

Strong Artin-Rees property in rings of dimension one and two

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To Fabio Rossi, for his enthusiasm.

SUMMARY. - *Let (R, \mathfrak{m}) be a local noetherian ring and let $N \subseteq M$ be two finitely generated R -modules such that $\dim M/N \leq 1$. We give simple proof of the fact that there exists an integer h such that $I^n M \cap N = I^{n-h}(I^h M \cap N)$, for all $n \geq h$ and for all ideals $I \subset R$. We give upper bounds for such an integer h . Moreover, we give two examples of rings of dimension two where the property fails.*

1. Introduction

Let R be a noetherian ring, $I \subset R$ an ideal of R , and let $N \subseteq M$ be two finitely generated R -modules. By the Artin-Rees Lemma there exists an integer h depending on I , M and N such that

$$I^n M \cap N = I^{n-h}(I^h M \cap N), \quad \text{for all } n \geq h. \quad (1)$$

A weaker property is often used in the applications, namely

$$I^n M \cap N \subset I^{n-h}N, \quad \text{for all } n \geq h. \quad (2)$$

Much work has been done to determine whether h can be chosen uniformly, in the sense that (1) or (2) would be satisfied simultaneously for every ideal of a given family, see ([2], [3], [5], [4], [6], [10]).

(*) Keywords: Artin-Rees Lemma

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Following the definitions in [3], we say that the pair (N, M) has the *strong Artin-Rees property with respect to \mathcal{W} with Artin-Rees number h* if (1) holds for all $I \in \mathcal{W}$. Notice that in this case, every integer bigger than h is an Artin-Rees number with respect to \mathcal{W} for the pair (N, M) . We denote by $\text{ar}_R(N, M; \mathcal{W})$ the least of such integers.

When \mathcal{W} is the family of all ideals, we say that the pair (N, M) has the *strong Artin-Rees property* and denote by $\text{ar}_R(N, M)$ the least of the Artin-Rees numbers.

Planas-Vilanova [6] proves that any pair (N, M) with $\dim M/N \leq 1$ has the strong Artin-Rees property if R is an excellent ring. The proof comes down to the case of local rings. In this note we give a simpler proof of the strong Artin-Rees property over a one-dimensional local ring, with particular attention paid to upper bounds for $\text{ar}_R(N, M)$. Such bounds find application in the study of other uniform Artin-Rees properties, see [7].

We also give an example of a family of ideals and two modules $N \subset M$ such that $\dim M/N = 2$ for which there exists no integer h such that (1) holds for all ideals in the family. If $\dim M/N = 3$ the strong Artin-Rees property was known to fail, see [10].

2. One-dimensional rings

In the rest of the paper $(R, \mathfrak{m}, \mathfrak{k})$ will denote a local noetherian ring with maximal ideal \mathfrak{m} and residue field \mathfrak{k} .

We first show that it is enough to study the strong Artin-Rees property with respect to the family of \mathfrak{m} -primary ideals. For this, we first need a lemma.

(2.1) **Lemma.** *Let M be an R -module and let N_1, N_2 be two submodules of M . There exists an $h = h(N_1 + N_2 \subseteq M)$ such that*

$$N_1 \cap (N_2 + \mathfrak{m}^n M) \subseteq (N_1 \cap N_2) + \mathfrak{m}^{n-h} N_1,$$

for every $n \geq h$.

Proof. By the Artin-Rees Lemma there exists h such that for every $n \geq h$ the following holds:

$$\mathfrak{m}^n M \cap (N_1 + N_2) = \mathfrak{m}^{n-h} (\mathfrak{m}^h M \cap (N_1 + N_2)) \subseteq \mathfrak{m}^{n-h} (N_1 + N_2).$$

Then the following holds for $n \geq h$:

$$\begin{aligned}
N_1 \cap (N_2 + \mathbf{m}^n M) &= N_1 \cap (N_2 + (\mathbf{m}^n M \cap (N_1 + N_2))) \\
&= N_1 \cap (N_2 + \mathbf{m}^{n-h}(\mathbf{m}^h M \cap (N_1 + N_2))) \\
&\subseteq N_1 \cap (N_2 + \mathbf{m}^{n-h}(N_1 + N_2)) \\
&\subseteq N_1 \cap (N_2 + \mathbf{m}^{n-h}N_1) \\
&\subseteq N_1 \cap N_2 + \mathbf{m}^{n-h}N_1. \quad \square
\end{aligned}$$

(2.2) **Remark.** Notice that if h is an integer that satisfies Lemma 2.1, then every bigger integer does as well.

(2.3) **Proposition.** Let M be an R -module and $N \subset M$ a submodule. Let \mathcal{W} be the family of \mathbf{m} -primary ideals. Assume that (N, M) has the strong uniform Artin-Rees property with respect to \mathcal{W} . Then $\text{ar}_R(N, M) \leq \text{ar}_R(N, M; \mathcal{W})$.

Proof. Let $h_0 = \text{ar}(N, M; \mathcal{W})$ and assume by contradiction that there exists $I \subset R$ and $n \geq h_0$ such that $I^{n-h_0}(I^{h_0}M \cap N) \neq I^n M \cap N$.

On the other hand, for all $h \gg 0$ and for such a fixed n and h_0 , the inclusions below hold. Inclusion (4) holds by the definition of h_0 , inclusions (5) and (6) hold by expanding the powers of $(I + \mathbf{m}^h)$.

$$I^n M \cap N \subseteq (I + \mathbf{m}^h)^n M \cap N, \quad (3)$$

$$\subseteq (I + \mathbf{m}^h)^{n-h_0}((I + \mathbf{m}^h)^{h_0} M \cap N), \quad (4)$$

$$\subseteq I^{n-h_0}((I + \mathbf{m}^h)^{h_0} M \cap N) + \mathbf{m}^h M, \quad (5)$$

$$\subseteq I^{n-h_0}((I^{h_0} + \mathbf{m}^h)M \cap N) + \mathbf{m}^h M, \quad (6)$$

$$= I^{n-h_0}((I^{h_0} M + \mathbf{m}^h M) \cap N) + \mathbf{m}^h M. \quad (7)$$

Let h_1 be an integer depending on $(I^{h_0} M + N) \subseteq M$ that satisfies Lemma 2.1 with $N_1 = N$, $N_2 = I^{h_0} M$. By Remark 2.2, we may assume $h_1 \geq n - h_0$ and obtain

$$(I^{h_0} M + \mathbf{m}^h M) \cap N \subseteq (I^{h_0} M \cap N) + \mathbf{m}^{h-h_1} M,$$

for every $h > h_1$. So for $n \geq h_0$ and $h > h_1$, with $h_1 \geq n - h_0$,

$$I^{n-h_0}((I^{h_0}M + \mathfrak{m}^h M) \cap N) + \mathfrak{m}^h M \quad (8)$$

$$\subseteq I^{n-h_0}(I^{h_0}M \cap N + \mathfrak{m}^{h-h_1}M) + \mathfrak{m}^h M, \quad (9)$$

$$\subseteq I^{n-h_0}(I^{h_0}M \cap N) + \mathfrak{m}^{h-h_1+n-h_0}M + \mathfrak{m}^h M, \quad (10)$$

$$\subseteq I^{n-h_0}(I^{h_0}M \cap N) + \mathfrak{m}^{h-h_1+n-h_0}M. \quad (11)$$

Putting together the right and the left end of the chain of inclusions (3)–(11), we obtain that

$$I^n M \cap N \subseteq I^{n-h_0}(I^{h_0}M \cap N) + \mathfrak{m}^{h-h_1+n-h_0}M,$$

for every $h > h_1$. By taking the intersection of the right side of the inclusion over all $h > h_1$, we can conclude $I^n M \cap N \subseteq I^{n-h_0}(I^{h_0}M \cap N)$. Since the reverse inclusion always holds, there is equality $I^{n-h_0}(I^{h_0}M \cap N) = I^n M \cap N$, which contradicts the assumption. \square

We also need another kind of reduction, see for example [3, (2.4)].

(2.4) **Lemma.** *Let $(R, \mathfrak{m}, \mathfrak{k})$ be a local noetherian ring. The extension $R \rightarrow R[x]_{\mathfrak{m}R[x]}$ is faithfully flat and $R[x]_{\mathfrak{m}R[x]}$ has an infinite residue field.*

Let $R \rightarrow S$ be a faithfully flat extension. Let M be an R -module and $N \subset M$ a submodule. If $(N \otimes_R S, M \otimes_R S)$ has the strong uniform Artin-Rees property then $\text{ar}_R(N \subseteq M) \leq \text{ar}_S(N \otimes_R S \subseteq M \otimes_R S)$.

(2.5) **Proposition.** *Suppose $(R, \mathfrak{m}, \mathfrak{k})$ is a one-dimensional local noetherian ring with infinite residue field. There exists an integer $r = r(R)$, such that for every \mathfrak{m} -primary ideal I there exists $y \in I$ so that $I^n = yI^{n-1}$, for every $n \geq r$.*

Proof. First suppose that R is Cohen-Macaulay and let e be the multiplicity of the ring. By [8, Chapter 3, (1.1)], we have that $\mu(I) \leq e$, where $\mu(I)$ denotes the minimal number of generators of I and I is an arbitrary \mathfrak{m} -primary ideal. Therefore, $\mu(I^e) \leq e < e + 1$. Hence, by [8, Chapter 2, (2.3)], there exists $y \in I$ such that $I^e = yI^{e-1}$, so that for every $n \geq e$ we have $I^n = yI^{n-1}$. Set r to be e .

Next suppose $\text{depth}(R) = 0$, and let $0 = q_1 \cap q_2 \cdots \cap q_{s+1}$ be a minimal primary decomposition of 0 where q_{s+1} is \mathfrak{m} -primary and set $J = q_1 \cap q_2 \cdots \cap q_s$. Then R/J is Cohen-Macaulay and there exists a h_0 such that $\mathfrak{m}^{h_0}J = 0$. Let e_1 be the multiplicity of R/J . Then, by the above case, there exists a $y \in I$ such that for every $n \geq e_1$ we have $I^n \subseteq yI^{n-1} + J$ and hence $I^n \subseteq yI^{n-1} + I^n \cap J$, for every $n > e_1$. By [3, (4.2)], there exists a h_1 , depending just on R and J , such that for every $n \geq h_1$ and every ideal $I \subset R$ we have $I^n \cap J \subseteq I^{n-h_1}J$. Hence, for every $n \geq r = \max\{e_1, h_0 + h_1\}$ one has the following inclusions:

$$\begin{aligned} I^n &\subseteq yI^{n-1} + I^n \subseteq yI^{n-1} + I^n \cap J \\ &\subseteq yI^{n-1} + I^{n-h_1}J \subseteq yI^{n-1} + \mathfrak{m}^{h_0}J = yI^{n-1}. \quad \square \end{aligned}$$

We are now ready to prove the main theorem. If M is a finite length module we denote by $\ell(M)$ its length.

(2.6) **Theorem.** *Let $(R, \mathfrak{m}, \mathfrak{k})$ be a one-dimensional local ring. Then every pair (N, M) , with $N \subset M$, has the strong uniform Artin-Rees property, and $\text{ar}_R(N, M) \leq \max\{r, \ell(\mathbf{H}_{\mathfrak{m}}^0(M/N))\} + \ell(\mathbf{H}_{\mathfrak{m}}^0(M/N))$, where $r = r(R)$ is an integer as in Proposition 2.5.*

Proof. By Lemma 2.4, $\text{ar}_R(N \subseteq M) \leq \text{ar}_S(N \otimes_R S \subseteq M \otimes_R S)$, for any ring extension $R \rightarrow S$; thus we may assume that R has infinite residue field. Let I be an \mathfrak{m} -primary ideal. Set $h_1 = \ell(\mathbf{H}_{\mathfrak{m}}^0(M/N))$ and $h = \max\{r, \ell(\mathbf{H}_{\mathfrak{m}}^0(M/N))\} + \ell(\mathbf{H}_{\mathfrak{m}}^0(M/N))$.

Assume first that M/N is Cohen-Macaulay. By Proposition 2.5 we can choose $y \in I$ such that y is a non-zero-divisor in M/N , such that for $n > h = r$,

$$\begin{aligned} I^n M \cap N &= yI^{n-1}M \cap N, \\ &\subseteq y(I^{n-1}M \cap N), \quad \text{by the property of } y, \\ &\subseteq I(I^{n-1}M \cap N), \quad \text{since } y \in I. \end{aligned}$$

Now suppose that M/N is not Cohen-Macaulay and let $M'/N =$

$H_m^0(M/N)$. For every $n \geq h$ and every $I \subseteq R$ we have:

$$\begin{aligned}
I^n M \cap N &= I^n M \cap M' \cap N, \text{ since } N \subset M', \\
&= I^{n-r}(I^r M \cap M') \cap N, \text{ since } M/M' \text{ is Cohen-Macaulay,} \\
&\subseteq I^{n-r}(I^r M \cap M'), \text{ since } n-r \geq h_1, \\
&= I^{n-r-h_1} I^{h_1}(I^r M \cap M'), \\
&= I^{n-r-h_1}(I^{h_1}(I^r M \cap M') \cap N), \text{ since } I^{h_1} M' \subset N, \\
&\subseteq I^{n-r-h_1}(I^{r+h_1} M \cap M' \cap N), \\
&\subseteq I^{n-h}(I^h M \cap N),
\end{aligned}$$

where the last containment follows since $r + h_1 \leq h$ and in general $I^n(I^h M \cap N) \subseteq I^{n-s}(I^{s+h} M \cap N)$ for all n, h , and $s \leq n$. \square

(2.7) **Relation Type.** Let $I = (x_1, \dots, x_n)$ be an ideal in R . Map the polynomial ring, with the standard grading, $R[X_1, \dots, X_n]$ onto the Rees algebra $R[It]$ by sending X_i to $x_i t$. Let L be the kernel of this map. Then L is an homogeneous ideal and the relation type of I is defined to be the minimum integer h such that the ideal L can be generated by elements of degree less than or equal to h . It is denoted by $\text{reltype}(I)$. This number does not depend on the choice of the minimal generators of the ideal I .

For an \mathfrak{m} -primary ideal I in a Cohen-Macaulay local $\text{reltype}(I) \leq e$, where e is the multiplicity of R , see [1].

The following lemma had been proved by Wang in [9] for parameters ideals. The same argument applies for every ideal, we include it here for simplicity.

(2.8) **Lemma.** Let $(R, \mathfrak{m}, \mathfrak{k})$ be a local ring and J be an ideal of R ; denote $\bar{R} = R/J$. Let $I = (x_1, \dots, x_m)$ be an ideal of R and suppose that $\text{reltype}(I\bar{R}) \leq h$, for some $h > 0$. Then for every $n > h$,

$$I^n \cap J = I^{n-h}(I^h \cap J).$$

Proof. Let $n \geq h$ and let $x \in I^n \cap J$. Then there exists a polynomial F in $R[X_1, \dots, X_m]$, homogeneous of degree n , such that $F(x_1, \dots, x_m) = x$. Modulo J , \bar{F} is a relation on the \bar{x}_i 's, so by

hypothesis there are polynomials G_i of degree h , and H_i , of degree $n - h$, such that $\bar{F} = \sum \bar{G}_i \bar{H}_i$ in $\bar{R}[X_1, \dots, X_m]$ and \bar{G}_i are relations on the \bar{x}_i . Therefore, $F = \sum G_i H_i + K$ for some $K \in R[X_1, \dots, X_m]$ of degree n and coefficients in J . Since:

$$\begin{aligned} K(x_1, \dots, x_m) &\in JI^n \subset I^{n-h}(I^h \cap J), \\ G_i(x_1, \dots, x_m) &\in I^h \cap J, \quad \text{and} \\ H_i(x_1, \dots, x_m) &\in I^{n-h}, \end{aligned}$$

this shows that the element $x = F(x_1, \dots, x_m)$ is in $I^{n-h}(I^h \cap J)$. \square

(2.9) **Lemma.** *Let $(R, \mathfrak{m}, \mathfrak{k})$ a noetherian local ring. If J is an ideal of R such that $\dim(R/J) \leq 1$ then (J, R) has the strong Artin-Rees property.*

Proof. If $\dim(R/J) = 0$ then there exists a power of the maximal ideal $\mathfrak{m}^h \subset J$. Therefore, for $n > h$ and for every ideal I we have the following:

$$I^n \cap J = I^n = II^{n-1} = I(I^{n-1} \cap J).$$

Assume $\dim(R/J) = 1$. By Lemma 2.1 it is enough to show that (J, R) has the strong Artin-Rees property with respect to the family of \mathfrak{m} -primary ideals. Suppose that R/J is Cohen-Macaulay; then the conclusion holds by 2.7 and by Lemma 2.8.

Suppose R/J has dimension one and it is not Cohen-Macaulay. Let $J \subset J'$ such that R/J' is Cohen-Macaulay and let h_0 such that $\mathfrak{m}^{h_0} J' \subset J$. By the Cohen-Macaulay case there exists an Artin-Rees number $h_1 = h_1(J' \subset R)$. We may assume $h_1 > h_0$. Let $h = h_1 + h_0$. For every $n \geq h$, the inequalities below follow.

$$\begin{aligned} I^n \cap J &= I^n \cap J' \cap J, && \text{since } J \subseteq J', \\ &= I^{n-h_1}(I^{h_1} \cap J') \cap J, && \text{by definition of } h_1, \\ &\subseteq I^{n-h_1}(I^{h_1} \cap J') \\ &= I^{n-h_1-h_0} I^{h_0}(I^{h_1} \cap J') \\ &= I^{n-h_1-h_0}(I^{h_0}(I^{h_1} \cap J') \cap J), && \text{since } I^{h_0} J' \subseteq J, \\ &\subseteq I^{n-h}(I^h \cap J' \cap J) \\ &= I^{n-h}(I^h \cap J). \end{aligned}$$

\square

(2.10) **Proposition.** *Let $(R, \mathfrak{m}, \mathfrak{k})$ be a local noetherian ring. Let M be an R -module and $N \subseteq M$ a submodule. Let $J \subset \text{ann}(M/N)$ be an ideal of R . If (J, R) and $(N/JM, M/JM)$ have the strong uniform Artin-Rees property, then*

$$\text{ar}_R(N, M) \leq \max\{\text{ar}_R(J, R), \text{ar}_{R/J}(N/JM, M/JM)\}.$$

In particular, if $\dim(M/N) = 1$ then $\text{ar}_R(N, M)$ is bounded above by

$$\max\{\text{ar}_R(J, R), \max\{r(R/J), \ell(\mathbf{H}_{\mathfrak{m}}^0(M/N))\} + \ell(\mathbf{H}_{\mathfrak{m}}^0(M/N))\}.$$

Proof. The second statement follows from the first and Theorem 2.6. For the first part, let $h = \max\{\text{ar}_R(J, R), \text{ar}_{R/J}(N/JM, M/JM)\}$. Let $\phi : R^m \rightarrow M$, a surjection of a free module onto M . Denote by $K = \ker(\phi)$ and by $L = \phi^{-1}(N)$, the pre-image of the submodule $N \subset M$. Then, as shown in [2], it is enough to show that there exists a h such that for every $n \geq h$ and for every ideal $I \subset R$, we have $I^n R^m \cap L = I^{n-h}(I^h R^m \cap L)$. Therefore, without loss of generality we may assume M is a free module.

Since $h \geq \text{ar}_{R/J}(N/JM, M/JM)$, for every $n \geq h$ and for every ideal I , we have $I^n M \cap N \subset I^{n-h}(I^h M \cap N) + JM$. Therefore,

$$I^n M \cap N \subset I^{n-h}(I^h M \cap N) + JM \cap I^n M = I^{n-h}(I^h M \cap N) + (I^n \cap J)M,$$

where the last equality holds since M is a free module. Since $h \geq \text{ar}_R(J, R)$, we have $I^n \cap J = I^{n-h}(I^h \cap J)$. Hence,

$$\begin{aligned} I^n M \cap N &= I^{n-h}(I^h M \cap N) + I^{n-h}(I^h \cap J)M \\ &= I^{n-h}(I^h M \cap N) + I^{n-h}(I^h M \cap JM) \\ &\subset I^{n-h}(I^h M \cap N), \quad \text{since } JM \subseteq N. \quad \square \end{aligned}$$

3. Two-dimensional rings

The following example (see [10]), shows that the uniform Artin-Rees property does not hold if $\dim M/N = 2$.

(3.1) **Example.** Let $R = \mathfrak{k}[x, y, z]/(z^2)$. Consider the following family of ideals in R :

$$I_n = (x^n, y^n, x^{n-1}y + z),$$

for every $n \in \mathbb{N}$. Let J the ideal generated by z .

We want to show that $I_n(I_n^{n-1} \cap J) \neq I_n^n \cap J$, for every $n \geq 2$. In particular we will show that

$$x^{(n-1)^2}y^{n-1}z \in I_n^n \cap J \quad \text{but} \quad x^{(n-1)^2}y^{n-1}z \notin I_n(I_n^{n-1} \cap J).$$

Denote $x^{(n-1)^2}y^{n-1}z$ by ξ .

The ideal I_n is a homogeneous ideal if we assign degree one to x and y and degree n to z . With such a grading, ξ has degree $(n-1)^2 + n - 1 + n = n^2$. Since $x^{(n-1)^2}y^{n-1}z = (x^{n-1}y + z)^n - (x^n)^{n-1}y^n \in I_n^n$ the first claim holds.

Suppose $x^{(n-1)^2}y^{n-1}z \in I_n(I_n^{n-1} \cap J)$. This remains true modulo $(x^{(n-1)^2+1}, y^n)$. The ideal I_n^{n-1} modulo $(x^{(n-1)^2+1}, y^n)R$ is generated by

$$\{x^{n(n-1-i)}(x^{n-1}y + z)^i \mid i = 0, 1, \dots, n-1\}.$$

Moreover,

$$\begin{aligned} x^{n(n-1-i)}(x^{n-1}y + z)^i &= x^{n(n-1-i)}(x^{(n-1)i}y^i + x^{(n-1)(i-1)}y^{i-1}z) \\ &= x^{n^2-n-i}y^i + x^{n^2-2n-i+1}y^{i-1}z. \end{aligned}$$

But $n^2 - n - i \geq (n-1)^2 + 1$ for $i \leq n-2$. Therefore, I_n^{n-1} modulo $(x^{(n-1)^2+1}, y^n)$ is generated by

$$\{x^{(n-1)^2}y^{n-1} + x^{(n-1)(n-2)}y^{(n-2)}z, \quad x^{n^2-2n-i+1}y^{i-1}z \mid i = 1, \dots, n-2\}.$$

Let

$$\begin{aligned} f &= x^{(n-1)^2}y^{n-1} + x^{(n-1)(n-2)}y^{(n-2)}z, \\ g_i &= x^{n^2-2n-i+1}y^{i-1}z. \end{aligned}$$

Let $hf + \sum h_i g_i$ be a homogeneous element of $I_n^{n-1} \cap J$ that appears in the expression of ξ as element of $I_n(I_n^{n-1} \cap J)$. By degree reasons we can assume h is not a constant polynomial.

Let $m(x, y, z)$ be a homogeneous monomial of h . If z does not divide m , then

$$\begin{aligned} m(x, y, z)f &= m'(x, y)x^{(n-1)(n-2)+1}y^{(n-2)}z \\ \text{or } m(x, y, z)f &= m'(x, y)x^{(n-1)(n-2)}y^{(n-2)+1}z; \end{aligned}$$

if z does divide m then $m(x, y, z)f = m'(x, y)x^{(n-1)^2}y^{n-1}z$, with m' possibly a unit. By a degree counting we can see that $\deg(hf) \geq n^2 - n + 1$. Therefore, for every element $a \in I_{n-1}$ we have $\deg(ahf) > n^2 = \deg(\xi)$. This shows a contradiction.

The following example refines the above in that now R is a reduced ring.

(3.2) **Example.** Let $R = \mathbb{k}[x, y, z]/xz$. Consider the following family of ideals:

$$I_n = (x^n, y^n, x^{n-1}y + z^n),$$

for every $n \in \mathbb{N}$. Let $J = (z)$. Again, we claim that $I_n(I_n^{n-1} \cap J) \neq I_n^n \cap J$ for every $n \geq 1$. We will show that

$$z^{n^2} \in I_n^n \cap J \quad \text{but} \quad z^{n^2} \notin I_n(I_n^{n-1} \cap J).$$

Indeed, $z^{n^2} = (x^{n-1}y + z^n)^n - (x^n)^{n-1}y^n \in I_n^n$ and trivially $z^{n^2} \in J$.

On the other hand I_n^{n-1} is generated by:

$$\{x^{n(n-1)}, x^{(n-1)^2}y^{n-1} + z^{n(n-1)}, y^n L, x^{(n-1)^2+i}y^{n-1-i} \mid i = 1, \dots, n-1\},$$

for some ideal L in R . Notice that if $z^{n^2} \in I_n(I_n^{n-1} \cap J)$ then this also holds modulo y^n . Moreover, if a homogeneous element

$$f(x, y)x^{n(n-1)} + g(x, y, z)(x^{(n-1)^2}y^{n-1} + z^{n(n-1)}) + \sum_{i=1}^{n-1} h_i(x, y)x^{(n-1)^2+i}y^{n-1-i}$$

is in J , writing $g(x, y, z) = g''(x, y) + zg'(x, y, z)$, we see that

$$f(x, y)x^{n(n-1)} + g''(x, y)x^{(n-1)^2}y^{n-1} + \sum h_i(x, y)x^{(n-1)^2+i}y^{n-1-i} = 0.$$

But if this is the case, since $xz = 0$ in R , we have

$$fx^{n(n-1)} + g(x^{(n-1)^2}y^{n-1} + z^{n(n-1)}) + \sum h_i x^{(n-1)^2+i} = zg'z^{n(n-1)}.$$

But $zg'z^{n(n-1)}$ is an homogeneous element of degree at least $n^2 - n + 1$ and multiplication by any element in I_n increases the degree by n . Therefore, any element in $I_n(I_n^{n-1} \cap J)$ has degree at least $n^2 + 1$ while z^{n^2} has degree strictly smaller.

REFERENCES

- [1] Rüdiger Achilles and Peter Schenzel, *A degree bound for the defining equations of one-dimensional tangent cones*, Seminar D. Eisenbud/B. Singh/W. Vogel, Vol. 2, Teubner-Texte zur Math., vol. 48, Teubner, Leipzig, 1982, pp. 77–87. MR 686461 (84g:14027)
- [2] A. J. Duncan and L. O’Carroll, *A full uniform Artin-Rees theorem*, J. Reine Angew. Math. **394** (1989), 203–207. MR 977443 (90c:13011)
- [3] Craig Huneke, *Uniform bounds in Noetherian rings*, Invent. Math. **107** (1992), no. 1, 203–223. MR 93b:13027
- [4] L. O’Carroll, *A note on Artin-Rees numbers*, Bull. London Math. Soc. **23** (1991), no. 3, 209–212. MR 92i:13001a
- [5] Liam O’Carroll, *A uniform Artin-Rees theorem and Zariski’s main lemma on holomorphic functions*, Invent. Math. **90** (1987), no. 3, 647–652. MR 914854 (89h:13013)
- [6] Francesc Planas-Vilanova, *The strong uniform Artin-Rees property in codimension one*, J. Reine Angew. Math. **527** (2000), 185–201. MR 2001g:13051
- [7] Janet Striuli, *Uniform Artin-Rees property for syzygies in rings of dimension one and two*, J. Pure Appl. Algebra **210** (2207) 577–588.
- [8] Judith D. Sally, *Numbers of generators of ideals in local rings*, Marcel Dekker Inc., New York, 1978. MR 0485852 (58 #5654)
- [9] Hsin-Ju Wang, *The relation-type conjecture holds for rings with finite local cohomology*, Comm. Algebra **25** (1997), no. 3, 785–801. MR 1433435 (98d:13017)
- [10] ———, *Some uniform properties of 2-dimensional local rings*, J. Algebra **188** (1997), no. 1, 1–15. MR 98e:13006

Received December 23, 2007.