Review Sheet (Exam 2) - updated

1.) Need to check if $A\vec{x}=0$ has only the trivial soln.

$[A|\begin{smallmatrix}0\\0\\0\end{smallmatrix}] \rightsquigarrow_{RREF} \left[\begin{array}{ccc|c}1&0&0&0\\0&1&0&0\\0&0&1&0\end{array}\right]$ So $\vec{x}=\vec{0}$ is the only soln. to $A\vec{x}=\vec{0}$.

Therefore the columns of A are LI.

2.) $\text{rref}(A)=\begin{bmatrix}1&0&0&.5\\0&1&0&.5\\0&0&1&-.5\end{bmatrix}$ free

$\text{rref}([A|\vec{0}])=\left[\begin{array}{cccc|c}1&0&0&.5&0\\0&1&0&.5&0\\0&0&1&-.5&0\end{array}\right]$

$\vec{x}=\begin{bmatrix}-.5\\-.5\\.5\\1\end{bmatrix}x_4$

a.) $Col(A)=Span\left\{\begin{bmatrix}2\\5\\1\end{bmatrix},\begin{bmatrix}3\\2\\7\end{bmatrix},\begin{bmatrix}3\\3\\2\end{bmatrix},\begin{bmatrix}1\\2\\3\end{bmatrix}\right\}$.

b.) $Null(A)=\left\{\begin{bmatrix}-.5\\-.5\\.5\\1\end{bmatrix}x_4 \,\middle|\, x_4\in\mathbb{R}\right\}$

c.) Basis for $Col(A)=\left\{\begin{bmatrix}2\\5\\1\end{bmatrix},\begin{bmatrix}3\\2\\7\end{bmatrix},\begin{bmatrix}3\\3\\2\end{bmatrix}\right\}$

d.) Basis for $Null(A)=\left\{\begin{bmatrix}-.5\\-.5\\.5\\1\end{bmatrix}\right\}$.

e.) $Rank(A)=3$ f.) $Nullity(A)=1$

3.) $T:\mathbb{R}^2\to\mathbb{R}^2$ by $T\left(\begin{bmatrix}x\\y\end{bmatrix}\right)=\begin{bmatrix}x+y\\x-3y\end{bmatrix}$

a) Let $\begin{bmatrix}x_1\\y_1\end{bmatrix},\begin{bmatrix}x_2\\y_2\end{bmatrix}\in\mathbb{R}^2$ and $r\in\mathbb{R}$.

$T\left(\begin{bmatrix}x_1\\y_1\end{bmatrix}+\begin{bmatrix}x_2\\y_2\end{bmatrix}\right)=T\left(\begin{bmatrix}x_1+x_2\\y_1+y_2\end{bmatrix}\right)=\begin{bmatrix}x_1+x_2+y_1+y_2\\x_1+x_2-3(y_1)-3y_2\end{bmatrix}$

$=\begin{bmatrix}x_1+y_1\\x_1-3y_1\end{bmatrix}+\begin{bmatrix}x_2+y_2\\x_2-3y_2\end{bmatrix}=T\left(\begin{bmatrix}x_1\\y_1\end{bmatrix}\right)+T\left(\begin{bmatrix}x_2\\y_2\end{bmatrix}\right)$

$$T\left(r\begin{bmatrix}x_1\\y_1\end{bmatrix}\right)=T\left(\begin{bmatrix}rx_1\\ry_1\end{bmatrix}\right)=\begin{bmatrix}rx_1+ry_1\\rx_1-3ry_1\end{bmatrix}=r\begin{bmatrix}x_1+y_1\\x_1-3y_1\end{bmatrix}=rT\left(\begin{bmatrix}x_1\\y_1\end{bmatrix}\right).$$

$\therefore$ T is a LT.

b.) $T\left(\begin{bmatrix}x\\y\end{bmatrix}\right)=\begin{bmatrix}1&1\\1&-3\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}$ So, $A=\begin{bmatrix}1&1\\1&-3\end{bmatrix}$.

c.) $A^{-1}=\begin{bmatrix}3/4&1/4\\1/4&-1/4\end{bmatrix}$ so $T^{-1}:\mathbb{R}^2\to\mathbb{R}^2$ by $T^{-1}\left(\begin{bmatrix}x\\y\end{bmatrix}\right)=\begin{bmatrix}3/4&1/4\\1/4&-1/4\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}$.

d) $\ker(T)=\{\vec{x}\mid A\vec{x}=0\}=\{\vec{0}\}$ since $A\vec{x}=\vec{0}$ only has the trivial soln

4.) If $T:\mathbb{R}^n\to\mathbb{R}^m$ and $T':\mathbb{R}^m\to\mathbb{R}^k$ are LT then
$T'\circ T:\mathbb{R}^n\to\mathbb{R}^k$ is a LT.

Pf: Let $T:\mathbb{R}^n\to\mathbb{R}^m$ and $T':\mathbb{R}^m\to\mathbb{R}^k$ be LT.
Suppose $\vec{x},\vec{y}\in\mathbb{R}^n$ and $r\in\mathbb{R}$.

$T'\circ T(\vec{x}+\vec{y})=T'(T(\vec{x}+\vec{y}))=T'(T(\vec{x})+T(\vec{y}))$ since T is a LT

$=T'(T(\vec{x}))+T'(T(\vec{y}))$ since T' is a LT

$=T'\circ T(\vec{x})+T'\circ T(\vec{y})$.

$T'\circ T(r\vec{x})=T'(T(r\vec{x}))=T'(rT(\vec{x}))$ since T is a LT

$=rT'(T(\vec{x}))$ since T' is a LT

$=rT'\circ T(\vec{x})$.

So, $T'\circ T$ is a LT.

5.) The property $1A=A$ is not met since if $A\neq\begin{bmatrix}0&0\\0&0\end{bmatrix}$
$1\cdot A=\begin{bmatrix}0&0\\0&0\end{bmatrix}\neq A$.

6. We need to find which vectors in S are indep.

$a(x^2+1)+b(x^2+x-1)+c(3x-6)+d(x^3+x^2+1)+ex^3=0$

matching the coefficients gives the System of eqns.

$(d+e)x^3+(a+b+d)x^2+(b+3c)x+(a-b-6c+d)=0$

$$\begin{array}{c|c} & \begin{array}{ccccc|c} a & b & c & d & e & \end{array} \\ \begin{array}{c} x^3: \\ x^2: \\ x: \\ 1 \end{array} & \left[\begin{array}{ccccc|c} 0 & 0 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 3 & 0 & 0 & 0 \\ 1 & -1 & -6 & 1 & 0 & 0 \end{array}\right] \end{array} \underset{\sim}{\text{RREF}} \begin{array}{c} \begin{array}{ccccc|c} a & b & c & d & e & \end{array} \\ \left[\begin{array}{ccccc|c} \boxed{1} & 0 & -3 & 0 & -1 & 0 \\ 0 & \boxed{1} & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & \boxed{1} & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \end{array}$$

So a basis for S is $\{x^2+1, x^2+x-1, x^3+x^2+1\}$

The $\dim(S)=3$.

7. a) $\{\sin(x), \sin(-x)\}$. vectors are LD since $\sin(-x)=-\sin(x)$.
so $1\sin(x)+1\sin(-x)=0 \quad \forall x$
(nontrivial soln)

There are other methods.

b.) $\{x^2-1, x^2+1, 4x, 2x-3\}$
See if there is only the trivial soln.
$* \; a(x^2-1)+b(x^2+1)+c(4x)+d(2x-3)=0$
$(a+b)x^2+(4c+2d)x+(-a+b-3d)=0$

$$\begin{array}{c|c} & \begin{array}{cccc|c} a & b & c & d & \end{array} \\ \begin{array}{c} x^2 \\ x \\ 1 \end{array} & \left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 4 & 2 & 0 \\ -1 & 1 & 0 & -3 & 0 \end{array}\right] \end{array} \underset{\sim}{\text{RREF}} \left[\begin{array}{cccc|c} \boxed{1} & 0 & 0 & 1.5 & 0 \\ 0 & \boxed{1} & 0 & -1.5 & 0 \\ 0 & 0 & \boxed{1} & .5 & 0 \end{array}\right]$$

free

There are nontrivial solns to the * eqn ⇒ vectors are LD.

(8) $S = \{f \mid f(0) = 0\} \subseteq F$.

pf Suppose $f, g \in S$ and $r \in \mathbb{R}$. So, $f(0) = g(0) = 0$.

$$(f+g)(0) = f(0) + g(0) = 0 + 0 = 0$$

So, $f+g \in S$.

and $(rf)(0) = r f(0) = r(0) = 0$.

So, $rf \in S$.

Therefore S is a subspace of F.

(9.) we need to find $a, b, c, d \in \mathbb{R}$ such that

$$a\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix} + c\begin{bmatrix} 1 & -1 \\ 0 & 3 \end{bmatrix} + d\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 3 & 4 \end{bmatrix}.$$

Adding componentwise we get:

$$\begin{bmatrix} c & a-b-c+d \\ a & 3c+d \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 3 & 4 \end{bmatrix}$$

So, $\boxed{c=1}$ $\boxed{a=3}$

$3c+d=4$

$3(1)+d=4$

$\boxed{d=1}$

$a-b-c+d=-2$

$3-b-1+1=-2$

$\boxed{5=b}$

Thus $$\begin{bmatrix} 1 & -2 \\ 3 & 4 \end{bmatrix}_B = \begin{bmatrix} 3 \\ 5 \\ 1 \\ 1 \end{bmatrix}$$