Anisotropic Banach Spaces and Thermodynamic Formalism for Dispersing Billiard Maps

Mark Demers
Fairfield University
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Overview of Course

Main Goal of Lectures: Introduce functional analytic framework to study transfer operators associated to hyperbolic systems, and use these tools to present recent progress regarding equilibrium states and topological pressure for dispersing billiards.

Plan for Lectures

1. Gentle introduction to Banach spaces for hyperbolic systems
   - Smooth expanding maps, contracting maps, Baker's map
2. Geometry of dispersing billiards
   - Hyperbolicity, singularities and complexity
3. Thermodynamic formalism for billiards
   - Geometric potentials and topological pressure, initial results
4. Thermodynamic formalism for billiards
   - Banach spaces and unique equilibrium states via spectral gap
5. Measure of maximal entropy
   - Topological entropy, motivation and exact exponential growth
6. Measure of maximal entropy
   - Banach spaces and unique equilibrium state via 'bare hands'

Transfer Operator or Ruelle-Perron-Frobenius Operator

Transformation $T : X \circlearrowleft$. **Transfer operator** $\mathcal{L}$ associated to T acts on a distribution μ by

$$\mathcal{L}\mu(\psi) = \mu(\psi \circ T), \qquad \psi \text{ a test function, say } C^\alpha.$$

If $d\mu = f dm$ is a measure abs. cont. w.r.t. m, then

$$\int \mathcal{L}f\, \psi\, dm = \int f\, \psi \circ T\, dm,$$

so that pointwise

$$\mathcal{L}f(x) = \sum_{y \in T^{-1}x} \frac{f(y)}{JT(y)},$$

where JT is the Jacobian of T with respect to m, represents the density of the measure $T_*\mu$, i.e. $d(T_*\mu) = \mathcal{L}f\, dm$.

$\mathcal{L} =$ Linear operator which governs evolution of measures, acting on some Banach space of functions, measures or distributions.

Weighted or Generalized Transfer Operator

Generalize the transfer operator by including a potential function g,

$$\mathcal{L}_g \mu(\psi) = \mu(e^g \, \psi \circ T) \, .$$

This allows the transfer operator to be used to study a variety of equilibrium states associated with some classes of potentials (often Hölder continuous). For example, the measure of maximal entropy.

In this case, one constructs an invariant measure μ using the left and right maximal eigenvectors of $\mathcal{L}_g$:

$\mathcal{L}_g \nu = \lambda \nu$ and $\mathcal{L}_g^* \tilde{\nu} = \lambda \tilde{\nu}$, where $\mathcal{L}_g^*$ is the dual to $\mathcal{L}_g$ on a suitable Banach space. Then

$$\mu(\psi) = \nu(\psi \tilde{\nu}),$$

is an invariant measure for T (Parry construction).

Today: Discuss the case $g = 0$.

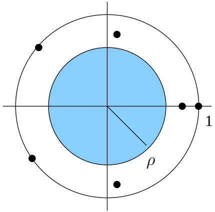
Spectral Decomposition of $\mathcal{L}$ via Quasi-Compactness

Goal: Use spectral properties of $\mathcal{L}$ acting on an appropriate Banach space to gain dynamical information about T.

Method: Prove $\mathcal{L}$ is **quasi-compact** on some Banach space $\mathcal{B}$:

$\exists \rho < 1$ s.t. the spectrum of $\mathcal{L}$ outside disk of radius ρ is finite-dimensional.

- Eigenspace corresponding to $1 =$ invariant measures
- Periodic behavior of $\mathcal{L}$ corresponds to eigenvalues other than 1 on the unit circle



- If 1 is a simple eigenvalue and we can eliminate periodicity, we can conclude that $\mathcal{L}$ has a spectral gap

Consequences of the Spectral Gap

$$\int f\,\psi \circ T^n\,dm = \mu_f(\psi \circ T^n) = \mathcal{L}^n \mu_f(\psi)\,, \quad \text{where } d\mu_f = f dm.$$

The presence of a spectral gap allows us to establish exponential decay of correlations and convergence to equilibrium, along with many limit theorems:

- Central Limit Theorem
- Large deviation estimates
- Almost-sure invariance principles

The functional analytic framework gives a unified (and often simplified) approach for handling perturbations as well, either through classical perturbation theory, or the weakened form due to [Keller, Liverani '99].

How can we apply this approach to specific systems?

Quasi-Compactness via Dynamical Inequalities

Dynamical method to estimate the essential spectral radius [Hennion '93] following [Doeblin, Fortet '37], [Ionescu-Tulcea, Marinescu '50], [Lasota, Yorke '73].

Essential ingredients:

- Two Banach spaces $(\mathcal{B}, \| \cdot \|)$ and $(\mathcal{B}_w, | \cdot |_w)$, with an embedding $\mathcal{B} \hookrightarrow \mathcal{B}_w$ such that $|f|_w \leq \|f\|$ for $f \in \mathcal{B}$
- The unit ball of $\mathcal{B}$ is compactly embedded in $\mathcal{B}_w$
- (Lasota-Yorke/Doeblin-Fortet inequalities)
  $\exists\, C > 0$ and $\rho < 1$ such that for all $f \in \mathcal{B}$, $n \geq 0$,

$$\|\mathcal{L}^n f\| \leq C\rho^n \|f\| + C|f|_w$$
$$|\mathcal{L}^n f|_w \leq C|f|_w$$

Then $\mathcal{L} : \mathcal{B} \circlearrowleft$ has essential spectral radius $\leq \rho$.

(Note: The above inequalities imply that the spectral radius is ≤ 1, but for reasonable choices of $\mathcal{B}$, it is actually 1.)

Ex 1: Expanding maps of the interval

[Lasota, Yorke '73]

$T : [0,1] \circlearrowleft$, piecewise C^2, $\exists \lambda < 1$ s.t. $|T'| \geq \lambda^{-1} > 1$
m denotes Lebesgue measure

- Weak space, $\mathcal{B}_w = L^1(m)$
- Strong space, $\mathcal{B} = BV$ with norm

$$\|f\|_{BV} = \sup_{\psi \in C^1, |\psi|_\infty \leq 1} \int f \, \psi' \, dm$$

- $\mathcal{L} f(x) = \sum_{y \in T^{-1}x} \frac{f(y)}{|T'(y)|}$ for $f \in L^1(m)$

One Lasota-Yorke inequality is immediate: $|\mathcal{L}^n f|_1 \leq |f|_1$ since

$$\int |\mathcal{L} f| \, dm \leq \int \mathcal{L}|f| \, dm = \int |f| \, dm \, .$$

Ex 1: Expanding maps of the interval

Estimate in the smooth case:

$$\int \mathcal{L}f\,\psi'\,dm = \int f\,\psi'\circ T\,dm = \int f\left(\frac{\psi\circ T}{T'}\right)' dm + \int f\frac{\psi\circ T}{(T')^2}T''\,dm$$
$$\leq \|f\|_{BV}|\psi|_\infty\lambda + |f|_1|\psi|_\infty C_{\mathsf{dist}}$$

Taking appropriate suprema,

$$\|\mathcal{L}f\|_{BV} \leq \lambda\|f\|_{BV} + C|f|_1$$

The case with discontinuities is handled similarly by splitting the integral over intervals of differentiability for T.

So $\mathcal{L}$ acting on BV is quasi-compact. If T is mixing, then $\mathcal{L}$ has a spectral gap.

Note: essential spectral radius bounded by $\lambda = \sup_{x\in I}\frac{1}{|T'(x)|} < 1$.

Ex 2: A contracting map of the interval

[Liverani '04]

$T : [0, 1] \circlearrowleft$, $T \in C^1$, $\exists \lambda < 1$ s.t. $|T'| \leq \lambda$, $\exists c \in I, T(c) = c$

- Expect convergence of measures to δ_c so usual function spaces will not work
- Consider spaces of distributions: Dual spaces to Hölder continuous test functions

$$|\psi|_{C^\alpha} = |\psi|_{C^0} + H^\alpha(\psi)\,, \qquad H^\alpha(\psi) = \sup_{x \neq y} \frac{|\psi(x) - \psi(y)|}{|x - y|^\alpha}$$

Let $f \in C^1(I)$ and let $d\mu = f dm$. Choose $\alpha < 1$ and define

$$|\mu|_w = \sup_{|\psi|_{C^1} \leq 1} |\mu(\psi)| \quad \text{and} \quad \|\mu\| = \sup_{|\psi|_{C^\alpha} \leq 1} |\mu(\psi)|$$

- $\mathcal{B}$ is the completion of C^1 in the $\|\cdot\|$-norm
- $\mathcal{B}_w$ is the completion of C^1 in the $|\cdot|_w$-norm
- Unit ball of $\mathcal{B}$ compact in $\mathcal{B}_w$ since C^1 compact in C^α

Ex 2: A contracting map of the interval

For $\psi \in C^\alpha$, let $\overline{\psi} = \int \psi \circ T \, dm$. Then

$$\mathcal{L}\mu(\psi) = \mu(\psi \circ T - \overline{\psi}) + \mu(\overline{\psi}) \le \|\mu\| |\psi \circ T - \overline{\psi}|_{C^\alpha} + |\mu|_w |\overline{\psi}|_{C^1}$$

Estimate $|\psi \circ T - \overline{\psi}|_{C^\alpha}$ by

$$|\psi \circ T(x) - \overline{\psi}(x)| = |\psi \circ T(x) - \psi \circ T(z)| \le |\psi|_{C^\alpha} \lambda^\alpha$$
$$|\psi \circ T(x) - \overline{\psi}(x) - \psi \circ T(y) + \overline{\psi}(y)| \le |\psi|_{C^\alpha} \lambda^\alpha |x - y|^\alpha$$

Also, $|\overline{\psi}|_{C^1} \le |\psi|_\infty = 1$, so that

$$\|\mathcal{L}\mu\| \le \lambda^\alpha \|\mu\| + |\mu|_w$$

A similar estimate shows that $|\mathcal{L}\mu|_w \le |\mu|_w$

• **Note: We cannot choose $\alpha = 0$ so $\mathcal{B}$ must be larger than the space of measures**

Norms for Hyperbolic systems

Conclusions we can draw from these simple examples:

- When T is expanding, $\mathcal{L}$ improves regularity of functions
- When T is contracting, $\mathcal{L}$ improves regularity in certain spaces of distributions

Moral: Hyperbolic systems have both contracting and expanding directions so by choosing spaces of distributions that are regular in the unstable direction and by averaging (integrating) along stable curves, we are able to define norms in which $\mathcal{L}$ improves regularity.

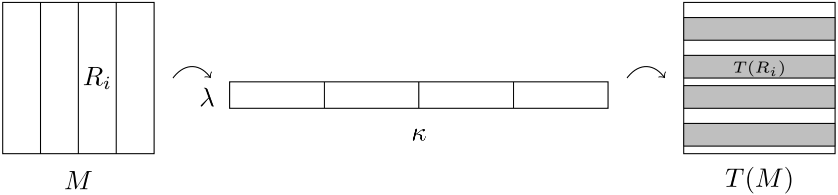
By integrating against Hölder continuous functions on stable curves, we are in spirit defining a notion that is dual to that of **standard pairs**, developed by Dolgopyat and Chernov, which considers the evolution of Hölder densities on unstable curves.

Ex 3: Generalized Baker's Map

$M = [0,1]^2$. Fix $\kappa \in \mathbb{N}$, $\kappa \geq 2$, and $\lambda \in \mathbb{R}$ such that $0 < \lambda \leq 1/\kappa$.

Define a generalized (κ, λ) Baker's transformation $T_{\kappa,\lambda}$:

- Subdivide M into κ vertical rectangles R_i of width $1/\kappa$.
- $T_{\kappa,\lambda}$ affine on each R_i: expands by factor κ horizontally, contracts by factor λ vertically
- $\{T_{\kappa,\lambda}(R_i)\}_i$ have disjoint interiors.



The map $T = T_{\kappa,\lambda}$ with $\kappa = 4$ and $\lambda < 1/4$.

If $\lambda = 1/\kappa$, then T is area preserving; otherwise, dissipative.

Ref: M Demers, *A gentle introduction to anisotropic Banach spaces*, Chaos, Solitons and Fractals (2018)

Transfer Operator

$\mathcal{W}^{s/u} :=$ {vertical/horizontal line segments of length 1 in M}
$\mathcal{W}^{s/u} =$ local stable/unstable manifolds for $T = T_{\kappa,\lambda}$

For $\alpha \in [0,1]$, define $|\psi|_{C^\alpha(\mathcal{W}^s)} = \sup_{W \in \mathcal{W}^s} |\psi|_{C^\alpha(W)}$.
If $\psi \in C^\alpha(\mathcal{W}^s)$, then $\psi \circ T \in C^\alpha(\mathcal{W}^s)$.

Now define $\mathcal{L}$ acting on $(C^\alpha(\mathcal{W}^s))^*$ by

$$\mathcal{L}f(\psi) = f(\psi \circ T), \quad \forall \psi \in C^\alpha(\mathcal{W}^s), f \in (C^\alpha(\mathcal{W}^s))^*$$

If $f \in C^1(M)$, then associate f with the measure fdm, $m =$ Lebesgue measure. Then pointwise,

$$\mathcal{L}f(x) = \frac{f \circ T^{-1}(x)}{\kappa\lambda}$$

Note $\mathcal{L}f = 0$ on $M \setminus T(M)$ and m is conformal, i.e. $\mathcal{L}^*m = m$.

Definition of Norms

Let $f \in C^1(\mathcal{W}^u)$.

Define the **weak norm** of f by

$$|f|_w = \sup_{W \in \mathcal{W}^s} \sup_{\substack{\psi \in C^1(W) \\ |\psi|_{C^1(W)} \le 1}} \int_W f \, \psi \, dm_W$$

$m_w =$ arclength measure on W.

Let $\alpha \in (0,1)$ and define the **strong stable norm** of f by

$$\|f\|_s = \sup_{W \in \mathcal{W}^s} \sup_{\substack{\psi \in C^\alpha(W) \\ |\psi|_{C^\alpha(W)} \le 1}} \int_W f \, \psi \, dm_W$$

On each $W \in \mathcal{W}^s$, these are simply the norms for the contracting map.

Definition of Norms

The strong norm should provide regularity in the unstable direction.

Write $W \in \mathcal{W}^s$ in coordinates:

$$W = \{(s,t) \in M : s = s_W, t \in [0,1]\}$$

Then define $d(W_1, W_2) = |s_{W_1} - s_{W_2}|$,
and for test functions $\psi_i \in C^1(W_i)$, define

$$d_0(\psi_1, \psi_2) = \sup_{t \in [0,1]} |\psi_1(s_{W_1}, t) - \psi_2(s_{W_2}, t)|$$

Choose $\beta \in (0,1)$ with $\beta \leq 1 - \alpha$.

Define the **strong unstable norm** of f by

$$\|f\|_u = \sup_{W_1, W_2 \in \mathcal{W}^s} \sup_{\substack{\psi_i \in C^1(W_i) \\ |\psi_i|_{C^1(W_i)} \leq 1 \\ d_0(\psi_1, \psi_2) = 0}} d(W_1, W_2)^{-\beta} \left| \int_{W_1} f\, \psi_1 - \int_{W_2} f\, \psi_2 \right|$$

Banach spaces $\mathcal{B}$ and $\mathcal{B}_w$

The **strong norm** of f is $\|f\|_{\mathcal{B}} = \|f\|_s + \|f\|_u$

Define the weak space $\mathcal{B}_w$ to be the completion of $C^1(\mathcal{W}^u)$ in the $|\cdot|_w$ norm.

Define the strong space $\mathcal{B}$ to be the completion of $C^1(\mathcal{W}^u)$ in the $\|\cdot\|_{\mathcal{B}}$ norm.

Lemma (Embedding Lemma)

We have the following sequence of continuous embeddings,

$$C^1(M) \hookrightarrow \mathcal{B} \hookrightarrow \mathcal{B}_w \hookrightarrow (C^1(\mathcal{W}^s))^*.$$

Moreover, the embedding $\mathcal{B} \hookrightarrow \mathcal{B}_w$ is relatively compact.

Proof of Relative Compactness

Fix $\varepsilon > 0$. Let $C_1^1(W)$ denote the unit ball of $C^1(W)$.

- Choose $\{\psi_i\}_{i=1}^{N_\varepsilon} \subset C^1([0,1])$ which forms an ε-cover of $C_1^1(W)$ in the $C^\alpha(W)$ norm for all $W \in \mathcal{W}^s$.
- Choose $\{W_j\}_{j=1}^{J_\varepsilon} \subset \mathcal{W}^s$ which forms an ε-cover of $\mathcal{W}^s$ in the metric $d(\cdot,\cdot)$.

Take $f \in C^1(\mathcal{W}^u)$, $W \in \mathcal{W}^s$, $\psi \in C_1^1(W)$. Choose ψ_i s.t. $|\psi - \psi_i|_{C^\alpha(W)} \le \varepsilon$ and W_j s.t. $d(W, W_j) \le \varepsilon$. Then,

$$\left| \int_W f\psi - \int_{W_j} f\psi_i \right| \le \left| \int_W f(\psi - \psi_i) \right| + \left| \int_W f\,\psi_i - \int_{W_j} f\,\psi_i \right|$$
$$\le \|f\|_s |\psi - \psi_i|_{C^\alpha} + d(W, W_j)^\beta \|f\|_u \le \varepsilon^\beta \|f\|_{\mathcal{B}}$$

Taking the supremum over W and ψ implies that

$$\min_{i,j} \big| |f|_w - \ell_{i,j}(f) \big| \le \varepsilon^\beta \|f\|_{\mathcal{B}}\,, \quad \text{where } \ell_{i,j}(f) = \int_{W_j} f\,\psi_i\,. \qquad \square$$

Dynamical Inequalities

Proposition

For any $n \geq 0$ and $f \in \mathcal{B}$,

$$\begin{aligned} \|\mathcal{L}^n f\|_s &\leq \lambda^{\alpha n}\|f\|_s + |f|_w, & (1) \\ \|\mathcal{L}^n f\|_u &\leq \kappa^{-\beta n}\|f\|_u, & (2) \\ |\mathcal{L}^n f|_w &\leq |f|_w. & (3) \end{aligned}$$

Proof: By density of $C^1(\mathcal{W}^u)$ in both $\mathcal{B}$ and $\mathcal{B}_w$, it suffices to prove the bounds for $f \in C^1(\mathcal{W}^u)$.

(1) Let $W \in \mathcal{W}^s$, $\psi \in C^\alpha(W)$, $|\psi|_{C^\alpha(W)} \leq 1$.
We must estimate

$$\int_W \mathcal{L}^n f \, \psi \, dm_W = \int_{T^{-n}W} f \, \psi \circ T^{-n} (\kappa\lambda)^{-n} J^s T^n \, dm_{T^{-1}W} \, ,$$

where $J^s T^n = \lambda^n$ is the stable Jacobian of T along $T^{-n}W$.

Proof of Strong Stable Norm Contraction

Note $T^{-n}W = \bigcup_{i=1}^{\kappa^n} W_i$, $W_i \in \mathcal{W}^s$.

Define $\overline{\psi}_i = \int_{W_i} \psi \circ T^n \, dm_{W_i}$.

$$\int_W \mathcal{L}^n f \, \psi \, dm_W = \kappa^{-n} \sum_{i=1}^{\kappa^n} \int_{W_i} f \, (\psi \circ T^n - \overline{\psi}_i) + \int_{W_i} f \, \overline{\psi}_i$$

As with the contracting map, $|\psi \circ T^n - \overline{\psi}_i|_{C^\alpha(W_i)} \le \lambda^{\alpha n}$, and $|\overline{\psi}_i|_{C^1(W_i)} \le 1$. Thus,

$$\int_W \mathcal{L}^n f \, \psi \, dm_W \le \lambda^{\alpha n} \|f\|_s + |f|_w \, ,$$

which proves (1).

The weak norm estimate (3) is similar, without subtracting $\overline{\psi}_i$.

Proof of Strong Unstable Norm Contraction

(2) W^1, $W^2 \in \mathcal{W}^s$, $|\psi_j|_{C^1(W^j)} \le 1$ s.t. $d_0(\psi_1, \psi_2) = 0$.

There is a 1-1 correspondence between elements of $T^{-n}W^1 = \cup_i W_i^1$ and $T^{-n}W^2 = \cup_i W_i^2$: For each i, W_i^1, W_i^2 lie in a vertical rectangle on which T^n is smooth.

Also, since T preserves horizontal lines, $d_0(\psi_1 \circ T^n, \psi_2 \circ T^n) = 0$ on each pair W_i^1, W_i^2.

$$\int_{W^1} \mathcal{L}^n f \, \psi_1 - \int_{W^2} \mathcal{L}^n f \, \psi_2 = \kappa^{-n} \sum_i \int_{W_i^1} f \, \psi_1 \circ T^n - \int_{W_i^2} f \, \psi_2 \circ T^n$$

$$\le \kappa^{-n} \sum_i d(W_i^1, W_i^2)^\beta \|f\|_u \le \kappa^{-\beta n} d(W^1, W^2)^\beta \|f\|_u$$

Dividing through by $d(W^1, W^2)^\beta$ and taking the appropriate suprema proves (2): $\|\mathcal{L}^n f\|_u \le \kappa^{-\beta n} \|f\|_u$. $\square$

Quasi-Compactness and a Spectral Gap

Theorem

$\mathcal{L}$ is quasi-compact as an operator of $\mathcal{B}$ with spectral radius 1 and essential spectral radius at most $\max\{\lambda^\alpha, \kappa^{-\beta}\} < 1$.
Moreover, $\mathcal{L}$ has a spectral gap on $\mathcal{B}$.

The upper bounds on the essential spectral radius and the spectral radius follow from the dynamical inequality:

$$\|\mathcal{L}^n f\|_{\mathcal{B}} = \|\mathcal{L}^n f\|_s + \|\mathcal{L}^n f\|_u \le \max\{\lambda^{\alpha n}, \kappa^{-\beta n}\}\|f\|_{\mathcal{B}} + |f|_w \, .$$

The fact that $\mathcal{L}^* m = m$ implies that the spectral radius is 1, (since 1 is in the spectrum of $\mathcal{L}^*$ and so also of $\mathcal{L}$) so that $\mathcal{L}$ is quasi-compact. Also, the peripheral spectrum contains no Jordan blocks since $\|\mathcal{L}^n\|_{\mathcal{B}}$ is uniformly bounded.

Peripheral Spectrum: Sketch of Proof

From quasi-compactness and the absence of Jordan blocks,

$$\mathcal{L} = \sum_{j=0}^{N} e^{2\pi\theta_j}\Pi_j + R, \quad \|R\|_{\mathcal{B}} < 1, \; \Pi_j\Pi_k = R\Pi_j = \Pi_j R = 0$$

Since there are no Jordan blocks, $\Pi_j = \lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} e^{-2\pi\theta_j k}\mathcal{L}^k$.

Set $\theta_0 = 0$ and $\mu_0 = \Pi_0 1$. Let $\mathbb{V}_j = \Pi_j(\mathcal{B})$.

a) **Elements of $\mathbb{V} = \oplus_j \mathbb{V}_j$ are measures abs. cont. wrt μ_0**

- Since $\Pi_j(C^1) = \mathbb{V}_j$, for each $\mu \in \mathbb{V}_j$, $\exists f \in C^1(M)$ s.t.

$$|\mu(\psi)| = |\Pi_j f(\psi)| \le \lim_n \sum_{k=0}^{n-1} |f(\psi \circ T^k)| \le |f|_\infty |\psi|_\infty \,,$$

and also $\mu(\psi) \le |f|_\infty \mu_0(\psi)$ if $\psi \ge 0$.

Peripheral Spectrum: Sketch of Proof

b) $\exists$ **finite** $\# \, q_k \in \mathbb{N}$ **st** $\cup_{j=0}^{N} \{\theta_j\} = \cup_k \{ \frac{p}{q_k} : 0 \le p < q_k, p \in \mathbb{N}\}$

- Let $\mu \in \mathbb{V}_j$. By (a) $\exists f_\mu \in L^\infty(\mu_0)$ s.t. $d\mu = f_\mu d\mu_0$. Then,

$$e^{2\pi\theta_j}\mu(\psi) = \mu(\psi \circ T) = \int \psi \circ T \, f_\mu d\mu_0 = \int \psi \, f_\mu \circ T^{-1} d\mu_0$$

so $f_\mu \circ T^{-1} = e^{2\pi\theta_j} f_\mu$. For $k > 1$, $\mu_k = (f_\mu)^k \mu_0$ satisfies $\mathcal{L}\mu_k = e^{2\pi\theta_j k}\mu_k$, i.e. $k\theta_j$ is in the peripheral spectrum of $\mathcal{L}$.

c) M **has a single ergodic component of pos.** μ_0 **measure.**

- Use the fact that $\mathcal{W}^s$ and $\mathcal{W}^u$ fully cross M and the definition of $\mu_0 = \Pi_0 1$ as a limit.

(c) implies 1 is a simple eigenvalue of $\mathcal{L}$.

If $\mu \in \mathbb{V}_j$, then $\theta_j = p/q$ by (b) so that $\mathcal{L}^q \mu = \mu$. But if $T = T_{\kappa,\lambda}$, then $T^q = T_{\kappa^q,\lambda^q}$ is another generalized Baker's map, so that 1 is a simple eigenvalue of $\mathcal{L}^q$ as well. Thus $\mu = \mu_0$ and $\theta_j = 0$, i.e. 1 is the only eigenvalue of modulus 1 and it is simple. □

Applications to Hyperbolic Maps

- Real-analytic hyperbolic diffeomorphisms [Rugh '94], [Fried '95]
- Anosov and Axiom A diffeomorphisms [Blank, Keller, Liverani '01], [Baladi '05], [Gouëzel, Liverani '06, '08], [Baladi, Tsujii '07], [Faure, Roy, Sjöstrand '08]
- Piecewise hyperbolic maps [D., Liverani '08], [Baladi, Gouëzel '09, '10]
- Planar billiard maps
  - Dispersing billiards and perturbations [D., Zhang '11,'13, '14]
  - Measure of maximal entropy [Baladi, D., '20]
  - Geometric potentials [Baladi, D., preprint '20]