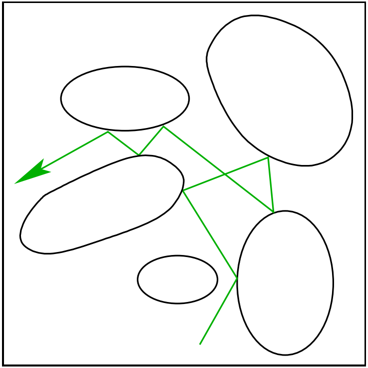
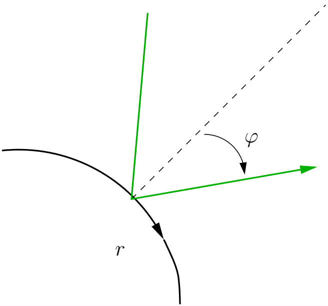
Periodic Lorentz Gas (Sinai Billiard) [Sinai '68]

- Billiard table $\mathcal{Q} = \mathbb{T}^2 \setminus \cup_i B_i$; scatterers B_i.
- Boundaries of scatterers are $\mathcal{C}^3$ and have strictly positive curvature.
- Billiard flow is given by a point particle moving at unit speed with elastic collisions at the boundary



Finite horizon condition: there is an upper bound on the free flight time between collisions. Otherwise **Infinite horizon**.

The Associated Billiard Map

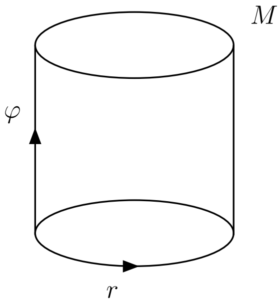


- $r =$ position coordinate oriented clockwise on boundary of scatterer ∂B_i
- $\varphi =$ angle outgoing trajectory makes with normal to scatterer

$M = \big(\cup_i \, \partial B_i \big) \times [-\frac{\pi}{2}, \frac{\pi}{2}]$, the natural "collision" cross-section for the billiard flow.

$T : (r, \varphi) \to (r', \varphi')$ is the first return map: the **billiard map**.

- a hyperbolic map with singularities



Statistical Properties with respect to SRB Measure

T preserves a smooth invariant measure on M, $\mu_{\text{SRB}} = \cos\varphi \, dr \, d\varphi$

With respect to this measure, many statistical properties have been proved using a variety of techniques.

- μ_{SRB} is ergodic [Sinai '70] and Bernoulli [Gallavotti, Ornstein '74]
- Countable Markov partitions and Markov "sieves" [Bunimovich, Sinai '80, '81], [Bunimovich, Chernov, Sinai '90, '91]
  - Central Limit Theorem
- Young Towers
  - exponential decay of correlations, [Young '98]
  - almost sure invariance principle [Melbourne, Nicol '05]
  - local moderate and large deviations, [Melbourne, Nicol '08], [Young, Rey-Bellet '08]
- Coupling arguments via standard pairs [Chernov '06], [Chernov, Dolgopyat '09]
- Transfer operator techniques [D., Zhang '11, '13, '14]

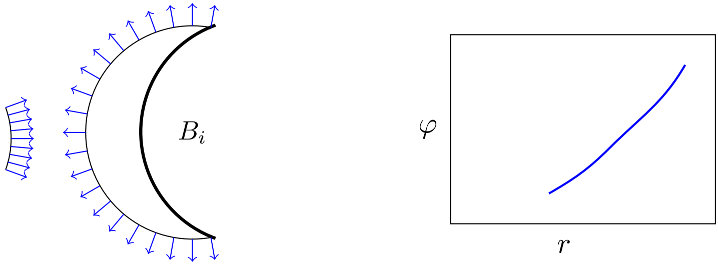
Goal of Lectures

Goal of present lectures: Describe recent work extending analysis to other invariant measures, namely, equilibrium states associated to geometric potentials, $g_t = -t \log J^u T$, $t \in [0, t_*)$ for some $t_* > 1$.

Goal for today: Recall some geometric facts about dispersing billiards that we will use in subsequent lectures: hyperbolicity, distortion, complexity, growth lemma.

Reference: N. Chernov and R. Markarian, *Chaotic Billiards*, Mathematical Surveys and Monographs **127** (2006), 330 pp.

Hyperbolicity away from Singularities



A dispersing wavefront before and after collision.

The wavefront projects to a curve with positive slope on B_i.

Positive slope in $M \implies$ unstable curve

Negative slope in $M \implies$ stable curve

Hyperbolicity: Stable and Unstable Cones

For both finite and infinite horizon, two global families of cones:

$$\mathcal{C}^u = \left\{ (dr, d\varphi) : \mathcal{K}_{\min} \le \frac{d\varphi}{dr} \le \mathcal{K}_{\max} + \frac{1}{\tau_{\min}} \right\}$$
$$\mathcal{C}^s = \left\{ (dr, d\varphi) : -\mathcal{K}_{\min} \ge \frac{d\varphi}{dr} \ge -\mathcal{K}_{\max} - \frac{1}{\tau_{\min}} \right\}$$

$\tau_{\min} > 0$ is minimum time between consecutive collisions

$\mathcal{K}_{\min\,/\,\max} =$ min/max curvature of scatterers

Strict invariance:

$$DT(x)\mathcal{C}^u \subsetneq \mathcal{C}^u \quad \text{and} \quad DT(x)^{-1}\mathcal{C}^s \subsetneq \mathcal{C}^s$$

Minimum expansion: $\Lambda := 1 + 2\mathcal{K}_{\min}\tau_{\min}$.

$$\exists C_0 > 0 \quad \text{s.t.} \quad \|DT^n(x)v\| \ge C_0\Lambda^n\|v\| \quad \forall v \in \mathcal{C}^u$$

and similarly for stable cone under $DT^{-n}(x)$.

Invariant Families of Stable/Unstable Curves

- Call a smooth curve $W \subset M$ **stable** (or cone-stable) if the tangent vector to W at each point belongs to $\mathcal{C}^s$.
- Define

$$\widehat{\mathcal{W}}^s = \{\text{stable curves with curvature bounded by } D_0 > 0 \\ \text{and length at most } \delta_0 > 0\ \}$$

  Since T is piecewise C^2 and uniformly hyperbolic away from its singularities, we can choose $D_0 > 0$ such that $\widehat{\mathcal{W}}^s$ is invariant under T^{-1}, up to subdivision of long curves.
- Define $\mathcal{W}^s \subset \widehat{\mathcal{W}}^s$ real local stable manifolds
- Similarly, define a T-invariant set $\widehat{\mathcal{W}}^u$ of (cone-) unstable curves, and local unstable manifolds $\mathcal{W}^u$.

Singularities

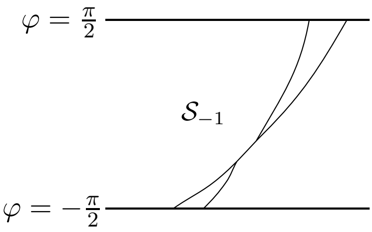
Tangential collisions create singularity curves for T.

- Let $\mathcal{S}_0 = \{\varphi = \pm\frac{\pi}{2}\}$.
- $\mathcal{S}_n = \bigcup_{i=0}^{n} T^{-i}\mathcal{S}_0$ is the singularity set for T^n, $n \geq 1$.
- $\mathcal{S}_{-n} = \bigcup_{i=0}^{n} T^{i}\mathcal{S}_0$ is the singularity set for T^{-n}, $n \geq 1$.

T^n is discontinuous at the set of decreasing curves $\mathcal{S}_n$ and T^{-n} is discontinuous at the set of increasing curves $\mathcal{S}_{-n}$.

Important fact: $\mathcal{S}_n$ is uniformly transverse to $\mathcal{C}^u$ and $\mathcal{S}_{-n}$ is uniformly transverse to $\mathcal{C}^s$.

- **Continuation of Singularities**
  Every curve in $\mathcal{S}_n \setminus \mathcal{S}_0$ is part of a monotonic piecewise smooth curve belonging to $\mathcal{S}_n$ which terminates on $\mathcal{S}_0$.



Linear Bound on Complexity

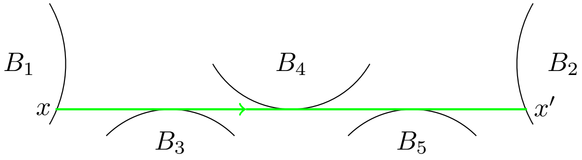
Want expansion due to hyperbolicity to beat cutting due to singularities. In the finite horizon case, there is a linear bound due to Bunimovich.

For $x \in M$, let $N(\mathcal{S}_n, x)$ denote the number of singularity curves in $\mathcal{S}_n$ that meet at x. Define $N(\mathcal{S}_n) = \sup_{x \in M} N(\mathcal{S}_n, x)$.

Lemma (Bunimovich, Chernov, Sinai '90)

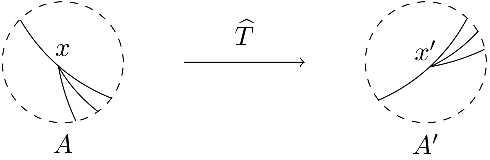
Assume finite horizon. There exists $K > 0$ depending only on the configuration of scatterers such that $N(\mathcal{S}_n) \leq Kn$ for all $n \geq 1$.

Idea of Proof. Let $x, x' \in M$ lie on a straight billiard trajectory with one or more tangential collisions between.



Linear Bound on Complexity

Let A, A' be neighborhoods of x, x' in M, partitioned into sectors $A_1, \dots A_k \subset A$ and $A'_1, \dots A'_k \subset A$ such that $T^{n_j} A_j = A'_j$.
Set $\widehat{T}|_{A_j} := T^{n_j}$



- Assume $N(\mathcal{S}_{n-1}) \leq K(n-1)$.
- Let $N(\mathcal{S}_i | A'_j, x')$ denote the number of curves in $\mathcal{S}_i$ passing through x' and lying in A'_j.
- $N(\mathcal{S}_n, x) \leq k + \sum_j N(\mathcal{S}_{n-n_j} | A'_j, x') \leq k + \sum_j N(\mathcal{S}_{n-1} | A'_j, x')$
- So $N(\mathcal{S}_n, x) \leq k + K(n-1) \leq Kn$ if $k \leq K$. $\square$

The proof uses that the flow is continuous.

Distortion Control and Extended Singularity Set

When $T(x)$ is near $\mathcal{S}_0$, $DT(x)$ becomes large:

$$\|DT(x)|_{E^u}\| \sim \frac{1}{\cos\varphi(Tx)} \sim d(x,\mathcal{S}_1)^{-1/2}, \quad \|DT(x)|_{E^s}\| \sim \cos\varphi(x)$$

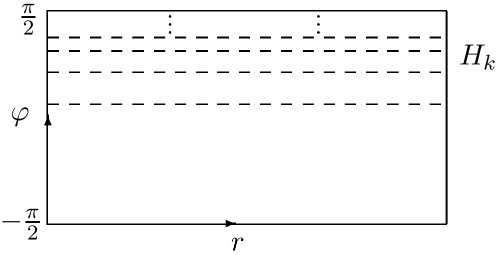
Indeed, $\det DT(x) = \frac{\cos\varphi(x)}{\cos\varphi(Tx)}$, which can be viewed as the product of stable and unstable Jacobians.

To control distortion, partition M into **homogeneity strips** $H_{\pm k}$,

$$H_k = \left\{ \frac{\pi}{2} - \frac{1}{k^q} < \varphi < \frac{\pi}{2} - \frac{1}{(k+1)^q} \right\}$$

and similarly for H_{-k}, $|k| \geq k_0$. Standard choice: $q = 2$

Distortion Control and Extended Singularity Set



- Define $\widehat{\mathcal{W}}^s_{\mathbb{H}}$ = homogeneous elements of $\widehat{\mathcal{W}}^s$
- Distortion depends on the exponent q: If $T^i W \subset \widehat{\mathcal{W}}^s_{\mathbb{H}}$, $i = 0, \ldots, n$, then

$$\log \frac{J_W T^n(x)}{J_W T^n(y)} \le C_d d(x,y)^{1/(q+1)} \qquad \forall x, y \in W$$

- But the singularity set becomes countable: $\mathcal{S}_0^{\mathbb{H}} := \mathcal{S}_0 \cup (\cup_{|k| \ge k_0} \partial H_k)$, and $\mathcal{S}_{\pm n}^{\mathbb{H}} = \bigcup_{i=0}^n T^{\mp i} \mathcal{S}_0^{\mathbb{H}}$.
- Need a new complexity bound.

One-step Expansion [Chernov '06]

Define an adapted metric in the tangent space $dx = (dr, d\varphi)$ by,

$$\|dx\|_* = \frac{\mathcal{K}(x) + |\mathcal{V}|}{\sqrt{1+\mathcal{V}^2}} \|dx\| \,,$$

where $\mathcal{V} = d\varphi/dr$ and $\mathcal{K}(x)$ is the curvature of the scatterer at x.

- $\|DT(x)^{-1} dx\|_* \geq \Lambda \|dx\|_*$ for all stable vectors dx, where $\Lambda = 1 + 2\mathcal{K}_{\min}\tau_{\min}$.

For $V \in \widehat{\mathcal{W}}^s_{\mathbb{H}}$, let $\{V_i\}_i$ = homogeneous connected comp. of $T^{-1}V$.

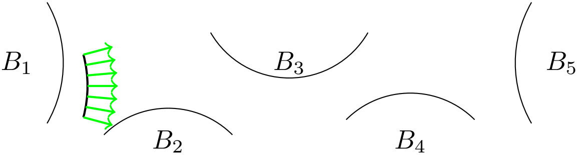
Lemma (One-step expansion)

There exists $\theta < 1$ such that for all $V \in \widehat{\mathcal{W}}^s_{\mathbb{H}}$,

$$\limsup_{\delta \downarrow 0} \sup_{|V| \leq \delta} \sum_{V_i} |J_{V_i} T|_* < \theta \,,$$

where $|\cdot|_*$ denotes the sup norm in the adapted metric.

Proof of One-step Expansion for Finite Horizon



- A short stable curve can be cut by at most $\tau_{\max}/\tau_{\min}$ tangential collisions under T^{-1}.
- All but one of these collisions is nearly grazing.
- Near the grazing collisions, $V_k \subset H_k$ and,

$$\sum_{k \geq k_0} |J_{V_k} T|_* \leq C \sum_{k \geq k_0} k^{-q} \leq C' k_0^{1-q} \quad \text{if } q > 1$$

- Fix $\varepsilon > 0$ and choose k_0 large enough that $k_0^{q-1} \frac{\tau_{\max}}{\tau_{\min}} \leq \varepsilon$.
- Choose $\delta_0 > 0$ so that a stable curve of length δ_0 must map into homogeneity strips of index $|k| \geq k_0$ at the nearly tangential collisions.
- Then $\theta = \Lambda^{-1} + \varepsilon$ satisfies the lemma. □

Growth Lemma

Consequence of one-step expansion is the following growth lemma.

- For $W \in \widehat{\mathcal{W}}^s$, partition $T^{-1}W$ into maximal connected homogeneous components. Subdivide any curve longer than δ_0 into curves of length between $\delta_0/2$ and δ_0. Call this collection $\mathcal{G}_1(W)$.
- Define inductively, $\mathcal{G}_n(W) = \{\mathcal{G}_1(W_i) : W_i \in \mathcal{G}_{n-1}(W)\}$.
- Let $L_n(W)$ denote those $W_i \in \mathcal{G}_n(W)$ such that $|W_i| \geq \delta_0/3$.
- Let $\mathcal{I}_n(W)$ denote those $W_i \in \mathcal{G}_n(W)$ such that $T^j W_i \subset V_j \in \mathcal{G}_{n-j}(W)$ with $|V_j| < \delta_0/3$ for all $j = 0, \ldots, n-1$. W is the **most recent long ancestor** of W_i.

Lemma

There exists $C_1 > 0$ such that for all $W \in \widehat{\mathcal{W}}^s$ and all $n \geq 1$,

$$\sum_{W_i \in \mathcal{G}_n(W)} |J_{W_i} T^n|_{C^0(W_i)} \leq C_1 \, .$$

Proof of Growth Lemma

Organize $W_i \in \mathcal{G}_n(W)$ by most recent long ancestor.

- If $W_i \in L_n(W)$, then W_i is its own most recent long ancestor.
- Otherwise, $W_i \in \mathcal{I}_j(V_k)$ for some $V_k \in L_{n-j}(W)$, $j \geq 1$.
- Or $W \in \mathcal{I}_n(W)$, whether W is long or short.

$$\begin{aligned}
\sum_{W_i \in \mathcal{G}_n(W)} |J_{W_i} T^n|_{C^0} &\leq \sum_{j=1}^{n} \sum_{V_k \in \mathcal{G}_{n-j}(W)} |J_{V_k} T^{n-j}|_{C^0} \sum_{W_i \in \mathcal{I}_j(V_k)} |J_{W_i} T^j|_{C^0} \\
&\leq C_* \theta^n + \sum_{j=1}^{n-1} \sum_{V_k \in \mathcal{G}_{n-j}(W)} |J_{V_k} T^{n-j}|_{C^0} C_* \theta^j \\
&\leq C_* \theta^n + \sum_{j=1}^{n-1} \sum_{V_k \in \mathcal{G}_{n-j}(W)} e^{C_d \delta_0^{1/q}} \frac{|T^{n-j} V_k|}{|V_k|} C_* \theta^j \\
&\leq C_* \theta^n + \sum_{j=1}^{n-1} C' \delta_0^{-1} |W| \theta^j \leq C_* \theta^n + C'' \delta_0^{-1} |W|
\end{aligned}$$