Extensions of modules and uniform bounds of Artin-Rees type

by

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Abstract

EXTENSIONS OF MODULES AND
UNIFORM BOUNDS OF ARTIN-REES TYPE

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This dissertation is about some aspects of commutative algebra. Two main themes are developed: extensions of modules and variations of the classical Artin-Rees Lemma.

In the first chapter we first explain some basic results on the Yoneda's correspondence, we present Miyata's theorem and we prove a generalization of it. We give many applications of this generalization and of its partial converse, particularly to Cohen-Macaulay rings of finite Cohen-Macaulay type. We introduce the notion of sparse modules, we study their properties and we show that $\mathrm{Ext}^1_R(M, N)$ is a sparse module if M and N are maximal Cohen-Macaulay modules over a ring of finite Cohen-Macaulay type.

The second and the third chapters are devoted to the study of variations of the classical Artin-Rees Lemma. First we give an easy proof that in one-dimensional excellent rings the strong uniform version of the Artin-Rees Lemma holds. We also give some examples to show that such property does not hold in two dimensional rings.

Finally we study a uniform Artin-Rees property for syzygies. We relate the uniform Artin-Rees property with uniform annihilators for a certain family of Tors. We show that in a one-dimensional Noetherian local ring any finitely generated R-module has the Artin-Rees property for syzygies.

to Mirco

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Contents

Introduction

In this dissertation we study the following topics: the Yoneda correspondence between short exact sequences and elements of the cohomology group $\operatorname{Ext}^1_R(-,-)$ and uniform versions of the classical Artin-Rees Lemma.

Chapter one is devoted to the study of the Yoneda correspondence and to some applications of the developed techniques. Let R be a Noetherian ring and M and N be finitely generated R-modules. There is a one to one correspondence between elements of $\operatorname{Ext}^1_R(M,N)$ and short exact sequences

$$\alpha : 0 \to N \to X_\alpha \to M \to 0,$$

modulo a certain equivalence relation. Under this correspondence the zero element is given by a split exact sequence. It is a theorem of Miyata [16] that a short exact sequence is split exact if and only if $X_\alpha \cong N \oplus M$. We extend Miyata's theorem to the following

Theorem 0.1. *Let R be a Noetherian ring and $I \subset R$ an ideal. Let α and β be two short exact sequences in $\operatorname{Ext}^1_R(M,N)$, with isomorphic middle modules. If $\alpha \in I \operatorname{Ext}^1_R(M,N)$ then $\beta \otimes R/I$ is a split exact sequence.*

We study some more properties of short exact sequences in $\mathrm{Ext}^1_R(M,N)$, for example when the minimal number of generators of the modules at the side sums up to the minimal number of generator of the module in the middle; in some situations, we can also say when a short exact sequence is a minimal generator of $\mathrm{Ext}^1_R(M,N)$.

Theorem 0.1 gives some interesting consequences on the structure of the module $\mathrm{Ext}^1_R(M,N)$ which are particularly helpful in the study of rings of finite Cohen-Macaulay type.

A Cohen-Macaulay ring is said to be of finite Cohen-Macaulay type if there are a finite number of isomorphism classes of maximal Cohen-Macaulay modules. Rings of finite Cohen-Macaulay type have been widely studied, see [24]. In particular it is know that if $(R,\mathbf{m})$ is a local ring of finite Cohen-Macaulay type and M and N are maximal Cohen-Macaulay modules then $\mathrm{Ext}^1_R(M,N)$ is a module of finite length, see [2] and [12] in this generality. We introduce the notion of sparse modules and derive some properties. We show that sparse modules are Artinian and that $\mathrm{Ext}^1_R(M,N)$ are sparse modules if M and N are maximal Cohen-Macaulay modules over a ring of finite Cohen-Macaulay type, giving a different proof from the one in [12]. Huneke-Leuschke, in the same paper, give a bound on the power of the maximal ideal that annihilates $\mathrm{Ext}^1_R(M,N)$. We are able to improve that bound and to give some information on the structure of the module of $\mathrm{Ext}^1_R(M,N)$. Specifically, we have the following theorem, see Proposition 1.59 and Proposition 1.60.

Theorem 0.2. *Let $(R,\mathbf{m},\mathsf{k})$ a Cohen-Macaulay local ring with infinite residue field. Let M and N be maximal Cohen-Macaulay R-modules and let h be the number of isomorphism classes of modules that can fit in the middle of a* *short exact sequence in* $\operatorname{Ext}^1_R(M,N)$. *Then* $\mathbf{m}^{h-1}\operatorname{Ext}^1_R(M,N)=0$. *Suppose that* $N = M_1$, *where* M_1 *is the first syzygy in a free resolution of* M, *and suppose that the residue field* k *is algebraically closed. If*

$$\mathbf{m}^{h-2}\operatorname{Ext}^1_R(M,N)\neq 0$$

then $\operatorname{Ext}^1_R(M,N)=\oplus_{i=1}^n C_i$, *where each* C_i *is a cyclic* R*-module and the length* $\lambda(\mathbf{m}^j C_i/\mathbf{m}^{j+1}C_i)$ *has value* 0 *or* 1 *for every integer* j.

In the *second chapter* we study a uniform version of the Artin-Rees Lemma. Let R be a Noetherian ring and $N \subset M$ be finitely generated R-modules. If $I \subset R$ is an ideal of R, the classical lemma of Artin-Rees states that there exists an integer k such that $I^nM \cap N = I^{n-k}(I^kM \cap N)$, for $n > k$. For a proof see [15]. In particular a weaker version of the lemma says that there exists a k such that $I^nM \cap N \subset I^{n-k}N$ for every $n > k$. Notice that the integer k depends, a priori, on the ideal I and the two modules N and M. The uniform version (the strong and the weak one) of the Artin-Rees Lemma asks whether we can find the integer k that works for all the ideals of a certain family. This kind of questions were first raised by Eisenbud-Hochster in [8], where the family of ideals they considered is the family of maximal ideals. Duncan-O'Carroll [6] gave a positive answer for the strong uniform Artin-Rees Lemma for the family of maximal ideals in excellent rings. Huneke [11] showed that in many rings a weak version of the uniform Artin-Rees Lemma holds for the family of all ideals. In general the strong uniform Artin-Rees Lemma does not hold. Wang [22] gave a counterexample in a three dimensional ring, and we give two counterexamples in rings of dimension two (see Examples 2.15 and 2.16). Planas-Vilanova [19] showed that in a one-dimensional excellent ring the strong uniform version of the Artin-Rees Lemma holds for the family of all ideals. We give a much simpler proof of this fact (see Corollary 2.14).

In the *third chapter* we study a uniform version of the Artin-Rees Lemma for syzygies in a free resolution. Let $(R, \mathbf{m})$ be a local Noetherian ring and M a finitely generated R-module. Let $\mathbb{F} = \{F_i\}$ be a free resolution of M. Denote by $M_i \subset F_i$ the i-th syzygy. We study if, given an ideal $I \subset R$, there exists an integer h such that

$$M_i \cap I^n F_i \subset I^{n-h} M_i,$$

for all $n > h$ and for all $i > 0$. If such an integer h exists we say that M has the uniform Artin-Rees property for syzygies with respect to an ideal I. This question was first raised by Eisenbud-Huneke [9], who proved that such h exists when the module M has finite projective dimension on the punctured spectrum.

We show that having the uniform Artin-Rees property for syzygies well behaves in short exact sequence. In particular if any of the two modules appearing in a short exact sequence have the property, then also the third one has it. We relate the uniform Artin-Rees property for syzygies with respect to an ideal I with uniform annihilators of the family of modules $\mathcal{T}_{i,n} = \operatorname{Tor}_i^R(M, R/I^n)$. In Cohen-Macaulay rings any finitely generated R-module have the uniform Artin-Rees property for syzygies with respect to $\mathbf{m}$-primary ideals. The proof of this fact was shown to me by Professor Katz.

We show that in a one dimensional ring any finitely generated R-module has the uniform Artin-Rees property with respect to the maximal ideal $\mathbf{m}$.

In analogy to the classical version of the Artin-Rees Lemma, we can ask whether given an ideal I there exists an integer h such that

$$M_i \cap I^n F_i = I(I^{n-1} F_i \cap M_i),$$

for every $n > h$ and for every $i > 0$. If such an integer h exists we say that the module M has the strong Artin-Rees property for syzygies. We show the following

Theorem 0.3. *Let* $(R, \mathbf{m}, \mathsf{k})$ *be a local Noetherian ring. Then* k *has the strong uniform Artin-Rees property for syzygies with respect to the maximal ideal* $\mathbf{m}$.

Chapter 1

Short exact sequences

In this chapter we will study closely the correspondence between short exact sequences and elements of $\operatorname{Ext}^1_R(-,-)$. All rings are assumed to be commutative with identity and Noetherian; moreover all modules are finitely generated.

In the first section we go through the Yoneda correspondence, setting up the notation. A more extended description of such correspondence can be found in [14].

Under this correspondence, Miyata's Theorem, see [16], gives a complete characterization of splitting exact sequences. In the second section we present an extension of this theorem.

We later study more properties of short exact sequences. In particular we give some situations when we are able to say if a short exact sequence is a minimal generator for $\operatorname{Ext}^1_R(M,N)$; we describe $y\operatorname{Ext}^1_R(M,N)$ as union of particular subsets of $\operatorname{Ext}^1_R(M,N)$, for y a non-zerodivisor on the ring R and on the module M; we also introduce the notion of sparse module and we derive some properties of such modules. The notion of sparse module rises from the behaviour of $\operatorname{Ext}^1_R(M,N)$, where R is a ring of finite Cohen-

Macaulay type and M and N are maximal Cohen-Macaulay modules. We give some applications of such a situation: for example we give a new proof of the fact that $\mathrm{Ext}^1_R(M,N)$ is a module of finite length if M and N are maximal Cohen-Macaulay over a ring of finite Cohen-Macaulay type, improving the bound given in [12] on the power of the maximal ideal that kills the Ext module.

Finally, we use the developed techniques to give a different proof of a theorem of O'Carroll and Popescu, [18].

1.1 The Yoneda correspondence

We describe the Yoneda correspondence between short exact sequences and elements of the group Ext^1_R. Let R be a Noetherian ring and M and N two finitely generated R-modules.

Definition 1.1. An *extension* of M by N is a short exact sequence

$$\alpha : 0 \longrightarrow N \longrightarrow X_\alpha \longrightarrow M \longrightarrow 0. \tag{1.1.1}$$

On the set of all extensions of M by N there is an equivalence relation such that the quotient set is naturally isomorphic to $\mathrm{Ext}^1_R(M,N)$.

Given two extensions of M by N, α as in the Definition 1.1 and

$$\beta : \quad 0 \to N \to X_\beta \to M \to 0, \tag{1.1.2}$$

we say $\alpha \sim \beta$ if there exists an R-homomorphism h making the following diagram commute

$$\begin{array}{ccccccccc} 0 & \longrightarrow & N & \longrightarrow & X_\alpha & \longrightarrow & M & \longrightarrow & 0 \\ & & \| & & \downarrow h & & \| & & \\ 0 & \longrightarrow & N & \longrightarrow & X_\beta & \longrightarrow & M & \longrightarrow & 0. \end{array}$$

By the Snake Lemma, h is an isomorphism and therefore the relation $\sim$ is an equivalence relation among extensions.

Denote by $\text{ext}(M,N)$ the set of all extensions of M by N modulo the equivalent relation $\sim$. In the following part of the section, we will go through the correspondence between $\text{ext}(M,N)$ and $\text{Ext}^1_R(M,N)$. Before doing so, it is useful to recall the construction of pushout.

Discussion 1.2. Let $\tilde{\alpha}$ be the following exact sequence:

$$\tilde{\alpha}: \quad M_2 \xrightarrow{f} M_1 \xrightarrow{g} M \longrightarrow 0,$$

where M_1, M_2 are finitely generated $R-$modules.

Suppose $h: M_2 \to N$ is an $R-$homomorphism, define P as the cokernel of the map:

$$M_2 \xrightarrow{(h,-f)} N \oplus M_1 \longrightarrow P \longrightarrow 0.$$

Consider the following diagram:

$$\begin{array}{lccccccc} \tilde{\alpha}: & M_2 & \xrightarrow{f} & M_1 & \xrightarrow{g} & M & \longrightarrow & 0 \\ & \downarrow h & & \downarrow j & & \| & & \\ h\tilde{\alpha}: & N & \xrightarrow{i} & P & \xrightarrow{\pi} & M & \longrightarrow & 0, \end{array}$$

where the maps are defined as follows:

$$\begin{aligned} i(n) &= \overline{(n,0)} \quad \text{for any } n \in N, \\ j(x) &= \overline{(0,x)} \quad \text{for any } x \in M_1, \\ \pi(\overline{(n,0)}) &= g(x). \end{aligned}$$

Then the diagram commutes and the bottom row is an exact sequence. Literally, the pushout of the diagram is the module P; we will abuse notation and call the exact sequence $h\tilde{\alpha}$ the *push out of* $\tilde{\alpha}$ *via the homomorphism* h.

Notice that if f is an injective map then so is i.

The pushout 1.2 has the following universal property:

Lemma 1.3. *Suppose the following commutative diagram is given:*

$$
\begin{array}{lccccccc}
\mu: & M_2 & \xrightarrow{f_2} & M_1 & \xrightarrow{f_1} & M & \longrightarrow & 0 \\
 & \downarrow{\scriptstyle \phi_2} & & \downarrow{\scriptstyle \phi_1} & & \downarrow{\scriptstyle \phi} & & \\
\nu: & N_2 & \xrightarrow{g_2} & N_1 & \xrightarrow{g_1} & N & \longrightarrow & 0.
\end{array}
$$

Then the diagram factors through the pushout of μ via ϕ_2. More precisely there exists the following commutative diagrams:

$$
\begin{array}{lccccccc}
\mu: & M_2 & \xrightarrow{f_2} & M_1 & \xrightarrow{f_1} & M & \longrightarrow & 0 \\
 & \downarrow{\scriptstyle \phi_2} & & \downarrow{\scriptstyle j} & & \| & & \\
\phi_2\mu: & N_2 & \xrightarrow{i} & P & \xrightarrow{\pi} & M & \longrightarrow & 0 \\
 & \| & & \downarrow{\scriptstyle \psi} & & \downarrow{\scriptstyle \phi} & & \\
\nu: & N_2 & \xrightarrow{g_2} & N_1 & \xrightarrow{g_1} & N & \longrightarrow & 0,
\end{array}
$$

where $\psi j = \phi_1$.

Proof. Define $\psi : P \to N_1$ such that $\psi(\overline{(n,m)}) = g_2(n) + \phi_1(m)$. It is a well-defined map since:

$$\psi(0) = g_2(\phi_2(m)) + \phi_1(-f_2(m)) = 0,$$

for any $m \in M_1$. Moreover for any $m \in M_2$ we have

$$\psi j(m) = \phi_1(m).$$

□

Immediately we have the following

Corollary 1.4. *Let μ be an exact sequence $M_2 \to M_1 \to M \to 0$ and let $\phi : M_2 \to N$ be an R-homorphism. If the pushout $\phi\mu$ is an extension of M by N then it is unique in* $\mathrm{ext}(M, N)$*.*

Lemma 1.5. *Let $\tilde{\alpha}$ an exact sequence as in the previous discussion, and let β_1 and β_2 two extensions of M by N, obtained respectively by pushing out $\tilde{\alpha}$ via h_1 and h_2. Then $\beta_1 = \beta_2$ in* $\mathrm{ext}(M, N)$ *if and only if there exists a map $\psi : M_1 \to N$ such that $\psi f = h_1 - h_2$.*

Proof. Let

$$\begin{array}{lccccccccc} \tilde{\alpha}: & & & M_2 & \xrightarrow{f} & M_1 & \xrightarrow{g} & M & \longrightarrow & 0, \\ & & & \downarrow{\scriptstyle h_1} & & \downarrow{\scriptstyle j_1} & & \| & & \\ \beta_1: & 0 & \longrightarrow & N & \xrightarrow{i_1} & P_1 & \xrightarrow{\pi_1} & M & \longrightarrow & 0, \end{array}$$

and

$$\begin{array}{lccccccccc} \tilde{\alpha}: & & & M_2 & \xrightarrow{f} & M_1 & \xrightarrow{g} & M & \longrightarrow & 0, \\ & & & \downarrow{\scriptstyle h_2} & & \downarrow{\scriptstyle j_2} & & \| & & \\ \beta_2: & 0 & \longrightarrow & N & \xrightarrow{i_2} & P_2 & \xrightarrow{\pi_2} & M & \longrightarrow & 0, \end{array}$$

obtained as in the discussion 1.2.

Suppose there is a map $\psi : M_1 \to N$ such that $\psi f = h_1 - h_2$. Define the map $\phi : P_1 \to P_2$ such that $\phi(\overline{(n, x)}) = \overline{(n + \psi(x), x)}$, for any $n \in N$ and any $x \in M_1$. Notice that it is a well-defined map since

$$\phi(0) = \phi(\overline{(h_1(n), -f(n))}) = \overline{(h_1(n) - \psi f(n), -f(n))} = \overline{(h_2(n), -f(n))} = 0.$$

The map ϕ is the one that satisfies the definition of $\beta_1 = \beta_2$.

On the other hand suppose that $\beta_1 = \beta_2$. Then, by definition, we have the following diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & N_1 & \xrightarrow{i_1} & P_1 & \xrightarrow{\pi_1} & M & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle \phi} & & \| & & \\ 0 & \longrightarrow & N_1 & \xrightarrow{i_2} & P_2 & \xrightarrow{\pi_2} & M & \longrightarrow & 0. \end{array}$$

For any $x \in M_1$, define

$$\overline{\psi}(x) = \phi j_1(x) - j_2(x) \in P_2.$$

Since $\pi_2(\overline{\psi}(x)) = 0$, there exists a map $\psi : M_1 \to N$ such that $i_2\psi = \overline{\psi}$. For any $n \in M_2$, we have:

$$\begin{aligned} i_2\psi f(n) &= \overline{\psi} f(n) = \phi j_1(f(n)) - j_2(f(n)) = \phi i_1 h_1(n) - i_2 h_2(n) \\ &= i_2 h_1(n) - i_2 h_2(n) = i_2(h_1 - h_2)(n). \end{aligned}$$

By assumption i_2 is an injective map and therefore $\psi f = h_1 - h_2$. $\square$

Let the following sequence be part of a projective resolution $\mathbb{F}$ of M:

$$\cdots \longrightarrow P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} M \longrightarrow 0.$$

Recall that $\operatorname{Ext}^1_R(M, N)$ is the first cohomology of the complex $\operatorname{Hom}_R(\mathbb{F}, N)$,

$$0 \longrightarrow \operatorname{Hom}_R(M, N) \xrightarrow{d_0^*} \operatorname{Hom}_R(P_0, N) \xrightarrow{d_1^*} \cdots$$

$$\cdots \longrightarrow \operatorname{Hom}_R(P_1, N) \xrightarrow{d_2^*} \operatorname{Hom}_R(P_2, N) \longrightarrow \cdots,$$

i.e.

$$\operatorname{Ext}^1_R(M, N) = \frac{\ker d_2^*}{\operatorname{Image} d_1^*}.$$

An element in $\operatorname{Ext}^1_R(M,N)$ is given by an equivalence class $\xi + \operatorname{Image} d_1^*$, where $\xi \in \ker d_2^*$.

Define a map $\Psi : \operatorname{Ext}^1_R(M,N) \to \operatorname{ext}(M,N)$ such that $\Psi(\xi + \operatorname{Image} d_1^*)$ is the short exact sequence γ obtained as a pushout of $P_1 \to P_0 \to M \to 0$ via ξ, as described in discussion 1.2:

$$\begin{array}{ccccccccccc}
\cdots & \longrightarrow & P_2 & \xrightarrow{d_2} & P_1 & \xrightarrow{d_1} & P_0 & \xrightarrow{d_0} & M & \longrightarrow & 0, \\
 & & & & \downarrow{\scriptstyle \xi} & & \downarrow{\scriptstyle j} & & \| & & \\
\gamma : & & 0 & \longrightarrow & N & \xrightarrow{i} & P & \longrightarrow & M & \longrightarrow & 0.
\end{array}$$

Remark 1.6. Notice that

(1) γ is a short exact sequence. Indeed if $i(n) = 0$ for some $n \in N$ then, by the definition of P,

$$(n, 0) = (\xi(p), -d_1(p)) \quad \text{for some } p \in P_1.$$

Since $\ker(d_1) = \operatorname{Image}(d_2)$, there exists a $q \in P_2$ such that $d_2(q) = p$. But $\xi d_2 = 0$, since $\xi \in \ker(d_2^*)$; therefore $n = \xi(p) = \xi d_2(q) = 0$.

(2) The definition of γ does not depend on the representative of $\xi + \operatorname{Image} d_1^*$, by Lemma 1.5.

To define a map $\Phi : \operatorname{ext}(M,N) \to \operatorname{Ext}^1_R(M,N)$, consider $\gamma \in \operatorname{ext}(M,N)$ and a projective resolution of M. Lift the identity map of M as in the following diagram:

$$\begin{array}{ccccccccccc}
\cdots & \longrightarrow & P_2 & \xrightarrow{d_2} & P_1 & \xrightarrow{d_1} & P_0 & \xrightarrow{d_0} & M & \longrightarrow & 0, \\
 & & \downarrow{\scriptstyle \xi_2} & & \downarrow{\scriptstyle \xi} & & \downarrow{\scriptstyle \xi_0} & & \| & & \\
\gamma : & & 0 & \longrightarrow & N & \xrightarrow{i} & P & \longrightarrow & M & \longrightarrow & 0.
\end{array}$$

and define $\Phi(\gamma) = \xi + \operatorname{Image} d_1^* \in \operatorname{Ext}^1_R(M,N)$.

Remark 1.7. Notice that

(1) $\xi \in \ker(d_2^*)$, since the squares are commutative.

(2) Φ is a well defined map by Lemma 1.5 and since any two lifting are homotopic.

The maps Φ and Ψ are inverse to each other. Easily we can see that $\Phi\Psi(\xi + \mathrm{Image}(d_1^*)) = \xi + \mathrm{Image}(d_1^*)$, since $(\ldots, 0, \ldots, 0, \xi, j)$ is a lifting of id_M. On the other hand $\Psi\Phi(\gamma) = \gamma$ by the uniqueness of the pushout, see Corollary 1.4.

Remark 1.8. Often in the following we will construct γ from

$$\sigma : \quad 0 \longrightarrow K \longrightarrow P_0 \xrightarrow{d_0} M \longrightarrow 0.$$

where $K = \ker d_0$. Indeed if $\xi \in \ker(d_2^*)$, $\mathrm{Image}\, d_2 \subset \ker \xi$ and therefore ξ induces a map

$$\overline{\xi} : \qquad \begin{array}{ccc} P_1/\,\mathrm{Image}\, d_2 & \longrightarrow & P_0 \\ \| & & \nearrow \\ P_1/\ker d_1 & & \\ \wr & \nearrow & \\ K & & \end{array}$$

Then it is easy to see that

$$\Psi(\xi + \mathrm{Image}\, d_1^*) = \overline{\xi}\sigma,$$

where $\overline{\xi}\sigma$ is the pushout of σ via $\overline{\xi}$.

The map Φ induces a structure of R-module on $\mathrm{ext}(M, N)$.

Remark 1.9. The zero element in $\text{ext}(M, N)$ corresponds to the pushout of σ via the 0 map, and by Lemma 1.5 via any map from K to N that can be extended to P_0. A representative extension for the zero element is the exact sequence $0 \to N \to N \oplus M \to M \to 0$ as the following diagram is commutative:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K & \longrightarrow & P_0 & \xrightarrow{\pi} & M_1 & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle (0,\pi)} & & \| & & \\ 0 & \longrightarrow & N & \longrightarrow & N \oplus M & \longrightarrow & M & \longrightarrow & 0. \end{array}$$

By the Snake Lemma, any representative of the zero element has the middle module isomorphic to $M \oplus N$ and it is a split exact sequence. On the other hand any short exact sequence with the middle module isomorphic to $M \oplus N$ is a representative of the zero element. The proof of this fact is a theorem by Miyata [16] which we will discuss in the next section.

We will finish the section by describing the first part of the long exact sequence induced by applying the functor Hom to an extension of M by N. Before doing so we need to introduce the pullback of a short exact sequence.

Discussion 1.10. Suppose

$$\alpha : 0 \longrightarrow N \xrightarrow{f} X_\alpha \xrightarrow{g} M \longrightarrow 0$$

is a given extension of M by N. Let $\phi : M_1 \to M$ an R-homorphism. Denote by Q the kernel of the following map:

$$0 \longrightarrow Q \longrightarrow X_\alpha \oplus M_1 \xrightarrow{g-\phi} M \longrightarrow 0.$$

Then the pullback of α via ϕ, which we will denote by $\alpha\phi$, is the exact top row, as in the following diagram:

$$
\begin{array}{lccccccccc}
\alpha\phi : & 0 & \longrightarrow & N & \xrightarrow{i} & Q & \xrightarrow{\pi} & M_1 & \longrightarrow & 0 \\
 & & & \| & & \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle \phi} & & \\
\alpha : & 0 & \longrightarrow & N & \xrightarrow{f} & X_\alpha & \xrightarrow{g} & M & \longrightarrow & 0,
\end{array}
$$

where the maps are

$$i(n) = (f(n), 0), \quad \text{for any } n \in N;$$

$$\pi(x, m) = g(x), \quad \text{for any } (n, x) \in Q;$$

$$p(n, x) = x, \quad \text{for any , } (n, x) \in Q.$$

The pullback of an extension has the following universal property, for the proof of which we refer the reader to [14].

Proposition 1.11. *Given the following commutative diagram:*

$$
\begin{array}{lccccccccc}
\alpha : & 0 & \longrightarrow & N & \xrightarrow{f_1} & X_\alpha & \xrightarrow{g_1} & M & \longrightarrow & 0 \\
 & & & \downarrow{\scriptstyle \phi_1} & & \downarrow{\scriptstyle \phi_2} & & \downarrow{\scriptstyle \phi_3} & & \\
\beta : & 0 & \longrightarrow & N & \xrightarrow{f} & X_\beta & \xrightarrow{g} & M & \longrightarrow & 0
\end{array}
$$

it factors through the pullback of β via ϕ_3, more precisely there exists a map ψ such that the $\psi p = \phi_2$ and the following diagram is commutative:

$$
\begin{array}{lccccccccc}
\alpha : & 0 & \longrightarrow & N & \xrightarrow{f_1} & X_\alpha & \xrightarrow{g_1} & M & \longrightarrow & 0 \\
 & & & \downarrow{\scriptstyle \phi_1} & & \downarrow{\scriptstyle \psi} & & \| & & \\
\beta\phi_3 : & 0 & \longrightarrow & N & \xrightarrow{i} & Q & \xrightarrow{\pi} & M & \longrightarrow & 0 \\
 & & & \| & & \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle \phi_3} & & \\
\beta : & 0 & \longrightarrow & N & \xrightarrow{f} & X_\beta & \xrightarrow{g} & M & \longrightarrow & 0.
\end{array}
$$

Lemma 1.12. *Given an extension α of M by N as in Discussion 1.10 and given a map $\phi : M_1 \to M$, $\alpha\phi$ is a split exact sequence if and only if there exists a map $\overline{\phi} : M_1 \to X_\alpha$ such that $g\overline{\phi} = \phi$.*

In the following we will drop the notation $\mathrm{ext}(M, N)$. We go throught the proof of the following proposition for later reference.

Proposition 1.13. *Let α be an extension of M by N*

$$0 \longrightarrow N \xrightarrow{f} X_\alpha \xrightarrow{g} M \longrightarrow 0.$$

Let A be a finitely generated R-module. There exists a long exact sequence:

$$0 \longrightarrow \mathrm{Hom}_R(A, N) \longrightarrow \mathrm{Hom}_R(A, X_\alpha) \longrightarrow \mathrm{Hom}_R(A, M) \xrightarrow{d} \cdots$$

$$\cdots \xrightarrow{d} \mathrm{Ext}^1_R(A, N) \xrightarrow{f^*} \mathrm{Ext}^1_R(A, X_\alpha) \xrightarrow{g^*} \mathrm{Ext}^1_R(A, M),$$

where

$$d(\phi) = \alpha\phi,$$

$$f^*(\beta) = f\beta,$$

$$g^*(\beta) = g\beta.$$

Proof. Exactness in $\mathrm{Hom}(A, M)$ is given by Lemma 1.12. Notice that by Corollary 1.4, $g(f(\alpha)) = (gf)(\alpha) = 0$. To prove exactness in $\mathrm{Ext}^1_R(A, X_\alpha)$ we need to show that for any $\beta \in \mathrm{Ext}^1_R(A, X_\alpha)$ such that $g\beta = 0$ there exists a $\gamma \in \mathrm{Ext}^1_R(A, N)$ such that $f\gamma = \beta$.

Consider the following diagram:

$$
\begin{array}{lcccccccccc}
 & & & N & & & & & & \\
 & & & \downarrow{\scriptstyle f} & & & & & & \\
\beta : & 0 & \longrightarrow & X_\alpha & \xrightarrow{h} & L & \xrightarrow{p} & A & \longrightarrow & 0 \\
 & & & \downarrow{\scriptstyle g} & \swarrow & \downarrow{\scriptstyle j} & & \| & & \\
g\beta : & 0 & \longrightarrow & M & \xrightarrow{i} & P & \xrightarrow{\pi} & A & \longrightarrow & 0
\end{array}
$$

By Lemma 1.5, the splitting of the sequence $g\beta$ implies the existence of a map $g' : L \to M$ such that $g'h = g$. Notice that $g'hf = gf = 0$, therefore Image $fh \subset \ker g' =: L'$. We claim that the sequence:

$$
\gamma : \qquad 0 \longrightarrow N \xrightarrow{fh} L' \xrightarrow{p|_{L'}} A \longrightarrow 0 \tag{1.1.3}
$$

is a short exact sequence and $f\gamma = \beta$. To prove the first claim we need to check first that the restriction of p to L' is still surjective. For it, let $a \in A$ and $l \in L$ such that $p(l) = a$. Let $x \in X_\alpha$ such that $g(x) = g'(l)$. Then $l - h(x) \in L'$ and it maps to a under p. To conclude the first claim we are left to prove exactness in L'. Since $ph = 0$ we have Image$(hf) \subset \ker(p|_{L'})$. For the other inclusion, take an $l \in L'$ such that $p(l) = 0$; since α is exact, there exists an $x \in X_\alpha$ such that $f(x) = l$. Moreover, $g(x) = g'h(x) = g'(l) = 0$ and therefore there exists an $n \in N$ such that $f(n) = x$, proving exactness in L'.

For the second claim notice that the following diagram commutes, and by the Lemma 1.4, we have that $f\gamma = \beta$.

$$
\begin{array}{lcccccccccc}
\gamma : & 0 & \longrightarrow & N & \xrightarrow{hf} & L' & \xrightarrow{p|_{L'}} & A & \longrightarrow & 0 \\
 & & & \downarrow{\scriptstyle f} & & \downarrow & & \| & & \\
\beta : & 0 & \longrightarrow & X_\alpha & \xrightarrow{h} & L & \xrightarrow{p} & A & \longrightarrow & 0
\end{array}
$$

□

Using a dual proof and the universal property of the pullback one can prove the following:

Proposition 1.14. *Let α be an extension of M by N*

$$0 \longrightarrow N \xrightarrow{f} X_\alpha \xrightarrow{g} M \longrightarrow 0.$$

Let A be a finitely generated R-module. There exists a long exact sequence:

$$0 \longrightarrow \mathrm{Hom}_R(M, A) \longrightarrow \mathrm{Hom}_R(X_\alpha, A) \longrightarrow \mathrm{Hom}_R(N, A) \xrightarrow{\delta} \cdots$$

$$\cdots \xrightarrow{\delta} \mathrm{Ext}^1_R(M, A) \xrightarrow{g^*} \mathrm{Ext}^1_R(X_\alpha, A) \xrightarrow{f^*} \mathrm{Ext}^1_R(M, A),$$

where

$$d(\phi) = \phi\alpha,$$

$$f^*(\beta) = \beta f,$$

$$g^*(\beta) = \beta g.$$

For the proof see [14], page 73-74.

1.2 Miyata's Theorem and its extension

In this section we present an extension of Miyata's Theorem, which characterizes when short exact sequences are split exact. In the following we refer to short exact sequences as elements of Ext, keeping in mind the correspondence of the previous section.

A very important tool used in this section is the following theorem. For the proof, in this context, we refer to [25]. For other proofs see [1], [3], [4], [10].

Theorem 1.15. *Let $(R, \mathbf{m})$ be a Noetherian local ring, and let M and N two finitely generated R-modules. Then $M \cong N$ if and only if*

$$\lambda_R(\mathrm{Hom}_R(M, L)) = \lambda_R(\mathrm{Hom}_R(N, L)),$$

for any R-module L such that $\lambda_R(L) < \infty$ if and only if

$$\lambda_R(M \otimes L) = \lambda_R(N \otimes L)$$

for any R-module L such that $\lambda_R(L) < \infty$.

Theorem 1.16 (Miyata). *Let R a Noetherian ring and let*

$$\alpha : \quad 0 \longrightarrow N \xrightarrow{f} X_\alpha \xrightarrow{g} M \longrightarrow 0,$$

be an element in $\mathrm{Ext}^1_R(M, N)$. Then α represents the zero element if and only if $X_\alpha \cong M \oplus N$.

The proof provided here is due to Huneke-Katz. We first state and prove separately a Lemma for easy reference.

Lemma 1.17. *Let R be a Noetherian ring. Let α be the following short exact sequence:*

$$\alpha : \quad 0 \longrightarrow N \xrightarrow{f} X_\alpha \xrightarrow{g} M \longrightarrow 0.$$

Denote by C the image of the connecting homomorphism δ (see the definition in Proposition 1.14), obtained by applying the functor $\mathrm{Hom}_R(\quad, N)$ to α:

$$0 \longrightarrow \mathrm{Hom}_R(M, N) \longrightarrow \mathrm{Hom}_R(X_\alpha, N) \longrightarrow \mathrm{Hom}_R(N, N) \xrightarrow{\delta} \cdots$$

$$\cdots \xrightarrow{\delta} \mathrm{Ext}^1_R(M, N) \xrightarrow{g^*} \mathrm{Ext}^1_R(X_\alpha, N) \xrightarrow{f^*} \mathrm{Ext}^1_R(N, N).$$

Then α is a split exact sequence if and only if $C = 0$.

Proof. Suppose $C = 0$, then since $\alpha = \delta(\mathrm{id}_N)$ we have that $\alpha = 0$ and hence α is a split exact sequence, see Remark 1.9.

On the other hand suppose that α is a split exact sequence, hence there exists an R-homomorphism $f' : X_\alpha \to N$, such that $f'f = \mathrm{id}_N$. To prove the lemma, we need to show that the map $f^* : \mathrm{Hom}(X_\alpha, N) \to \mathrm{Hom}(N, N)$ is a surjective map. But for any $l \in \mathrm{Hom}(N, N)$ we have $l = f^*(lf')$. □

Now the proof of Theorem 1.16.

Huneke-Katz. One direction of the Theorem has been already discussed in Remark 1.9: any representative of the zero element in $\mathrm{Ext}^1_R(M, N)$ has middle module isomorphic to $M \oplus N$. For the other direction, by Lemma 1.17, it is enough to prove that $C = 0$. By way of contradiction, assume $C \neq 0$ and let $P \in \mathrm{Spec}(R)$ be a minimal prime over $\mathrm{Ann}(C)$. By localizing at P, we may assume $(R, \mathbf{m})$ local, and C is finite length. Since $X_\alpha \cong M \oplus N$, we have the following exact sequence:

$$0 \longrightarrow \mathrm{Ext}^1_R(M, N)/C \longrightarrow \mathrm{Ext}^1_R(M, N) \oplus \mathrm{Ext}^1_R(N, N) \longrightarrow \mathrm{Ext}^1_R(N, N),$$

and therefore

$$0 \longrightarrow \mathrm{H}^0_{\mathbf{m}}\left(\frac{\mathrm{Ext}^1_R(M,N)}{C}\right) \longrightarrow \mathrm{H}^0_{\mathbf{m}}(\mathrm{Ext}^1_R(M, N)) \oplus \mathrm{H}^0_{\mathbf{m}}(\mathrm{Ext}^1_R(N, N)) \longrightarrow$$

$$\cdots \longrightarrow \mathrm{H}^0_{\mathbf{m}}(\mathrm{Ext}^1_R(N, N)).$$

Since $\mathrm{H}^0_{\mathbf{m}}(\mathrm{Ext}^1_R(M, N)/C) = \mathrm{H}^0_{\mathbf{m}}(\mathrm{Ext}^1_R(M, N))/C$, by counting lengths we obtain $C = 0$, which is a contradiction. □

Locally, the following is an equivalent statement:

Theorem 1.18. *Let* $(R, \mathbf{m})$ *be a local Noetherian ring and let*

$$\alpha : \quad 0 \longrightarrow N \xrightarrow{f} X_\alpha \xrightarrow{g} M \longrightarrow 0,$$

be an element in $\operatorname{Ext}^1_R(M, N)$*. Then* α *represents the zero element if and only if there exists a surjection*

$$X_\alpha \twoheadrightarrow M \oplus N.$$

Proof. The only thing to show is that in this case the surjection implies the existence of an isomorphism. Indeed, for any R-module of finite length L, we have the following inequalities:

$$\begin{aligned} \lambda(X_\alpha \otimes L) &\leq \lambda(M \otimes L) + \lambda(N \otimes L), \quad \text{by tensoring } \alpha \text{ by } L, \\ &\leq \lambda(X_\alpha \otimes L), \quad \text{by tensoring the surjection } \pi \text{ by } L. \end{aligned}$$

Therefore $\lambda(X_\alpha \otimes L) = \lambda((M \oplus N) \otimes L)$, for any R-module of finite length. Theorem 1.15 implies $X_\alpha \cong M \oplus N$. □

The dual statement of Theorem 1.18, where we replace the surjection by an injection, does not hold. For example if x, y are a regular sequence on $(R, \mathbf{m})$, then we have the short exact sequence:

$$\alpha : \qquad 0 \longrightarrow R \longrightarrow R^2 \longrightarrow (x, y) \longrightarrow 0.$$

By looking at the number of minimal generators of each module we conclude that α cannot be a split exact sequence. On the other hand there exists the injection $0 \to R \oplus (x, y) \to R^2$.

We can further relax the assumption on α. In particular we have the following

Lemma 1.19. *Let* $(R, \mathbf{m})$ *be a Noetherian ring and let* $\tilde{\alpha}$ *the following exact sequence of finitely generated* R*-modules:*

$$\tilde{\alpha}: \qquad N \xrightarrow{f} X_\alpha \xrightarrow{g} M \longrightarrow 0.$$

Suppose there exists a surjection

$$\pi: \qquad X_\alpha \twoheadrightarrow M \oplus N.$$

Then $\tilde{\alpha}$ *is a split short exact sequence.*

Proof. By Theorem 1.18, it is enough to show that the map f is an injection. Suppose by way of contradiction that there exists a non zero element $x \in N$ such that $f(x) = 0$. By the Krull Intersection Theorem there exists an $n \in \mathbb{N}$ such that $x \notin \mathbf{m}^n N$. For such a choice of n, tensor the exact sequence $\tilde{\alpha}$ by $R/\mathbf{m}^n$ to obtain:

$$0 \longrightarrow C \longrightarrow N/\mathbf{m}^n N \xrightarrow{\bar{f}} X_\alpha/\mathbf{m}^n X_\alpha \xrightarrow{\bar{g}} M/\mathbf{m}^n M \longrightarrow 0,$$

where $C \neq 0$ since $x \bmod \mathbf{m}^n N \in C$. Notice that by tensoring the surjection we still have a surjection:

$$\bar{\pi}: \qquad X_\alpha/\mathbf{m}^n X_\alpha \twoheadrightarrow M/\mathbf{m}^n M \oplus N/\mathbf{m}^n N.$$

By counting the lengths of the modules we obtain

$$\begin{aligned} \lambda(X_\alpha/\mathbf{m}^n X_\alpha) + \lambda(C) &= \lambda(N/\mathbf{m}^n N) + \lambda(M/\mathbf{m}^n M) \\ &\leq \lambda(X_\alpha/\mathbf{m}^n X_\alpha), \quad \text{from the surjection } \bar{\pi}. \end{aligned}$$

Hence $C = 0$, which is a contradiction. □

We are ready to state and prove the main Theorem of the section.

Theorem 1.20. *Let R be a Noetherian ring and let $I \subset R$ be an ideal. Let M and N be finitely generated R-modules. Let $\alpha : 0 \to N \to X_\alpha \to M \to 0$ be a short exact sequence and denote by $\alpha \otimes R/I$ the complex (not necessarily exact) $0 \to N/IN \to X_\alpha/IX_\alpha \to M/IM \to 0$. If $\alpha \in I\operatorname{Ext}^1_R(M,N)$ then $\alpha \otimes R/I$ is a split exact sequence.*

Proof. We first show that it is enough to prove a local statement. Assume that the local version of the theorem holds, and let

$$\alpha : 0 \longrightarrow N \xrightarrow{f} X_\alpha \xrightarrow{g} M \longrightarrow 0$$

be a short exact sequence in $I\operatorname{Ext}^1_R(M,N)$.
By Corollary 1.4, $\alpha/1 \in \operatorname{Ext}^1_{R_\mathbf{m}}(M_\mathbf{m}, N_\mathbf{m})$ is given by the sequence

$$0 \longrightarrow N_\mathbf{m} \xrightarrow{f/1} (X_\alpha)_\mathbf{m} \xrightarrow{g/1} M_\mathbf{m} \longrightarrow 0,$$

for any maximal ideal $\mathbf{m}$. Let C be the kernel of $f \otimes \operatorname{id}_{R/I}$, then $\operatorname{Supp}(C) \subset V(I)$. Let $\mathbf{m} \in V(I)$. $\alpha \in I\operatorname{Ext}^1_R(M,N)$ implies $\alpha/1 \in IR_\mathbf{m}\operatorname{Ext}^1_{R_\mathbf{m}}(M_\mathbf{m}, N_\mathbf{m})$ and therefore $\alpha/1 \otimes R_\mathbf{m}/IR_\mathbf{m}$ is a split exact sequence. In particular

$$C_\mathbf{m} = \ker(f \otimes \operatorname{id})_\mathbf{m} = \ker(f/1 \otimes \operatorname{id}_{R_\mathbf{m}/IR_\mathbf{m}}) = 0,$$

proving that $\alpha \otimes R/I$ is a short exact sequence. We need to show that $\alpha \otimes R/I$ is actually a split exact sequence. By Lemma 1.17 it is enough to prove that $\operatorname{Image}(\delta) = 0$, where δ is the connecting homomorphism in the following sequence (for its definition see Proposition 1.14):

$$\operatorname{Hom}_R(\tfrac{X_\alpha}{IX_\alpha}, \tfrac{N}{IN}) \longrightarrow \operatorname{Hom}_R(\tfrac{N}{IN}, \tfrac{N}{IN}) \xrightarrow{\delta} \operatorname{Ext}^1_R(\tfrac{M}{IM}, \tfrac{N}{IN}).$$

Call $C = \operatorname{Image}(\delta)$, again $\operatorname{Supp}(C) \subset V(I)$. Let $\mathbf{m}$ any maximal ideal in $V(I)$, then $C_\mathbf{m} = \operatorname{Image}(\overline{\delta})$, where $\overline{\delta}$ is defined in Proposition 1.14:

$$\operatorname{Hom}_{R_\mathbf{m}}(N_\mathbf{m}/IN_\mathbf{m}, N_\mathbf{m}/IN_\mathbf{m}) \xrightarrow{\overline{\delta}} \operatorname{Ext}^1_{R_\mathbf{m}}(M_\mathbf{m}/IM_\mathbf{m}, N_\mathbf{m}/IN_\mathbf{m}).$$

But, by Lemma 1.17, Image$(\overline{\delta}) = 0$ since $\alpha/1 \otimes R_{\mathbf{m}}/IR_{\mathbf{m}}$ is a split exact sequence. Therefore $C_{\mathbf{m}} = 0$ for any $\mathbf{m} \in V(I)$, which implies that $C = 0$.

We can assume $(R, \mathbf{m})$ is a local Noetherian ring. By Lemma 1.19, it is enough to show that $X_\alpha/IX_\alpha \cong M/IM \oplus N/IN$. Since $\alpha \in I\operatorname{Ext}^1_R(M,N)$, following the construction of the map Φ as in the previous section, one can choose a representative short exact sequence which is the pushout of the presentation of M

$$\sigma: \quad P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} M \longrightarrow 0$$

via a map $\xi \in I\operatorname{Hom}(P_1, N)$, more precisely via a map $\xi \in I\ker(d_2^*)$. We have the following diagram

$$\beta: P_1 \xrightarrow{(d_1,-\xi)} P_0 \oplus N \longrightarrow X_\alpha \longrightarrow 0.$$

Let Ω be a finitely generated module of finite length such that $I\Omega = 0$. Tensor both the sequences β and σ with Ω and denote by Image_1 and by Image_2 the images $d_1 \otimes \text{id}(P_1 \otimes \Omega)$ and $(d_1 \otimes \text{id}, -\xi \otimes \text{id})(P_1 \otimes \Omega)$. Since Image$(\xi) \subset IN$, it follows that $\text{Image}_1 = \text{Image}_2 \subset P_0 \otimes \Omega$.

If $\lambda_R(M)$ denotes the length of an R-module M, we have:

$$\begin{aligned}
\lambda_R(X_\alpha \otimes \Omega) &= \lambda_R(P_0 \otimes \Omega) + \lambda_R(N \otimes \Omega) - \lambda_R(\text{Image}_1) \\
&= \lambda_R(M \otimes \Omega) + \lambda_R(\text{Image}_2) + \lambda_R(N \otimes \Omega) - \lambda_R(\text{Image}_1) \\
&= \lambda_R(M \otimes \Omega) + \lambda_R(N \otimes \Omega).
\end{aligned}$$

Notice that if Y is any R-module, which is also an R/I-module, then we have that $\lambda_R(Y) = \lambda_{R/I}(Y)$, hence the equality above is just:

$$\lambda_{R/I}(M/IM \otimes \Omega) + \lambda_{R/I}(N/IN \otimes \Omega) = \lambda_{R/I}(X_\alpha/IX_\alpha \otimes \Omega).$$

In this equality above we can choose any R/I module of finite length Ω, therefore by Theorem 1.15 we have:

$$M/IM \oplus N/IN \simeq X_\alpha/IX_\alpha.$$

□

An immediate corollary is the following

Theorem 1.21. *Let R be a Noetherian ring and $I \subset R$ an ideal. Let α and β be two short exact sequences in $\operatorname{Ext}^1_R(M,N)$, with isomorphic middle modules. If $\alpha \in I\operatorname{Ext}^1_R(M,N)$ then $\beta \otimes R/I$ is a split exact sequence.*

Notice that this is an extension of Miyata's Theorem. For, if I is the zero ideal then α is a split exact sequence and the middle module is isomorphic to $M \oplus N$.

Proof. Let

$$\alpha : 0 \longrightarrow N \longrightarrow X_\alpha \longrightarrow M \longrightarrow 0,$$

and

$$\beta : 0 \longrightarrow N \longrightarrow X_\beta \longrightarrow M \longrightarrow 0,$$

where $X_\alpha \cong X_\beta$. Since $\alpha \in I\operatorname{Ext}^1_R(M,N)$, $\alpha \otimes R/I$ is a split exact sequence, by Theorem 1.20. Therefore,

$$\begin{aligned} X_\beta/IX_\beta &\cong X_\alpha/IX_\alpha \\ &\cong M/IM \oplus N/IN. \end{aligned}$$

By Lemma 1.19, the complex $N/IN \to X_\beta/IX_\beta \to M/IM \to 0$ is a split exact sequence. □

In general, given a short exact sequence $\alpha : 0 \to N \to X \to M \to 0$, tensoring by R/I gives a long exact sequence with homology groups. As we have seen, if $\alpha \in I \operatorname{Ext}^1_R(M, N)$ implies that the first three modules in this long exact sequence form a split short exact sequence. We can ask the following

Question 1.22. If $\alpha \in I \operatorname{Ext}^1_R(M, N)$, does the long exact sequence

$$\cdots \to \operatorname{Tor}_i(R/I, N) \to \operatorname{Tor}_i(R/I, X_\alpha) \to \operatorname{Tor}_i(R/I, M) \to \dots$$

split in short split exact sequences

$$0 \to \operatorname{Tor}_i(R/I, N) \to \operatorname{Tor}_i(R/I, X_\alpha) \to \operatorname{Tor}_i(R/I, M) \to 0$$

for every i?

The converse of Theorem 1.20 does not hold in general, as the following example shows.

Example 1.23. Let $R = \mathsf{k}[|x^2, x^3|]$. Every non-zero element $\alpha \in \operatorname{Ext}^1_R(\mathsf{k}, R)$ is a minimal generator and hence is not in $\mathbf{m} \operatorname{Ext}^1_R(\mathsf{k}, R)$.

Let $\alpha : 0 \to R \to P \to \mathsf{k} \to 0$, where P is the pushout of the following diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbf{m} & \xrightarrow{i} & R & \longrightarrow & \mathsf{k} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \psi} & & \downarrow & & \| & & \\
0 & \longrightarrow & R & \longrightarrow & P & \longrightarrow & \mathsf{k} & \longrightarrow & 0,
\end{array}$$

where ψ is the R-homomorphism sending x^2 to x^3 and x^3 to x^4. The short exact sequence α is not split exact because there is no map from R that extends ψ, hence α is not in $\mathbf{m} \operatorname{Ext}^1_R(\mathsf{k}, R)$. On the other hand, the minimal number of generators of P is 2 and hence $P/\mathbf{m}P \simeq \mathsf{k} \oplus \mathsf{k}$.

However, the following holds

Proposition 1.24. *Let M and N be finitely generated R-modules, let $y \in R$ be a non-zerodivisor on R, M, N and let α be the short exact sequence*

$$\alpha : 0 \longrightarrow N \xrightarrow{f} X_\alpha \longrightarrow M \longrightarrow 0,$$

Suppose that $X_\alpha / yX_\alpha \simeq M/yM \oplus N/yN$. Then $\alpha \in y \operatorname{Ext}^1_R(M, N)$.

Proof. Since y is a non-zerodivisor on N we have the following exact sequences:

$$0 \longrightarrow N \xrightarrow{y} N \xrightarrow{\pi} N/yN \longrightarrow 0,$$

and, by applying the functor $\operatorname{Hom}_R(M, \quad)$,

$$\cdots \longrightarrow \operatorname{Ext}^1_R(M, N) \xrightarrow{y} \operatorname{Ext}^1_R(M, N) \xrightarrow{\pi^*} \operatorname{Ext}^1_R(M, N/yN).$$

By exactness, to show that $\alpha \in y \operatorname{Ext}^1_R(M, N)$, it is enough to show that $\pi^*(\alpha) = 0$.

Call $\phi : \operatorname{Ext}^1_R(M, N/yN) \to \operatorname{Ext}^1_{R/yR}(M/yM, N/yN)$ the isomorphism that takes a short exact sequence $\beta : 0 \to N/yN \to Y \to M \to 0$ to the short exact sequence $\beta \otimes R/yR : 0 \to N/yN \to Y/yY \to M/yM \to 0$, which is exact because y is a non-zerodivisor on M. Since $\pi^*(\alpha)$ is

$$0 \longrightarrow N/yN \longrightarrow X_\alpha / yf(N) \longrightarrow M \longrightarrow 0,$$

$\phi\pi^*(\alpha)$ is the short exact sequence:

$$0 \longrightarrow N/yN \longrightarrow X_\alpha / yX_\alpha \longrightarrow M/yM \longrightarrow 0,$$

which is split exact. Hence $\pi^*(\alpha) = 0$. □

Question 1.25. Does Proposition 1.24 hold for a sequence regular on R, M and N, of length greater than one?

1.3 More results on short exact sequences

In this section we present some results about short exact sequences that will be used in further chapters. Assume that $(R, \mathbf{m}, \mathsf{k})$ is a local Noetherian ring, unless otherwise specified. Recall that, as general notation, we denote by X_α the module that appears in the middle of a short exact sequence $\alpha \in \operatorname{Ext}^1_R(M, N)$. Let us start with the following

Definition 1.26. Let $\alpha : 0 \to N \to X_\alpha \to M \to 0$ be a short exact sequence. We say that α is additive if

$$\mu(M) + \mu(N) = \mu(X_\alpha).$$

The following Lemma gives a necessary and sufficient condition for a short exact sequence to be additive.

Lemma 1.27. *Let $\xi : 0 \to M_1 \to F \to M \to 0$ the initial part of a minimal free resolution of M and let α be the pushout of ξ via ϕ, where $\phi \in \operatorname{Hom}_R(M_1, N)$, (see 1.8). Then, α is additive if and only if $\phi(M_1) \subset \mathbf{m}N$.*

Proof. First of all notice that the map ϕ is not uniquely determined by the short exact sequence α; on the other hand, by Lemma 1.5, if ϕ_1 and ϕ_2 are two maps which give the same short exact sequence α, there exists a map $f : F_0 \to N$ such that $\phi_1 = \phi_2 + fi$. Since $M_1 \subset \mathbf{m}F$, we have that

$$\phi_1(M_1) \subset \mathbf{m}N \quad \text{if and only if} \quad \phi_2(M_1) \subset \mathbf{m}N.$$

Denote by i the inclusion of M_1 in F. By definition of the pushout, X_α is the cokernel of the map $(i, -\phi)$, as in the following short exact sequence:

$$\nu_\alpha : \quad 0 \longrightarrow M_1 \xrightarrow{(i,-\phi)} F_0 \oplus N \xrightarrow{\pi} X_\alpha \longrightarrow 0.$$

Tensor ν_α by $R/\mathbf{m}$ and denote the operation $\_\_ \otimes R/\mathbf{m}$ by $\bar{\ }$. We have:

$$\overline{X_\alpha} = \overline{F_0} \oplus (\overline{N}/\langle \overline{\phi}(\bar{l}) \,|\, \bar{l} \in \overline{M_1} \rangle),$$

where $\overline{\phi} : \overline{M_1} \longrightarrow \overline{N}$ is the map induced by ϕ.
If $\phi(M_1) \subset \mathbf{m}N$ then the module $\langle \overline{\phi}(\bar{l}) \,|\, \bar{l} \in \overline{L} \rangle = 0$, hence:

$$\mu(X_\alpha) = \mu(N) + \mu(F_0) = \mu(N) + \mu(M).$$

On the other hand, suppose there exists an $l \in M_1$ such that $\phi(l) \notin \mathbf{m}N$, then $\langle \overline{\phi}(\bar{l}) \,|\, \bar{l} \in \overline{L} \rangle \neq 0$ implies $\mu(\overline{N}/\langle \overline{\phi}(\bar{l}) \,|\, \bar{l} \in \overline{L} \rangle) < \mu(\overline{N})$. Therefore,

$$\mu(X_\alpha) < \mu(F_0) + \mu(N) = \mu(M) + \mu(N).$$

□

Corollary 1.28. *In the previous lemma, assume $N \simeq \mathsf{k}^n$ in the short exact sequence $\alpha : \quad 0 \to N \to X_\alpha \to M \to 0$. Then, α is split exact if and only if α is additive.*

Proof. If α is split exact then α is additive.

On the other hand, keeping the notation of the above lemma, suppose that α is additive, then by the previous lemma we have that $\phi(M_1) \subset \mathbf{m}N$, where ϕ is as in the previous lemma. But N is a vector space and $\mathbf{m}N = 0$. This says that the short exact sequence α is induced as a pushout by the 0 map from M_1 to N. Therefore α is split exact. □

1.3.1 Minimal generators in $\mathrm{Ext}^1_R(M,N)$

Before studying more in detail additive short exact sequences, we give some situations where we can describe minimal generators for $\mathrm{Ext}^1_R(M,N)$. First, as a corollary of Theorem 1.20, we have

Corollary 1.29. *Let α be a short exact sequence in $\mathrm{Ext}^1_R(M,N) \neq 0$. and let M_1 be the first syzygy of M in a minimal free resolution. If $\mathrm{Ext}^1_R(X_\alpha, M_1) = 0$ then α is a minimal generator of $\mathrm{Ext}^1_R(M,N)$.*

Proof. By way of contradiction, assume $\alpha \in \mathbf{m}\,\mathrm{Ext}^1_R(M,N)$. By Theorem 1.20, $\alpha \otimes R/\mathbf{m}$ is a split exact sequence and therefore

$$\mu(M) + \mu(N) = \mu(X_\alpha).$$

On the other hand, if $0 \to M_1 \to F \to M \to 0$ is the beginning of a minimal free resolution for M, there exists an R-homomorphism $\phi : M_1 \to N$ such that X_α is the cokernel as in the following short exact sequence:

$$\beta : 0 \longrightarrow M_1 \xrightarrow{(i,-\phi)} F \oplus N \longrightarrow X_\alpha \longrightarrow 0,$$

where i is the inclusion. Since $\mathrm{Ext}^1_R(X_\alpha, M_1) = 0$, β is a split exact sequence and $X_\alpha \oplus M_1 \cong F \oplus N$. Therefore we have the following contradiction:

$$\mu(M) + \mu(N) = \mu(X_\alpha) \leq \mu(X_\alpha) + \mu(M_1) = \mu(F) + \mu(N) = \mu(M) + \mu(N).$$

It follows that $\mu(M_1) = 0$ whixh implies that M is free and $\mathrm{Ext}^1_R(M,N) = 0$, which is a contradiction. $\square$

The same conclusion holds if we have $\mathrm{Ext}^1_R(X_\alpha, N) = 0$ instead of the condition $\mathrm{Ext}^1_R(X_\alpha, M_1) = 0$. For this, we need some preparatory propositions.

For easy reference, recall the following:

Lemma 1.30 (Schanuel). *Suppose that*

$$\alpha: \quad 0 \to N_\alpha \to X_\alpha \to M \to 0$$

$$\beta: \quad 0 \to N_\beta \to X_\beta \to M \to 0$$

are two short exact sequences where X_α and X_β are two free modules. Then,

$$N_\beta \oplus X_\alpha \cong X_\beta \oplus N_\alpha.$$

The previous lemma can be generalized in the following way:

Lemma 1.31. *Suppose that*

$$\alpha: \quad 0 \to N_\alpha \to X_\alpha \to M \to 0$$

$$\beta: \quad 0 \to N_\beta \to X_\beta \to M \to 0$$

are two short exact sequences such that $\operatorname{Ext}^1_R(X_\alpha, N_\beta) = \operatorname{Ext}^1_R(X_\beta, N_\alpha) = 0$. *Then,*

$$N_\beta \oplus X_\alpha \cong X_\beta \oplus N_\alpha.$$

Proof. Apply the functor $\operatorname{Hom}(\quad, N_\beta)$ to α to obtain

$$\cdots \to \operatorname{Hom}(N_\alpha, N_\beta) \to \operatorname{Ext}^1_R(M, N_\beta) \to \operatorname{Ext}^1_R(X_\alpha, N_\beta) = 0.$$

The surjectivity of the last map implies that there exists an R-homorphism ϕ such that β is given by the following pushout

$$\begin{array}{ccccccccc} \alpha: 0 & \longrightarrow & N_\alpha & \longrightarrow & X_\alpha & \longrightarrow & M & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle\phi} & & \downarrow & & \| & & \\ \beta: 0 & \longrightarrow & N_\beta & \longrightarrow & X_\beta & \longrightarrow & M & \longrightarrow & 0, \end{array}$$

where X_β is given as in the following short exact sequence:

$$\nu : \quad 0 \to N_\alpha \to N_\beta \oplus X_\alpha \to X_\beta \to 0.$$

Since ν is an element of $\operatorname{Ext}_R^1(X_\beta, N_\alpha) = 0$, we have that

$$N_\beta \oplus X_\alpha \cong X_\beta \oplus N_\alpha.$$

□

Proposition 1.32. *Let M and N finitely generated R-modules and assume that $\operatorname{Ext}_R^1(M,N) \neq 0$. Let α be a non zero element in $\operatorname{Ext}_R^1(M,N)$. If $\operatorname{Ext}_R^1(X_\alpha, N) = 0$ then α is a minimal generator of $\operatorname{Ext}_R^1(M,N)$.*

Proof. In this proof we keep the notation as in Corollary 1.29. By way of contradiction, assume $\alpha \in \mathbf{m}\operatorname{Ext}_R^1(M,N)$. By Theorem 1.20, $\alpha \otimes R/\mathbf{m}$ is a split exact sequence. Therefore, $\mu(X_\alpha) = \mu(M) + \mu(N)$. On the other hand, by Lemma 1.31 $N \oplus F_0 \cong M_1 \oplus X_\alpha$, which implies the following equalities:

$$\mu(X_\alpha) + \mu(M_1) = \mu(N) + \mu(F_0) = \mu(N) + \mu(M) = \mu(X_\alpha).$$

It follows that $M_1 = 0$, which implies that M is a free module and hence $\operatorname{Ext}_R^1(M,N) = 0$, contradicting the assumptions. □

1.3.2 Additive sequences

We now study additive sequences; we will prove first that they form a submodule of $\operatorname{Ext}_R^1(M,N)$.

Proposition 1.33. *Let α and β be two short exact sequences in $\operatorname{Ext}_R^1(M,N)$. If α and β are additive then $\alpha + \beta$ is additive.*

Proof. Assume that α and β are given respectively as pushouts of

$$\xi : 0 \longrightarrow M_1 \longrightarrow F \longrightarrow M \longrightarrow 0,$$

via the homomorphism ϕ_1, $\phi_2 \in \mathrm{Hom}_R(M_1, N)$. Then $\alpha+\beta$ is the pushout of ξ via the homomorphisms $\phi_1+\phi_2$. Since $\phi_i(M_1) \subset \mathbf{m}N$, $(\phi_1+\phi_2)(M_1) \subset \mathbf{m}N$ and hence, by Lemma 1.27, $\alpha+\beta$ is additive.

□

Proposition 1.34. *Let $\alpha \in \mathrm{Ext}^1_R(M,N)$ be an additive short exact sequence. Let N' be a finitely generated R-module and let $\phi \in \mathrm{Hom}_R(N,N')$. If $\beta \in \mathrm{Ext}^1_R(M,N')$ is the pushout of α via ϕ, then β is additive.*

Similarly, let M' be a finitely generated R-module and let $\phi \in \mathrm{Hom}_R(M',M)$. If β is the pullback of α via ϕ, then β is additive.

Proof. For the first part of the proposition, we have the following diagram

$$\begin{array}{ccccccccc} \alpha : 0 & \longrightarrow & N & \longrightarrow & X_\alpha & \longrightarrow & M & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \phi} & & \downarrow & & \| & & \\ \beta : 0 & \longrightarrow & N' & \longrightarrow & X_\beta & \longrightarrow & M & \longrightarrow & 0, \end{array}$$

where X_β is the cokernel of the short exact sequence

$$0 \longrightarrow N \longrightarrow N' \oplus X_\alpha \longrightarrow X_\beta \longrightarrow 0.$$

If β is not additive then we have the following contradiction:

$$\begin{aligned} \mu(X_\beta) + \mu(N) &< \mu(M) + \mu(N') + \mu(N) \\ &= \mu(X_\alpha) + \mu(N') \\ &\leq \mu(X_\beta) + \mu(N). \end{aligned}$$

The proof goes the same way for the second statment in the proposition. Indeed, we have the following diagram:

$$\begin{array}{ccccccccc}
\beta : 0 & \longrightarrow & N & \longrightarrow & X_\beta & \longrightarrow & M' & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow \phi & & \\
\alpha : 0 & \longrightarrow & N & \longrightarrow & X_\alpha & \longrightarrow & M & \longrightarrow & 0,
\end{array}$$

where X_β is the kernel of the short exact sequence

$$0 \longrightarrow X_\beta \longrightarrow M' \oplus X_\alpha \longrightarrow M \longrightarrow 0.$$

If β is not additive then we have the following contradiction

$$\begin{aligned}
\mu(X_\beta) + \mu(M) &< \mu(M) + \mu(N) + \mu(M') \\
&= \mu(X_\alpha) + \mu(M') \\
&\leq \mu(X_\beta) + \mu(M).
\end{aligned}$$

□

Definition 1.35. Denote by $< M, N > \subset \operatorname{Ext}^1_R(M, N)$ the set of all elements in $\operatorname{Ext}^1_R(M, N)$ which are additive.

Corollary 1.36. *$< M, N >$ is an R-module.*

Proof. Closure under multiplication by an element of R is assured by Proposition 1.34; indeed given a short exact sequence $\alpha \in \operatorname{Ext}^1_R(M, N)$ and an element $r \in R$, $r\alpha$ corresponds to the pushout of α via multiplication by r. Finally, closure under sum is assured by Proposition 1.33. □

If we apply Proposition 1.13 to an additive short exact sequence, then the maps can be restricted to additive short exact sequences. In particular,

Proposition 1.37. *Let $\alpha \in < M, N >$. Write*

$$\alpha :\ 0 \longrightarrow N \xrightarrow{f} X_\alpha \xrightarrow{g} M \longrightarrow 0.$$

Let A be a finitely generated R-module. There exists a long exact sequence:

$$0 \longrightarrow \mathrm{Hom}_R(A, N) \longrightarrow \mathrm{Hom}_R(A, X_\alpha) \longrightarrow \mathrm{Hom}_R(A, M) \xrightarrow{d} \cdots$$

$$\cdots \xrightarrow{d} < A, N > \xrightarrow{f^*} < A, X_\alpha > \xrightarrow{g^*} < A, M >,$$

where

$$d(\phi) = \alpha\phi,$$

$$f^*(\beta) = f\beta,$$

$$g^*(\beta) = g\beta.$$

Proof. The maps d and f^* are well-defined by Proposition 1.34. By Proposition 1.13, we have that $\mathrm{Image}\, d \subset \ker f^*$ and $\mathrm{Image}\, f^* \subset \ker g^*$. To prove the proposition we need to check that $\ker g^* \subset \mathrm{Image}\, f^*$.

For, let $\beta \in < A, X_\alpha >$ be such that $g^*(\beta) = 0$. Recall that in Proposition 1.13 we constructed the short exact sequence γ (1.1.3) such that $f^*(\gamma) = \beta$. In particular, we have the following situation:

$$\gamma : 0 \longrightarrow N \longrightarrow L' \longrightarrow A \longrightarrow 0,$$

$$\beta : 0 \longrightarrow X_\alpha \longrightarrow L \longrightarrow A \longrightarrow 0,$$

where L and L' are related by the following short exact sequence:

$$0 \longrightarrow L' \longrightarrow L \longrightarrow M \longrightarrow 0.$$

Assume that γ is not additive, then we have the following contradiction:

$$\begin{aligned}
\mu(N)+\mu(A) &= \mu(X_\alpha)-\mu(M)+\mu(A), && \text{since } \alpha \text{ is additive,}\\
&= \mu(L)-\mu(M), && \text{since } \beta \text{ is additive}\\
&\leq \mu(L'), && \text{by the relation between } L \text{ and } L'.
\end{aligned}$$

Since the other inequality always holds, we have that γ has the right number of generators. □

With a similar proof, the following Proposition holds (see Proposition 1.14):

Proposition 1.38. *Let α be an extension of M by N*

$$0 \longrightarrow N \xrightarrow{f} X_\alpha \xrightarrow{g} M \longrightarrow 0.$$

Let A be a finitely generated R-module. There exists a long exact sequence:

$$0 \longrightarrow \mathrm{Hom}_R(M,A) \longrightarrow \mathrm{Hom}_R(X_\alpha,A) \longrightarrow \mathrm{Hom}_R(N,A) \xrightarrow{d} \cdots$$

$$\cdots \xrightarrow{d} < M,A > \xrightarrow{g^*} < X_\alpha, A > \xrightarrow{f^*} < M,A >,$$

where

$$d(\phi)=\phi\alpha,$$

$$f^*(\beta)=\beta f,$$

$$g^*(\beta)=\beta g.$$

1.3.3 Structure of $\operatorname{Ext}^1_R(M,N)$

In this section we describe $y\operatorname{Ext}^1_R(M,N)$, where y is a non-zerodivisor on R, M and N as a disjoint union of subsets. This description will be particularly useful in studying rings of finite Cohen-Macaulay type. Recall that for any short exact sequence α, we denote by X_α the module which appears in the middle. Also, given an R-module X we denote by $[X]$ the isomorphism class of X.

For any R-module X that can fit in the middle of a short exact sequence in $\operatorname{Ext}^1_R(M,N)$, define the set:

$$E_{[X]} := \{\alpha \in \operatorname{Ext}^1_R(M,N) \quad \text{such that} \quad X_\alpha \cong X\}.$$

For any element $y \in \mathbf{m}$ define the set:

$$\mathcal{S}_y := \{[X] \quad \text{such that} \quad \exists \alpha \in y\operatorname{Ext}^1_R(M,N) \quad \text{and} \quad X_\alpha \cong X\}.$$

with these definitions in mind, we are ready to state and prove the following:

Theorem 1.39. *Let* $(R,\mathbf{m},\mathsf{k})$ *be a Noetherian local ring with strictly positive depth. Assume that* y *is a non-zerodivisor on* R, M *and* N. *Then*

$$y\operatorname{Ext}^1_R(M,N) = \bigcup_{X\in\mathcal{S}_y} E_{[X]}.$$

Proof. The fact that $y\operatorname{Ext}^1_R(M,N) \subset \bigcup_{X\in\mathcal{S}_y} E_{[X]}$ is clear by the definition of the sets $\mathcal{S}_y$. For the other inclusion, it is enough to show that if

$$E_{[X]} \cap y\operatorname{Ext}^1_R(M,N) \neq \emptyset,$$

then $E_{[X]} \subset y\operatorname{Ext}^1_R(M,N)$. Let $\alpha \in E_{[X]} \cap y\operatorname{Ext}^1_R(M,N)$ and $\beta \in E_{[X]}$. Since $X_\alpha \cong X_\beta \cong X$ we can apply Theorem 1.18 to conclude that $\beta \otimes R/(y)$ is a split exact sequence. Since y is a non-zerodivisor on M, N and R we can apply Proposition 1.24 to conclude that $\beta \in y \operatorname{Ext}^1_R(M,N)$. $\square$

Notice that the sets $E_{[X]}$ are not submodules in general.

Proposition 1.40. *For every $E_{[X]} \subset \operatorname{Ext}^1_R(M,N)$, where $X \ncong M \oplus N$, there exists an integer n such that $\mathbf{m}^n \operatorname{Ext}^1_R(M,N) \cap E_{[X]} = \emptyset$.*

To prove the proposition we need to recall the following Theorem due to Guralnick, [10].

Theorem 1.41. *Let $(R,\mathbf{m})$ be a Noetherian local ring. Let M and N be finite R-modules. Then there exists a nonnegative integer $h := h(M,N)$ such that for any $n \in \mathbb{N}$ and for any R-linear map $\sigma \in \operatorname{Hom}_R(M/\mathbf{m}^{n+h}M, N/\mathbf{m}^{n+h}N)$, there exists an R-homomorphism $\tau \in \operatorname{Hom}_R(M,N)$ such that σ and τ induce the same homorphism in $\operatorname{Hom}_R(M/\mathbf{m}^nM, N/\mathbf{m}^nN)$.*

Corollary 1.42. *Let $\alpha \in \operatorname{Ext}^1_R(M,N)$. There exists an integer $h(X_\alpha, N)$ such that if $\alpha \otimes R/\mathbf{m}^{h+1}$ is a split short exact sequence then α is a split exact sequence.*

Proof. Write

$$\alpha : 0 \longrightarrow N \xrightarrow{f} X_\alpha \longrightarrow M \longrightarrow 0.$$

Let $h(X_\alpha, N)$ the integer given by Theorem 1.41. For every R-module X and Y, given a map $l \in \operatorname{Hom}_R(X,Y)$, denote by l_n the map

$$l \otimes_R \operatorname{id}_{R/\mathbf{m}^n} : X/\mathbf{m}^nX \to Y/\mathbf{m}^nY.$$

Assume that $\alpha \otimes R/\mathbf{m}^{h+1}$ is a split exact sequence, then there exists an $R/\mathbf{m}^{h+1}$-homorphism

$$g' : X_\alpha/\mathbf{m}^{h+1}X_\alpha \to N/\mathbf{m}^{h+1}N$$

such that $f_{h+1}g' = \mathrm{id}_{h+1}$. By Theorem 1.41, we can lift g' to an R-homorphism $g \in \mathrm{Hom}_R(X_\alpha, N)$ in such a way that g and g' induce the same homorphism g_1 in $\mathrm{Hom}_R(X_\alpha/\mathbf{m}X_\alpha, N/\mathbf{m}N)$. Therefore we have the following chain of equalities:

$$\begin{aligned}
(gf)_1 &= g_1 f_1 \\
&= (g_{h+1} \otimes \mathrm{id}_{R/\mathbf{m}})(f_{h+1} \otimes \mathrm{id}_{R/\mathbf{m}}) \\
&= (g_{h+1} f_{h+1}) \otimes \mathrm{id}_{R/\mathbf{m}} \\
&= \mathrm{id}_{h+1} \otimes \mathrm{id}_{R/\mathbf{m}} \\
&= \mathrm{id}_1
\end{aligned}$$

This implies that $gf(N) = N + \mathbf{m}N$ and hence, by Nakayama's Lemma, gf is a surjective map from N to itself. Since N is Noetherian, gf is an isomorphism and α is a split exact sequence. □

Proof. of Proposition 1.40. Let $X \ncong M \oplus N$ and let $\alpha \in E_{[X]} \subset \mathrm{Ext}^1_R(M, N)$. Let $h(X_\alpha, N)$ be as in Theorem 1.41. We claim that for any $n > h(X_\alpha, N)$,

$$E_{[X]} \cap \mathbf{m}^n \,\mathrm{Ext}^1_R(M, N) = \emptyset.$$

Assume, by way of contradiction, that the intersection is not empty and let β be an element of it. Since $X_\alpha \cong X_\beta \cong X$, by Theorem 1.20, $\beta \in \mathbf{m}^n \,\mathrm{Ext}^1_R(M, N)$ implies $\alpha \otimes R/\mathbf{m}^n$ is a split exact sequence. Since $n \geq h+1$, $\alpha \otimes R/\mathbf{m}^{h+1}$ is a split exact sequence and by the above corollary it follows that α is a split exact sequence. This is a contradiction, since

$$X_\alpha \cong X \ncong M \oplus N.$$

□

It would be nice to know when the sum of two sequences in $E_{[X]}$ is still in $E_{[X]}$. In general this is hard to know, even in the case when X is a free module. Before giving an example, we prove the following

Lemma 1.43. *Let $\alpha \in E_{[F]} \cap \operatorname{Ext}^1_R(M,N)$, where F is a free R-module such that $\mu(F) = \mu(M)$. For any $\phi \in \operatorname{Hom}_R(N,N)$, $\phi\alpha \in E_{[F]}$ if and only if ϕ is an automorphism.*

Proof. Write

$$\begin{array}{ccccccccc} \alpha : 0 & \longrightarrow & N & \xrightarrow{f} & F & \longrightarrow & M & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \phi} & & \downarrow & & \| & & \\ \phi\alpha : 0 & \longrightarrow & N & \longrightarrow & P & \longrightarrow & M & \longrightarrow & 0, \end{array}$$

where P is the cokernel of the following exact sequence:

$$\nu : 0 \longrightarrow N \xrightarrow{(\phi,-f)} N \oplus F \longrightarrow P \longrightarrow 0.$$

If ϕ is an automorphism, then by Snake Lemma, we have $P \cong F$. If P is a free module the short exact sequence ν is split exact. Therefore there exist a map $\psi = (\psi_1, \psi_2)$, such that $\psi_1\phi - \psi_2 f = \mathrm{id}_N$. Since $f(N) \subset \mathbf{m}F$, by Nakayama's lemma $\psi_1\phi$ is a surjective map. Since N is Noetherian, $\psi_1\phi$ is an isomorphism. It follows that ϕ is an isomorphism. $\square$

Example 1.44. In this example we show that $E_F \subset \operatorname{Ext}^1_R(M, M_1)$ is not closed under addition modulo $\mathbf{m}\operatorname{Ext}^1_R(M,N)$. Assume M_1 is a syzygy in a minimal free resolution, $\alpha : 0 \to M_1 \to F \to M \to 0$. Assume that $M_1 = A \oplus B$ and let β be the pushout of α via ϕ

$$\begin{pmatrix} \mathrm{id}_A & 0 \\ 0 & -\mathrm{id}_B \end{pmatrix}.$$

Then $\alpha + \beta$ is the pushout of α via $1 + \phi$

$$\begin{pmatrix} \mathrm{id}_A & 0 \\ 0 & 0 \end{pmatrix}.$$

By Lemma 1.43, $\beta \in E_{[F]}$ but $\alpha+\beta \notin E_{[F]}$. Moreover, since $(1+\phi)(N) \nsubseteq \mathbf{m}N$, $\alpha + \beta$ is not additive and therefore not in $\mathbf{m}\operatorname{Ext}^1_R(M, N)$.

1.3.4 Annihilators of short exact sequences

We end the section with a proposition about the annihilators of short exact sequences. We will use this proposition later in the chapter, in Proposition 1.60.

Proposition 1.45. *Let α and β be two short exact sequences in $\operatorname{Ext}^1_R(M, N)$. If $\operatorname{Ext}^1_R(X_\alpha, N) = \operatorname{Ext}^1_R(X_\beta, N) = 0$ then $\operatorname{Ann}(\alpha) = \operatorname{Ann}(\beta)$.*

Proof. Apply the functor $\operatorname{Hom}_R(\quad, N)$ to the short exact sequence

$$\alpha : \; 0 \longrightarrow N \longrightarrow X_\alpha \longrightarrow M \longrightarrow 0,$$

to obtain the long exact sequence

$$\cdots \longrightarrow \operatorname{Hom}_R(X_\alpha, N) \longrightarrow \operatorname{Hom}_R(N, N) \longrightarrow \operatorname{Ext}^1_R(M, N) \longrightarrow 0.$$

Since $\operatorname{Hom}_R(N, N)$ surjects onto $\operatorname{Ext}^1_R(M, N)$ and $\beta \in \operatorname{Ext}^1_R(M, N)$, then there exists a $\phi \in \operatorname{Hom}_R(N, N)$ such that β is the pushout of α via ϕ. Therefore $\operatorname{Ann}(\alpha) \subset \operatorname{Ann}(\beta)$. Since we can repeat the same argument replacing α by β, we have the required equality. ☐

The following examples were shown to me by Giulio Caviglia and they show two short exact sequences with the same modules but with different annihilators.

Example 1.46. Let α and β the following sequences of $\mathbb{Z}$-modules:

$$\alpha : 0 \longrightarrow \mathbb{Z}^2 \oplus \mathbb{Z}_2 \xrightarrow{f_\alpha} \mathbb{Z}^2 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_2 \xrightarrow{g_\alpha} \mathbb{Z}_4 \oplus \mathbb{Z}_2^2 \longrightarrow 0,$$

where $f_\alpha(x, y, z) = (4x, y, 2z, 0)$ and $g_\alpha(x, y, z, t) = (x, z, t)$.

$$\beta : 0 \longrightarrow \mathbb{Z}^2 \oplus \mathbb{Z}_2 \xrightarrow{f_\beta} \mathbb{Z}^2 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_2 \xrightarrow{g_\beta} \mathbb{Z}_4 \oplus \mathbb{Z}_2^2 \longrightarrow 0,$$

where $f_\beta(x, y, z) = (2x, 2y, 0, z)$ and $g_\beta(x, y, z, t) = (z, x, y)$.
Notice that $\beta = \beta_1 \oplus \beta_2 \oplus \beta_2$, where β_1 is the split exact sequence

$$0 \to \mathbb{Z}_2 \to \mathbb{Z}_2 \oplus \mathbb{Z}_4 \to \mathbb{Z}_4 \to 0$$

and β_2 is the generator of $\operatorname{Ext}^1_{\mathbb{Z}}(\mathbb{Z}_2, \mathbb{Z})$. The annihilator of β is therefore $2\mathbb{Z}$. On the other hand α is the direct sum of the split exact sequence

$$0 \to \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z}_2 \to \mathbb{Z}_2 \to 0$$

and the generators of $\operatorname{Ext}^1_{\mathbb{Z}}(\mathbb{Z}_4, \mathbb{Z})$ and $\operatorname{Ext}^1_{\mathbb{Z}}(\mathbb{Z}_2, \mathbb{Z}_2)$. Therefore $\operatorname{Ann}(\alpha) = 4\mathbb{Z}$. The example shows also that $\operatorname{Ann} \alpha$ and $\operatorname{Ann} \beta$ do not have the same integral closure (see exercise A3.29, page 656, [7]).

1.4 Sparse modules

In this section we introduce the notion of sparse modules and we study them. Sparse modules are introduced to better understand the structure of the modules $\operatorname{Ext}^1_R(M, N)$, where M and N are maximal Cohen-Macaulay modules over a ring of finite Cohen-Macaulay type (see the following section).

Definition 1.47. Let M be a finitely generated module over a local Noetherian ring $(R, \mathbf{m})$ of positive depth. M is said to be sparse if there is only a finite number of submodules xM, where $x \in \mathbf{m}$ is a non-zerodivisor on R.

Being sparse is closed under taking direct sums and quotients. On the other hand, submodules of sparse modules need not to be sparse, as the following example shows.

Example 1.48. Let $R = \mathsf{k}[|x, y|]$, with k infinite. Let M be the cokernel of the following map:

$$R^4 \xrightarrow{\beta} R^2$$

where β is given by the following matrix:

$$\begin{pmatrix} 0 & x & 0 & y^2 \\ x & y & y^2 & 0 \end{pmatrix}.$$

Let m_1, m_2 be elements of M corresponding to $(1,0), (0,1) \in R^2$. Notice that $\mathbf{m}^2 M = 0$. A basis for $\mathbf{m}M$ is given by ym_1 and ym_2. The multiplication by an element $ax+by \in \mathbf{m}\backslash\mathbf{m}^2$ takes $u_1m_1+u_2m_2 \in M\backslash\mathbf{m}M$ to $v_1ym_1+v_2ym_2$, where v_1 and v_2 are given as follows:

$$\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} b & 0 \\ -a & b \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}.$$

It follows that the only submodules, which are not zero, of the form lM with $l \in \mathbf{m}$ are xM and $(x+y)M$. If $N = Rm_1$, $\mathbf{m}N$ is a two dimensional vector space and therefore it cannot be sparse.

Proposition 1.49. *Let $(R, \mathbf{m})$ be a local ring of positive depth. If M is a sparse module then M is an Artinian module.*

Proof. We can choose a set of generators of the maximal ideal which are non-zero-divisors on the ring R, say $x_1, \ldots, x_n$. For each $i = 1, \ldots, n$, there exists h and s such that $x_i^h M = x_i^{h+s} M$, since M is sparse and powers of regular elements are regular. By Nakayama's lemma, $x_i^h M = 0$. Therefore there exists a power of the maximal ideal that annihilates M. □

Proposition 1.50. *Let* $(R, \mathbf{m}, \mathsf{k})$ *be a local ring of positive* depth. *Assume that the residue field is infinite. If* M *is a sparse module then there exists an element* $l \in \mathbf{m}$ *such that* $lM = \mathbf{m}M$.

Proof. Let $l_1M, \dots, l_hM$ the list of all possible modules of the form xM, where $x \in \mathbf{m}$ is a regular element on R. This list is finite since M is sparse. Let $x_1, \dots, x_n$ be a list of minimal generators for the maximal ideal $\mathbf{m}$. Choose a sequence of vectors $\mathtt{v}_i = (\mathtt{v}_i^1, \dots, \mathtt{v}_i^n) \in \mathsf{k}^n$ such that $\mathtt{v}_1, \dots, \mathtt{v}_n$ are linearly independent and for $i > n$, $\mathtt{v}_i$ is linearly independent with any subset of $n-1$ vectors of $\mathtt{v}_1, \dots, \mathtt{v}_{i-1}$. After at most $hn+1$ steps, there exists a j $(1 \leq j \leq h)$ and n linearly independent vectors (which for simplicity we rename $\mathtt{v}_1, \dots, \mathtt{v}_n$) such that

$$l_jM = y_sM, \quad \text{for any} \quad s = 1, \dots, n,$$

where $y_s = \sum_{i=1}^n \mathtt{v}_s^i x_i$. Since the vectors $\mathtt{v}_i$ are linearly independent, the y_i are a system of generators for $\mathbf{m}$. We claim that $\mathbf{m}M = l_jM$. For, if $a \in \mathbf{m}$ then $a = \sum r_i y_i$ and $aM \subset y_1M + \cdots + y_hM \subset l_jM$ so that $\mathbf{m}M \subset l_jM$. Since the other inclusion always holds, we have equality. □

Corollary 1.51. *Let* $(R, \mathbf{m}, \mathsf{k})$ *be a local Noetherian ring of positive depth and infinite residue field. Let* M *a finitely generated sparse* R*-module. Denote by* μ *the number of minimal generators. Then* $\mu(\mathbf{m}^h M) \leq \mu(M)$, *for any* $h \geq 0$.

Proof. By Proposition 1.50, for any integer k, we have that $\mathbf{m}^k M = l^k M$, from which the proposition follows. □

It would be very interesting to find a characterization of sparse modules. The following example shows what can happen in a very specific situation.

Example 1.52. Let $(R, \mathbf{m}, \mathsf{k})$ be a local ring with algebraically closed residue field k. Suppose that the maximal ideal is generated by three elements, l_1, l_2, l_3 and that M is a R-module generated by two elements, m_1, m_2. Assume that M is sparse and that $\mathbf{m}^2 M = 0$ so that $\mathbf{m}M$ is a vector space. By Corollary 1.51, $\mu(\mathbf{m}M) \leq 2$. Assume that $\mu(\mathbf{m}M) = 2$ and fix a basis $\mathtt{v}_1, \mathtt{v}_2$. We can describe the multiplication of M by an element of the maximal ideal, by a 2×2 matrix with entries homogeneous linear polynomials with coefficients in k. Indeed, for any element a in $\mathbf{m}$, we can write $a = xl_1 + yl_2 + zl_3 + \xi$, where $\xi \in \mathbf{m}^2$ and $x, y, z \in \mathsf{k}$. Since $\mathbf{m}^2 M = 0$, for any $m \in M$ we have $am = (xl_1 + yl_2 + zl_3)m$. If $m = b_1 m_1 + b_2 m_2$, then $am = c_1 \mathtt{v}_1 + c_2 \mathtt{v}_2$, where

$$\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} \alpha_{11}x + \beta_{11}y + \gamma_{11}z & \alpha_{12}x + \beta_{12}y + \gamma_{12}z \\ \alpha_{21}x + \beta_{21}y + \gamma_{21}z & \alpha_{22}x + \beta_{22}y + \gamma_{22}z \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \end{pmatrix},$$

and the columns of the matrix represents am_i in terms of the $\mathtt{v}_1, \mathtt{v}_2$. Denote the 2×2 matrix by $\mathcal{A}(x, y, z)$. Studying when the module is sparse is equivalent to study when the matrix $\mathcal{A}(x, y, z)$ has a finite number image spaces, while x, y, z vary.

The entries of the matrix represents four lines in $\mathbb{P}^2$, therefore we have the following cases.

(1) There exists no point $P(x_0, y_0, z_0)$ where three of the four lines intersect. We will show that in this case M cannot be sparse.

After a change of coordinates we can write the matrix $\mathcal{A}(x, y, z)$ as

$$\begin{pmatrix} x & y \\ z & ax + by + cz \end{pmatrix},$$

where none of the a, b, c are zero. We are looking for conditions forcing this matrix to have a finite number of images. If the rank of the matrix is two then the image is the all vector space $\mathbf{m}M$. Therefore, it is enough to study what happen when the rank of the matrix is one. The determinant being zero gives us the conic $\mathcal{C} : ax^2 + bxy + cxz - yz = 0$. Either the conic $\mathcal{C}$ is irreducible or not. In the first case $a + bc \neq 0$. For any point $P = (x_0, y_0, z_0) \neq (0, 1, 0)$ the image of $\mathcal{A}(x_0, y_0, z_0)$ is generated by the vector (x_0, z_0). On the other hand we claim that the map that takes a point $P(x_0, y_0, z_0) \neq (0, 1, 0)$ of the conic to the vector (x_0, z_0) is injective. If the claim is true the points of the conic different from $(0, 1, 0)$ are in one to one correspondence with one dimensional subspaces of k^2. Since the conic is irreducible and the field algebraically closed, there are infinitely many different possible images of the matrix $\mathcal{A}(x, y, z)$ and therefore the module cannot be sparse. To prove the claim, notice that for any point $(x_0, y_0, z_0) \neq (0, 1, 0)$, y_0 is uniquely given by $\frac{-ax_0^2 - cx_0z_0}{bx_0 - z_0}$, where $bx_0 - z_0 \neq 0$. For it, if $x_0 = 0$ and $bx_0 - z_0 = 0$ then $z_0 = 0$ which is a contradiction since our point is different from $(0, 1, 0)$. If $x_0 \neq 0$ and $bx_0 - z_0 = 0$ then

$$x_0^2(a + bc) = x_0^2a + x_0^2bc = x_0^2a + x_0cz_0 = y_0(bx_0 - z_0) = 0$$

which leads to the contradiction $a + bc = 0$ since we are assuming the the conic $\mathcal{C}$ is irreducible.

Suppose that the conic is reducible in two lines

$$cx - y = 0 \quad \text{and} - bx + z = 0.$$

The vectors (s, t) for any value of s, t are all possible images of the matrix $\mathcal{A}(P)$ for some P in the conic, more in particular in the first line.

(2) There exists a point $P \in P^2$ where at least three of the four lines rapresented by the entries of the matrix, intersect. There are two subcases:

(a) The fourth line goes through P. We will show that in this case there are always a infinite number of image spaces.

A change of coordinates the matrix $\mathcal{A}(x, y, z)$ to the form:

$$\begin{pmatrix} x & y \\ x + y & ax + by + cz \end{pmatrix},$$

where $c \neq 0$. Again, we need to study what happen when the rank of the matrix is one which means that we have to study the family of matrix obtained by specializing x, y, z with any point $P \in \mathcal{C}$ where $\mathcal{C}$ is given by the equation $ax^2+bxy+cxz-xy-y^2 = 0$ and it is always irreducible. For any $P(x_0, y_0, z_0) \in \mathcal{C}$ but $P \neq (0, 0, 1)$ the image of the matrix $\mathcal{A}(P)$ is generated by the vector $(x_0, x_0 + y_0)$. The map the takes a point $P \in \mathcal{C}$ different from $(0, 0, 1)$ to the vector generating the image, is a one to one map. For it, it is enough to notice that given x_0 and y_0 not both equal to zero, there exists a unique z_0 such that $(x_0, y_0, z_0) \in \mathcal{C}$: $z_0 = \frac{x_0y_0+y_0^2-ax_0^2-bxy}{cx_0}$. Notice that the denominator is always different from zero since c is and

$x_0 = 0$ would imply $y_0 = 0$ which is a contradiction.

(b) The fourth lines does not go through P. We will show that the module M is sparse.

After a change of coordinates, we can write the matrix $\mathcal{A}(x, y, z)$ as follows:

$$\begin{pmatrix} x & y \\ x+y & ax+by \end{pmatrix}$$

As previously, we reduce ourselves to study what happen to the images along the conic $\mathcal{C}$, which in this case is given by the equation $ax^2 + (b-1)xy - y^2 = 0$. It is easy to see that the number of images are finite. For example, if $a = 0$ and $b \neq 1$, then $\mathcal{C}$ is the union of the two lines $y = 0$ and $y = (b-1)x$. For any point P in the first line different from $(0, 0, 1)$ the image of $\mathcal{A}(P)$ is generated by the vector $(1, 1)$. For any point P in the second line different from $(0, 0, 1)$ the image of $\mathcal{A}(P)$ is generated by the vector $(1, b-1)$. Notice that the two lines meet exactly in $(0, 0, 1)$ in which the rank of the matrix $\mathcal{A}$ is zero.

Suppose that R is a standard graded ring over k. Assume k is infinite and R has depth at least one. Let M be a finitely generated graded R-module. Then, we can formulate the following

Definition 1.53. *M is said to be homogeneously sparse if there are a finite number of submodules xM where $x \in \mathbf{m}$ is a homogeneous non-zerodivisor of R.*

Question 1.54. In the case of graded modules over graded ring, being sparse implies being homogeneously sparse. Is the converse true?

Theorem 1.55. *Let R be a standard graded ring over an algebraically closed field k and assume that R_1 is generated by two elements. Let M be a standard graded module over R. Then, M is homogeneously sparse if and only if M is Artinian and there exists a linear form $l \in \mathbf{m}$ such that $lM = \mathbf{m}M$, where $\mathbf{m}$ is the homogeneous maximal ideal.*

Proof. The proof for the "only if" direction is the graded version of Proposition 1.49 and Proposition 1.50.

For the other direction, suppose that M is Artinian and that there exists an $l \in \mathbf{m}$ such that $lM = \mathbf{m}M$. Since M is Artinian, it is enough to show that there are finitely many submodules of the form fM, where f is a homogeneous polynomial of degree less than d, for some $d > 0$. Since k is algebraically closed, any homogeneous polynomial in two variables can be factored into linear terms; hence it is enough to show that there are finitely many submodules of the form tM, where t is a linear form. Moreover, since $M = \oplus M_i$, it is enough to show that there are finitely many tM_i, where t is a linear form and M_i is the i-th degree component of the module M. It is enough to show that there are finitely many k vector spaces of the form tM_0, for t a linear form. Indeed, if we want to show that there are finitely many tM_i for $i > 0$, replace the module M by the module generated by M_i. Without loss of generality, let R_1 be generated by l and s. Since $lM = \mathbf{m}M$, $\mu(\mathbf{m}M) \leq \mu(M)$. Suppose $m_1, ..., m_h$ are minimal generators in degree zero, such that $lm_1, ..., lm_h$ generate $\mathbf{m}M$. Complete $m_1, ..., m_h$ to a minimal system of generators of M, $m_1, ..., m_n$ say. Let $\alpha l + \beta s$ be a general linear form.

Let A to be the matrix where the columns are the images in M_1 of $(\alpha l+\beta s)m_i$ in terms of the lm_i, $i=1,\dots,h$:

$$\begin{pmatrix} \alpha+\beta a_{11} & \beta a_{12} & \dots & \beta a_{1h} & \dots & \beta a_{1n} \\ \beta a_{21} & \alpha+\beta a_{22} & \dots & \beta a_{2h} & \dots & \beta a_{1n} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \beta a_{h1} & \dots & \dots & \alpha+\beta a_{hh} & \dots & \beta a_{hn} \end{pmatrix}.$$

If there are only finitely many (α,β) such that the first $h-$columns have a zero determinant, then the module M is sparse. Since the determinant of the first h-columns is a homogeneous polynomial of degree h in two variables, it is enough to show that it is not identically zero. But the values $\alpha=1$, $\beta=0$ give a nonzero determinant. □

1.5 Application to rings of finite Cohen-Macaulay type

In this section we apply the previous material to rings of finite Cohen-Macaulay type. These rings are widely studied; Yoshino's monograph [24] is a comprehensive source for information about rings of finite Cohen-Macaulay type. We use the theory of sparse modules developed in the previous section to prove a theorem of Auslander [2], which states that if R is a complete local ring of finite Cohen-Macaulay type then R is isolated singularity. We are able to prove this fact without assuming that the ring is complete (see also the work of Huneke-Leuschke [12]). Specifically, we prove that the modules $\operatorname{Ext}^1_R(M,N)$ are sparse. We give a bound for the power of the maximal ideal that annihilates $\operatorname{Ext}^1_R(M,N)$ which improves the one given in [12].

In the following of the section, $(R,\mathbf{m},\mathsf{k})$ is a Notherian local ring. Fix two finitely generated R-modules M and N. For any $\alpha\in\operatorname{Ext}^1_R(M,N)$ denote X_α

the module which appears in the middle of the short exact sequence α. Recall the following

Definition 1.56. A local Noetherian ring $(R, \mathbf{m})$ is said to be of finite Cohen-Macaulay type if it is Cohen-Macaulay and if there exists a finite number of isomorphism classes of indecomposable maximal Cohen-Macaulay modules.

Proposition 1.57. *Let $(R, \mathbf{m})$ be a ring of finite Cohen-Macaulay type and let M and N be finitely generated maximal Cohen-Macaulay R-modules. Then $\operatorname{Ext}^1_R(M, N)$ is a sparse R-module.*

Proof. If R is of finite Cohen-Macaulay type then there is a finite number of modules that can fit in the middle of a short exact sequence in $\operatorname{Ext}^1_R(M, N)$. Therefore there is a finite number of possible sets $E_{[X]}$ and hence, by Proposition 1.39, there is a finite number of submodules of the type $y \operatorname{Ext}^1_R(M, N)$, where y is a non-zerodivisor on R (and on M and N, since they are maximal Cohen-Macaulay). □

Corollary 1.58. *Let $(R, \mathbf{m})$ a ring of finite Cohen-Macaulay type. Then it has a isolated singularity.*

Proof. It is enough to show that $\operatorname{Ext}^1_R(M, N)$ are modules of finite length for any maximal Cohen-Macaulay modules M and N. For suppose this is true and set $d = \dim(R)$. For every prime ideal $\mathbf{p} \neq \mathbf{m}$, consider a free resolution of $R/\mathbf{p}$. At the d-th step we get a short exact sequence, which is an element of $\operatorname{Ext}^1_R(\Omega_d(R/\mathbf{p}), \Omega_{d+1}(R/\mathbf{p}))$

$$0 \to \Omega^{d+1}_R(R/\mathbf{p}) \to F_d \to \Omega^d_R(R/\mathbf{p}) \to 0.$$

Both the modules at the side are maximal Cohen-Macaulay therefore after localizing at $\mathbf{p}$ this is a split exact sequence. This implies that both $\Omega_R^{d+1}(R/\mathbf{p})_{\mathbf{p}}$ and $\Omega_R^d(R/\mathbf{p})_{\mathbf{p}}$ are free $R_{\mathbf{p}}$-modules. Therefore, the residue field of $R_{\mathbf{p}}$ has a finite free resolution, and so $R_{\mathbf{p}}$ is a regular local ring. By Proposition 1.57, $\operatorname{Ext}_R^1(M,N)$ is a sparse module and hence Artinian, by Proposition 1.49. □

The fact that rings of finite Cohen-Macaulay type have isolated singularities was proved first by Auslander [2] in the case when R is complete. In this generality it is a theorem due to Huneke-Leuschke [12]. Notice that another proof of this fact follows from Proposition 1.40. Indeed, assume there is a finite number of isomorphic classes of modules X that can fit in the middle of a short exact sequence in $\operatorname{Ext}_R^1(M,N)$, $[X_1],\dots,[X_t],[X_{t+1}]$, where $X_{t+1}\cong M\oplus N$; as in Proposition 1.40, take

$$n=\max\{h(X_i,N)+1 \mid i=1,\dots,t\}.$$

Then $E_{[X_i]}\cap \mathbf{m}^n\operatorname{Ext}_R^1(M,N)=\emptyset$, for $i=1,\dots,t$. Since

$$\operatorname{Ext}_R^1(M,N)=\cup_{i=1}^n E_{[X_i]}\cup E_{[M\oplus N]},$$

we have

$$\mathbf{m}^n\operatorname{Ext}_R^1(M,N)=(\mathbf{m}^n\operatorname{Ext}_R^1(M,N))\cap(\cup_{i=1}^n E_{[X_i]}\cup E_{[M\oplus N]})=E_{[M\oplus N]}=0.$$

In their paper Huneke and Leuschke gave a bound on the power of the maximal ideal that kills the R-module $\operatorname{Ext}_R^1(M,N)$. In particular, if h is the number of modules that can fit in the middle of a short exact sequence in $\operatorname{Ext}_R^1(M,N)$ then $\mathbf{m}^h\operatorname{Ext}_R^1(M,N)=0$. We are able to improve the bound.

Proposition 1.59. *Let* $(R, \mathbf{m}, \mathsf{k})$ *a Cohen-Macaulay local ring with infinite residue field. Let* M *and* N *be maximal Cohen-Macaulay* R*-modules and let* h *be the number of isomorphism classes of modules that can fit in the middle of a short exact sequence in* $\operatorname{Ext}^1_R(M, N)$*. Then* $\mathbf{m}^{h-1} \operatorname{Ext}^1_R(M, N) = 0$*.*

Proof. Assume by way of contradiction that $\mathbf{m}^{h-1} \operatorname{Ext}^1_R(M, N) \neq 0$ and let

$$X_1, \dots, X_{h-1},$$

$$X_h \cong M \oplus N$$

be a list of representatives for all the isomorphism classes of modules that can fit in the middle of a short exact sequence in $\operatorname{Ext}^1_R(M, N)$. To simplify notation, denote by E_i the set $E_{[X_i]}$. Define two sets of positive integers, $\mathcal{S}_1 := \{0, \dots, h-1\}$ and $\mathcal{S}_2 := \{1, \dots, h-1\}$. Define a map ϕ from $\mathcal{S}_1$ to the power set of $\mathcal{S}_2$, in such a way that $r \in \phi(i)$ if and only if there exists a minimal generator for $\mathbf{m}^i \operatorname{Ext}^1_R(M, N)$ which is in E_r. We claim that the sets $\phi(i)$ are disjoint subsets of $\mathcal{S}_2$. If the claim holds, we have the desired contradiction since the cardinality of $\mathcal{S}_2$ is strictly smaller than the cardinality of $\mathcal{S}_1$. Before proving the claim, recall that, by Proposition 1.50, there exists a non-zerodivisor $l \in \mathbf{m}$ such that $\mathbf{m} \operatorname{Ext}^1_R(M, N) = l \operatorname{Ext}^1_R(M, N)$. To prove the claim, assume by way of contradiction that there exist $i < j$ such that $r \in \phi(i) \cap \phi(j)$. Pick α and β in E_r to be minimal generators for $l^j \operatorname{Ext}^1_R(M, N)$ and $l^i \operatorname{Ext}^1_R(M, N)$ respectively. Since $X_\alpha \cong X_\beta \cong X_r$, we can apply Theorem 1.18 to conclude that $\beta \otimes R/(l^j)$ is a split exact sequence and Proposition 1.24 to prove that $\beta \in l^j \operatorname{Ext}^1_R(M, N) = \mathbf{m}^j \operatorname{Ext}^1(M, N) \subset \mathbf{m} l^i \operatorname{Ext}^1_R(M, N)$, contradicting the fact that β is a minimal generator for $l^i \operatorname{Ext}^1_R(M, N)$. ∎

Using the decomposition of the Ext modules in subset E_i we can give more information on the structure of the Ext.

Proposition 1.60. *In the same setup of the previous proposition, assume that* $N = M_1$ *where* M_1 *is the first syzygy in a free resolution of* M*. Assume that the residue field* k *is algebraically closed and that*

$$\mathbf{m}^{h-2}\operatorname{Ext}^1_R(M,N) \neq 0.$$

Then $\operatorname{Ext}^1_R(M,N)$ *is the direct sum of cyclic* R*-modules, whose Hilbert functions have as only values* 0 *and* 1*.*

Proof. In this proof we will use the notation of the proof of the above proposition. In particular, $\mathcal{S}_1 := \{0,\dots,h-2\}$ and $\mathcal{S}_2 := \{1,\dots,h-1\}$ are two sets of positive integers. In the above proof we showed that the sets $\phi(i)$, where ϕ is the map defined above between $\mathcal{S}_1$ and the power set of $\mathcal{S}_2$, are disjoint for any $i \in \mathcal{S}_2$. Therefore, since $\mathbf{m}^{h-2}\operatorname{Ext}^1_R(M,M_1) \neq 0$, we have that the cardinality of $\phi(i)$ is one, for every $i \in \mathcal{S}_1$. In particular, all the short exact sequences which are minimal generators for the R-modules $\mathbf{m}^i\operatorname{Ext}^1_R(M,M_1)$ have isomorphic modules in the middle. By Proposition 1.32, the initial part of the minimal free resolution of M, $\alpha : 0 \to M_1 \to F \to M \to 0$, is a minimal generator for the R-module $\operatorname{Ext}^1_R(M,M_1)$ and hence any other minimal generator of the R-module $\operatorname{Ext}^1_R(M,M_1)$ belongs to E_1, where we define E_1 to be the set of all short exact sequences having a free module isomorphic to F in the middle. By Proposition 1.50, there exists an element $l \in \mathbf{m}\backslash\mathbf{m}^2$, which is a non-zero divisor on R, such that

$$l^{h-2}\operatorname{Ext}^1_R(M,M_1) = \mathbf{m}^{h-2}\operatorname{Ext}^1_R(M,M_1) \neq 0;$$

hence there exists a $\beta \in E_1$ such that $l^j\beta \neq 0$, for $j = 1, \dots, h-2$. By Proposition 1.45, $l^j\gamma \neq 0$, for every $\gamma \in E_1$ and $j = 1, \dots, h-2$.

Let $\alpha_1, \dots, \alpha_m$ be a minimal set of generators for $\operatorname{Ext}^1_R(M, M_1)$. We first claim that for any $j \in \mathcal{S}_2$, $l^j\alpha_1, \dots, l^j\alpha_m$ are minimal generators for $\mathbf{m}^j \operatorname{Ext}^1_R(M, M_1)$. Assume there exists a linear combination $\sum_{i=1}^m \lambda_i l^j \alpha_i = 0$, where $\lambda_i \in \mathsf{k}$, and not all equal to zero, hence $l^j(\sum_{i=1}^m \lambda_i\alpha_i) = 0$. Since $\sum_{i=1}^m \lambda_i\alpha_i \in E_1$, applying Proposition 1.45 we get that $l^j\gamma = 0$ for every $\gamma \in E_1$, contradicting the last statement of the above paragraph.

The last step is to prove that we can complete l to a minimal system of generators for $\mathbf{m}$, say $l, l_1, \dots, l_n$, such that $l_i \operatorname{Ext}^1_R(M, N) = 0$, for every $i = 1, \dots, n$. To prove this, we claim that if $l, l_1, \dots, l_n$ is a system of minimal generators for $\mathbf{m}$, then there exists a $\lambda_i \in \mathsf{k}$ such that $(l_i - \lambda_i l)\gamma = 0$ for every $\gamma \in E_1$ and for every $i = 1, \dots, n$. To prove the claim, notice that the multiplication by l_i:

$$\frac{\operatorname{Ext}^1_R(M,M_1)}{\mathbf{m}\operatorname{Ext}^1_R(M,M_1)} \xrightarrow{l_i} \frac{\mathbf{m}\operatorname{Ext}^1_R(M,M_1)}{\mathbf{m}^2\operatorname{Ext}^1_R(M,M_1)},$$

is a k-linear map between k^m and itself. Since k is algebraically closed, there exists an eigenvalue λ_i and an eigenvector γ_i. we can write $l_i\gamma_i = \lambda_i l\gamma_i$. This means that $(l_i - \lambda_i l)\gamma_i = 0$, since $\gamma_i \in E_1$, by Proposition 1.45, we have $(l_i - \lambda_i l)\gamma = 0$ for every $\gamma \in E_1$. By replacing l_i with $l_i - \lambda_i l$, we have the claim.

This shows that we can write $\operatorname{Ext}^1_R(M, M_1) = \oplus_{i=1}^m R\alpha_i$, and each $R\alpha_i$ has the property of the Hilbert function to have possible as only values 0 and 1. □

Remark 1.61. Notice that the same conclusion of the above proposition holds if all the minimal generators for $\mathrm{Ext}^1_R(M,N)$ have a free module in the middle.

1.6 Other applications

1.6.1 Efficient systems of parameters

Recall the following definitions:

Definition 1.62. Let R be a T algebra and let μ the multiplication map from $R \otimes_T R \to R$. The Noetherian differ is $\mathcal{N}^R_T = \mu(\mathrm{Ann}_{R\otimes_T R}(\ker(\mu)))$.

Definition 1.63. A system of parameters $x_1, \ldots, x_n$ is an efficient system of parameters if for any $i = 1, \ldots, n$ there is a regular subring T_i of R over which R is finite and such that x_i belongs to the Noetherian different $\mathcal{N}^R_{T_i}$.

The proofs above can be used to give an improvement of the following proposition (Proposition 6.17, [24]):

Proposition 1.64. *Let* $\mathbf{x} = \{x_1, \ldots, x_n\}$ *be an efficient system of parameters. Let* $\alpha : 0 \to N \to X \to M \to 0$ *be a short exact sequence of maximal Cohen-Macaulay modules. Denote by* $\mathbf{x}^2$ *the ideal generated by* $x_1^2, \ldots, x_n^2$ *and let* $\overline{\alpha}$ *be the short exact sequence* $0 \to N/\mathbf{x}^2N \to X/\mathbf{x}^2X \to M/\mathbf{x}^2M \to 0$*. If* $\overline{\alpha}$ *is split exact then* α *is split exact.*

We can substitute $\mathbf{x}^2$ simply by $\mathbf{x}$ and prove the following Proposition:

Proposition 1.65. *Let* $\mathbf{x} = \{x_1, \ldots, x_n\}$ *be an efficient system of parameters. Let* $\alpha : 0 \to N \to X \to M \to 0$ *be a short exact sequence of maximal Cohen-Macaulay modules and let* $\overline{\alpha}$ *be* $0 \to N/\mathbf{x}N \to X/\mathbf{x}X \to M/\mathbf{x}M \to 0$*. If* $\overline{\alpha}$ *is split exact then* α *is split exact.*

Proof. Denote by $\mathbf{x}_i = \{x_1, \ldots, x_i\}$ and by α_i the following short exact sequence:

$$0 \longrightarrow N/\mathbf{x}_i N \longrightarrow X/\mathbf{x}_i X \longrightarrow M/\mathbf{x}_i M \longrightarrow 0.$$

We will show by descending induction on i that α_i is split exact. The case $i = 0$ will give the thesis of the improved Proposition 1.65. The assumption say that α_n is a split exact sequence, which is the case $i = n$. Suppose that the case $i = k$ is true. Then, by Proposition 1.24,

$$\alpha_{k-1} \in x_k \operatorname{Ext}^1_{R/\mathbf{x}_{k-1}R}(M/\mathbf{x}_{k-1}M, N/\mathbf{x}_{k-1}N) \simeq x_k \operatorname{Ext}^1_R(M, N/\mathbf{x}_{k-1}N).$$

By definition of efficient systems of parameters, there exists a regular subring S_k such that the extension $S_k \subset R$ is finite and x_k is in $\mathcal{N}^R_{S_k}$, the Noetherian different. By a result proved by Wang ([21], Proposition 5.9), we have that

$$\mathcal{N}^R_{S_k} \operatorname{Ann}(\operatorname{Ext}^1_{S_k}(M, N/\mathbf{x}_{k-1}N)) \subset \operatorname{Ann}(\operatorname{Ext}^1_R(M, N/\mathbf{x}_{k-1}N)).$$

But M is a free S_k-module so that $\operatorname{Ann}(\operatorname{Ext}^1_{S_k}(M, N/\mathbf{x}_{k-1}N))$ is the unit ideal of S_k and hence $x_k \in \operatorname{Ann}(\operatorname{Ext}^1_R(M, N/\mathbf{x}_{k-1}N))$, which implies that α_{k-1} is a split exact sequence. □

1.6.2 Splitting of syzygies

In this section $(R, \mathbf{m})$ is a local Noetherian ring, and M is a finitely generated R-module. We denote by $\Omega^i_R(M)$ the i-th syzygy of M in a minimal free resolution over the ring R. In this section we give a simpler proof of the following theorem

Theorem 1.66. *Let* $(R, \mathbf{m})$ *be a local Noetherian ring and let* $z = (z_1, \dots, z_n)$ *be a regular sequence on the ring and on a finitely generated* R*-module* M*. Then,* $z \operatorname{Ext}^i_R(M, \quad) = 0$*, for* $i \geq 1$*, if and only if*

$$\Omega^j_R(M/(z_1, \dots, z_n)M) \cong \bigoplus_{i=0}^{j} \Omega^i_R(M)^{\binom{j}{i}},$$

for every $j = 1, \dots, n$*.*

This theorem was first proved by O'Carroll and Popescu in [18], Theorem 2.1.

Let

$$\xi : \quad 0 \longrightarrow \Omega^1_R(M) \longrightarrow F_0 \xrightarrow{\pi} M \longrightarrow 0 \tag{1.6.1}$$

be part of a minimal free resolution of M. If z is a regular element on R and M, then $\Omega^1_R(M/zM) = \Omega^1_R(M) + zF_0$.

Remark 1.67. Let z be a regular element on the ring R and on a module M, then there exists the following short exact sequence:

$$\gamma(M, z) : 0 \longrightarrow \Omega^1_R(M) \longrightarrow \Omega^1_R(M/zM) \xrightarrow{f} M \longrightarrow 0,$$

where $f(a + zb) = \pi(b)$, for $a \in \Omega^1_R(M)$ and $b \in F_0$. Notice that the map is well defined, in fact if $a + zb = a_1 + zb_1$, for $a, a_1 \in \Omega^1_R(M)$ and for $b, b_1 \in F_0$, then $0 = \pi(a) - \pi(a_1) = \pi(a - a_1) = z\pi(b_1 - b) \in M$. Since z is a non-zerodivisor on M, we have that $\pi(b_1 - b) = 0$, showing that the map f is well-defined. Moreover for every element $a + zb$ in $\ker f$ we have that $\pi(b) = 0$ and hence b and $a + zb \in \Omega^1_R(M)$.

Lemma 1.68. *With the same notation as in Remark 1.67, the following holds*

$$\gamma(M,z) \in z\operatorname{Ext}^1_R(M,\Omega^1_R(M)),$$

and

$$\gamma(M,z)=0 \quad \text{if and only if} \quad z\operatorname{Ext}^1_R(M,\quad)=0.$$

Proof. It is enough to show that $\gamma(M,z)=z\xi$, where ξ is as in 1.6.1. The following diagram shows that $\gamma(M,z)$ is the pushout of ξ and therefore proves the claim:

$$\begin{array}{rcccccccc}
\xi : 0 & \longrightarrow & \Omega^1_R(M) & \longrightarrow & F_0 & \longrightarrow & M & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle z} & & \downarrow{\scriptstyle z} & & \| & & \\
\gamma(M,z) : 0 & \longrightarrow & \Omega^1_R(M) & \longrightarrow & \Omega^1_R(M)+zF_0 & \xrightarrow{f} & M & \longrightarrow & 0.
\end{array}$$

□

Remark 1.69. Notice that $z\operatorname{Ext}^1_R(M,\quad)=0$ implies $z\operatorname{Ext}^i_R(M,\quad)=0$ for every $i\geq 2$. Indeed, by the previous lemma, we have that $M\oplus\Omega^1_R(M)=\Omega^1_R(M/zM)$ and hence

$$\begin{aligned}
0 &= z\operatorname{Ext}^i_R(M/zM,\quad)=z\operatorname{Ext}^{i-1}_R(\Omega^1_R(M/zM),\quad) \\
&= z\operatorname{Ext}^{i-1}_R(M,\quad)\oplus z\operatorname{Ext}^{i-1}_R(\Omega^1_R(M),\quad),
\end{aligned}$$

so that each summand at the right hand side is zero.

Remark 1.70. Lemma 1.68 and Remark 1.69 give the proof of Theorem 1.66 for $n=1$. Indeed, $M\oplus\Omega^1_R(M)\cong\Omega^1_R(M/zM)$ if and only if $\gamma(M,z)=0$ (by Miyata's Theorem), iff and only if $z\operatorname{Ext}^1_R(M,\quad)=0$ (by Lemma 1.68), if and only if $z\operatorname{Ext}^i_R(M,\quad)=0$ (by Remark 1.69).

Remark 1.71. The above proof shows that

$$\gamma(M,z) \in z\,\mathrm{Ext}^1_R(M,\Omega^1_R(M)) \subset \mathbf{m}\,\mathrm{Ext}^1_R(M,\Omega^1_R(M)).$$

Therefore by 1.20 $\mu(M)+\mu(\Omega^1_R(M)) = \mu(\Omega^1_R(M/zM))$.

Before proving the theorem we need another lemma. If z is a non-zerodivisor, we denote M/zM by $\overline{M}$.

Lemma 1.72. *Let z be a non-zerodivisor on the ring and on a finitely generated R-module M, then $\Omega^i_R(\Omega^1_{\overline{R}}(\overline{M})) = \Omega^{i+1}_R(\overline{M})$.*

Proof. Since z is a non-zero divisor on M, we have $zF_0 \cap \Omega^1_R(M) = z\Omega^1_R(M)$. Therefore there exists the following short exact sequence:

$$0 \longrightarrow zF_0 \longrightarrow \Omega^1_R(M)+zF_0 \longrightarrow \Omega^1_R(M)/z\Omega^1_R(M) \longrightarrow 0.$$

Notice that zF_0 is a free module and hence, using the Horseshoe Lemma we have:

$$\Omega^i_R(\Omega^1_R(M+zF_0)) \cong \Omega^i_R(\Omega^1_R(M)/z\Omega^1_R(M)),$$

for every $i>0$. In particular we have:

$$\begin{aligned} \Omega^{i+1}_R(\overline{M}) &\cong \Omega^i_R(\Omega^1_R(\overline{M})) \\ &\cong \Omega^i_R(\Omega^1_R(M)+zF_0) \\ &\cong \Omega^i(\Omega^1_R(M)/z\Omega^1_R(M)) \\ &\cong \Omega^i(\Omega^1_{\overline{R}}(\overline{M})). \end{aligned}$$

∎

We are ready to give a proof of Theorem 1.66

Proof. One direction of the theorem is easy to prove. Indeed, if

$$\Omega_R^n(M/(zM)) \cong \bigoplus_{i=0}^{n} \Omega_R^i(M)^{\binom{n}{i}}$$

then

$$\begin{aligned}(z) &\subset \operatorname{Ann}_R(\operatorname{Ext}_R^{i+n}(M/zM, \quad) \\ &= \operatorname{Ann}_R(\operatorname{Ext}_R^{i}(\Omega_R^n(M/zM), \quad)) \\ &= \operatorname{Ann}_R(\oplus_{j=0}^{j=n} \operatorname{Ext}_R^{i}(\Omega_R^j(M), \quad)) \\ &\subset \operatorname{Ann}_R(\operatorname{Ext}_R^{i}(M, \quad)), \quad \text{for every} \quad i \geq 1.\end{aligned}$$

We will prove the other direction of the statement by induction on n. The case $n = 1$ is given by Remark 1.70.

Assume that the conclusion holds for $i \leq n$.

Denote by $\overline{M} = M/(z_1 \dots z_n)M$ and consider the sequence $\gamma(\Omega_R^n(\overline{M}), z_{n+1})$, where z_{n+1} is a non-zero divisor on $\overline{M}$ and therefore on $\Omega_R^n(\overline{M})$.

Assume that $(z_1, \dots, z_{n+1}) \operatorname{Ext}_R^i(M, \quad) = 0$, in particular we have

$$(z_1, \dots, z_n) \operatorname{Ext}_R^i(M, \quad) = 0$$

and , by induction,

$$\Omega_R^n(\overline{M}) \cong \bigoplus_{i=0}^{n} \Omega_R^i(M)^{\binom{n}{i}}. \tag{1.6.2}$$

By Lemma 1.68,

$$z_{n+1} \operatorname{Ext}_R^1(\Omega_R^n(\overline{M}), \quad) = 0 \quad \text{if and only if} \quad \gamma(\Omega_R^n(\overline{M}), z_{n+1}) = 0.$$

Moreover, $\gamma(\Omega_R^n(\overline{M}), z_{n+1}) = 0$ if and only if

$$\Omega_R^1\left(\frac{\Omega_R^n(\overline{M})}{z_{n+1}\Omega_R^n(\overline{M})}\right) \cong \Omega_R^n(\overline{M}) \oplus \Omega_R^{n+1}(\overline{M})$$

For the left hand side, we have

$$\begin{aligned}\Omega^1_R(\frac{\Omega^n_R(\overline{M})}{z_{n+1}\Omega^n_R(\overline{M})}) &\cong \Omega^1_R(\Omega^n_{R/z_{n+1}R}(\frac{M}{(z_1,\ldots z_n,z_{n+1})M}))\\ &\cong \Omega^{n+1}_R(\frac{M}{(z_1,\ldots,z_{n+1})M}),\end{aligned}$$

where the last congruence is true by Lemma 1.72. Putting this together with equation 1.6.2 we obtain

$$\begin{aligned}\Omega^{n+1}_R(M/(z_1,\ldots,z_{n+1})M) &\cong \Omega^n_R(\overline{M})\oplus\Omega^{n+1}_R(\overline{M})\\ &\cong \Omega^n_R(\overline{M})\oplus\Omega^1_R(\Omega^n_R(\overline{M}))\\ &\cong \oplus^n_{j=0}\Omega^j_R(M)^{\binom{n}{j}}\oplus\Omega^1_R(\oplus^n_{j=0}\Omega^j_R(M)^{\binom{n}{j}})\\ &\cong \oplus^{n+1}_{j=0}\Omega^j_R(M)^{\binom{n+1}{j}}.\end{aligned}$$

On the other hand, again by the induction hypothesis,

$$z_{n+1}\operatorname{Ext}^1_R(\Omega^n_R(\overline{M}),\quad)=0$$

if and only if

$$z_{n+1}\operatorname{Ext}^1_R(\Omega^j_R(M),\quad)\cong z_{n+1}\operatorname{Ext}^{j+1}_R(M,\quad)=0,\qquad\text{for}\quad j=0,\ldots,n.$$

□

Chapter 2

Artin Rees Numbers

We say that a Noetherian ring R has the uniform Artin-Rees property with respect to the set of ideals $\mathcal{W}$ if for any given two finitely generated R-modules $N \subseteq M$ there exists an integer h, depending only on M and N, such that for any $I \in \mathcal{W}$ and for any $n \geq h$ we have $I^n M \cap N \subseteq I^{n-h}N$. Such h is said to be a uniform Artin-Rees number, and we denote the minimum of such h by $\mathrm{ar}(N \subset M, \mathcal{W})$.

We say that a Noetherian ring R has the strong uniform Artin-Rees property with respect to a set of ideals $\mathcal{W}$ if for any given two modules $N \subseteq M$ there exists an integer s, depending on N and M, such that for any $I \in \mathcal{W}$ and for any $n \geq s$ we have $I^n M \cap N = I(I^{n-1}M \cap N)$. We call such a s a strong uniform Artin-Rees number and, more precisely, we denote it as $\mathrm{AR}(N \subseteq M, \mathcal{W})$ or $\mathrm{AR}(N \subseteq M)$ whenever the set $\mathcal{W}$ is the set of all ideals. In this last case we say that the ring R has the strong uniform Artin-Rees property.

The question of whether an excellent ring has the strong uniform Artin-

Rees property with respect to the family of maximal ideals, was raised by Eisenbud-Hochster in [8] and positively answered by Duncan-O'Carroll in [6]. O'Carroll [17] proved that the family of principal ideals has the strong uniform Artin-Rees property. Huneke [11] proved that many Noetherian rings have the uniform Artin-Rees property for the family of all ideals in the ring. Planas-Vilanova [19] proved that one dimensional excellent rings have the strong uniform Artin-Rees property; here we give a simple proof of this fact. We also give some examples in two-dimensional rings. A three-dimensional ring fails in general to have a strong uniform Artin-Rees number as it is shown in [22].

2.1 One dimensional rings

In this section we prove that excellent rings of dimension one have the strong uniform Artin-Rees property. Our main reduction is to show that it is enough to find a uniform bound for the family of all $\mathbf{m}$-primary ideals, as in the case of the uniform Artin-Rees property. To do this we first need a lemma.

Lemma 2.1. *Let $(R, \mathbf{m})$ be a Noetherian local ring and let $N_1, N_2 \subseteq M$ be finitely generated R-modules. Then there exists $h_0 = h_0((N_1 + N_2) \subseteq M)$ such that for any $h \geq h_0$ we have:*

$$N_1 \cap (N_2 + \mathbf{m}^h M) \subseteq (N_1 \cap N_2) + \mathbf{m}^{h-h_0} N_1.$$

Proof. Let h_0 be chosen such that for any $h \geq h_0$ we have

$$\mathbf{m}^h M \cap (N_1 + N_2) = \mathbf{m}^{h-h_0}(\mathbf{m}^{h_0} M \cap (N_1 + N_2)) \subset \mathbf{m}^{h-h_0}(N_1 + N_2).$$

Then the following holds for $h > h_0$:

$$\begin{aligned} N_1 \cap (N_2 + \mathbf{m}^h M) &= N_1 \cap (N_2 + (\mathbf{m}^h M \cap (N_1 + N_2)) \\ &\subseteq N_1 \cap (N_2 + \mathbf{m}^{h-h_0}(\mathbf{m}^{h_0} M \cap (N_1 + N_2))) \\ &\subseteq N_1 \cap (N_2 + \mathbf{m}^{h-h_0}(N_1 + N_2)) \\ &\subseteq N_1 \cap N_2 + \mathbf{m}^{h-h_0} N_1. \end{aligned}$$

□

Remark 2.2. Notice that if h_0 is an integer that satisfies Lemma 2.1, any bigger integer does also.

Proposition 2.3. *Let R be a Noetherian ring and $N \subseteq M$ two finitely generated R-modules. Suppose that there exists a strong uniform Artin-Rees number $s(N \subseteq M, \mathcal{W})$ where $\mathcal{W} = \{I \subseteq R \mid \text{nilrad}(I) = \mathbf{m} \text{ is a maximal ideal}\}$. Then there exists a strong uniform Artin-Rees number $s(N \subseteq M)$, with respect to the family of all ideals of R.*

Proof. We actually prove that the Artin-Rees number $s = s(N \subseteq M, \mathcal{W})$ works for all ideals. Suppose by contradiction that there exists $I \subset R$ and $n \geq s$ such that $I^{n-s}(I^s M \cap N) \neq I^n M \cap N$; since this inequality is preserved after localizing at some maximal ideal $\mathbf{m}$, we may assume $(R, \mathbf{m})$ local.
On the other hand, for all $h >> 0$ and for such a fixed n and s, we have:

$$\begin{aligned} I^n M \cap N &\subseteq (I + \mathbf{m}^h)^n M \cap N \\ &\subseteq (I + \mathbf{m}^h)^{n-s}((I + \mathbf{m}^h)^s M \cap N), \quad \text{by the definition of } s, \\ &\subseteq I^{n-s}((I + \mathbf{m}^h)^s M \cap N) + \mathbf{m}^h M, \quad \text{by expanding the powers,} \\ &\subseteq I^{n-s}((I^s + \mathbf{m}^h) M \cap N) + \mathbf{m}^h M \\ &= I^{n-s}((I^s M + \mathbf{m}^h M) \cap N) + \mathbf{m}^h M. \end{aligned}$$

Let h_0 be an integer depending on $(I^sM + N) \subseteq M$ that satisfies Lemma 2.1 with $N_1 = N$, $N_2 = I^sM$. By Remark 2.2, we may assume $h_0 \geq n - s$. Applying Lemma 2.1, we have $(I^sM + \mathbf{m}^hM) \cap N \subseteq (I^sM \cap N) + \mathbf{m}^{h-h_0}M$, for any $h > h_0$. Therefore, the following holds:

$$\begin{aligned} I^{n-s}((I^sM + \mathbf{m}^hM) \cap N) + \mathbf{m}^hM &\subseteq I^{n-s}(I^sM \cap N + \mathbf{m}^{h-h_0}M) + \mathbf{m}^hM, \\ &\subseteq I^{n-s}(I^sM \cap N) + \mathbf{m}^{h-h_0+n-s}M \\ &\quad +\mathbf{m}^hM, \\ &\subseteq I^{n-s}(I^sM \cap N) + \mathbf{m}^{h-h_0+n-s}M. \end{aligned}$$

Putting together the right and the left end of the chain of inclusions, we have

$$I^nM \cap N \subseteq I^{n-s}(I^sM \cap N) + \mathbf{m}^{h-h_0+n-s}M,$$

for any $h > h_0$. By taking the intersection of the right side of the inclusion over all $h > h_0$, we can conclude $I^nM \cap N \subseteq I^{n-s}(I^sM \cap N)$. Since the reverse inclusion always holds, we conclude $I^{n-s}(I^sM \cap N) = I^nM \cap N$, contradicting the assumption. □

The following are standard reductions.

Proposition 2.4. *Let $R \to S$ be a faithfully flat extension. If the strong uniform Artin-Rees property holds for S, then it holds for R.*

Proof. Let $N \subseteq M$ be two finitely generated R-modules, let I be an ideal of R and let s be a strong uniform Artin-Rees number for $N \otimes_R S \subset M \otimes_R S$.

Then for any $n \geq s$ and for any ideal I of R, we have:

$$\begin{aligned}(I^nM \cap N) \otimes_R S &= (I^nM \otimes_R S) \cap (N \otimes_R S) \\ &= ((IS)^nM \otimes_R S) \cap (N \otimes_R S) \\ &= (IS)^{n-s}(((IS)^sM \otimes_R S) \cap (N \otimes_R S)) \\ &= I^{n-s}(I^sM \cap N) \otimes_R S.\end{aligned}$$

Since S is faithfully flat, it follows that $I^nM \cap N = I^{n-s}(I^sM \cap N)$. □

Remark 2.5. Note that if $\dim R = 0$ then R has the strong uniform Artin-Rees property. In fact there are finitely many maximal ideals. Let h be an integer such that $\mathbf{m}^h R_{\mathbf{m}}$, for all maximal ideals in R. Since for every primary ideal, h is a strong uniform Artin-Rees number, then by Proposition 2.3 h is a strong uniform Artin-Rees number for all the ideals of R.

Remark 2.6. Let $(R, \mathbf{m})$ be a local Noetherian ring. Then $R \longrightarrow R[x]_{\mathbf{m}R[x]}$ is a faithfully flat extension and $R[x]_{\mathbf{m}R[x]}$ has an infinite residue field.

Proposition 2.7. *Suppose* $(R, \mathbf{m})$ *is a one-dimensional Noetherian local ring with infinite residue field. Then there exists an integer* h*, depending only on the ring, such that for any* $\mathbf{m}$*-primary ideal* I *there exists* $y \in I$ *such that* $I^n = yI^{n-1}$*, for any* $n \geq h$*.*

Proof. First suppose that R is Cohen-Macaulay and let e be the multiplicity of the ring. By [20], Theorem 1.1, Chapter 3, for every $\mathbf{m}$-primary ideal, we have that $\mu(I) \leq e$, where $\mu(I)$ denotes the minimal number of generators of I. Therefore, we have $\mu(I^e) \leq e < e + 1$. Hence, by [20], Theorem 2.3, Chapter 2, there exists $y \in I$ such that $I^e = yI^{e-1}$, so that for any $n \geq e = h$ we have $I^n = yI^{n-1}$.

Next suppose depth$(R) = 0$, and let $0 = q_1 \cap q_2 \cdots \cap q_{s+1}$ be a minimal primary decomposition of 0 where q_{s+1} is m-primary and set $J = q_1 \cap q_2 \cdots \cap q_s$. Then R/J is Cohen-Macaulay and there exists a h_0 such that $m^{h_0} J = 0$. Let e_1 be the multiplicity of R/J then, by the above case, there exists a $y \in I$ such that for any $n \geq e_1$ we have $I^n \subseteq yI^{n-1} + J$. By Huneke [11] Theorem 4.12, there exists a h_1, depending just on R and J such that for any $n \geq h_1$ and for any ideal $I \subset R$ we have

$$I^n \cap J \subset I^{n-1} J.$$

Hence, for any $n \geq h = max\{e_1, h_0 + 1, h_1\}$:

$$I^n \subseteq yI^{n-1} + I^n \cap J \subset yI^{n-1} + I^{n-1} J \subset yI^{n-1} + m^{n-1} \cap J = yI^{n-1}.$$

□

Proposition 2.8. *Let (R, m) be a one-dimensional Noetherian local ring. Then R has the strong uniform Artin-Rees property.*

Proof. Without loss of generality, by Proposition 2.4 and Remark 2.6, we may assume that R has infinite residue field. Let t be the integer as in Proposition 2.7 and let I be an m-primary ideal. If M/N is Cohen-Macaulay, then we can choose $y \in I$ such that y is a non-zerodivisor in M/N, so that for $n \geq t$,

$$\begin{aligned} I^n M \cap N &= yI^{n-1} M \cap N, \\ &\subseteq y(I^{n-1} M \cap N), \quad \text{since } y \text{ is a non-zerodivisor on } M/N, \\ &\subseteq I(I^{n-1} M \cap N), \quad \text{since } y \in I, \end{aligned}$$

proving that t is a strong Artin-Rees number.

Now suppose that M/N is not Cohen-Macaulay and let $M'/N = H^0_m(M/N)$.

We have that M/M' is Cohen-Macaulay, so there exists a uniform strong Artin-Rees number s for $M' \subset M$, and there exists l such that $m^l M' \subseteq N$. Let $h = \max\{l, s\}$ and let $t = h + l$. For any $n \geq t$ and for any $I \subseteq R$ we have:

$$\begin{aligned}
I^n M \cap N &= I^n M \cap M' \cap N, && \text{since } N \subset M', \\
&= I^{n-h}(I^h M \cap M') \cap N, && \text{since } M/M' \text{ is Cohen-Macaulay}, \\
&\subseteq I^{n-h}(I^h M \cap M'), && \text{since } n-h > l \text{ and } I^l M' \subset N, \\
&= I^{n-h-l} I^l (I^h M \cap M'). \\
&= I^{n-t}(I^l(I^h M \cap M') \cap N), && \text{since } I^l M' \subset N, \\
&\subseteq I^{n-t}(I^{h+l} M \cap M' \cap N), \\
&= I^{n-t}(I^t M \cap N),
\end{aligned}$$

proving that t is a strong Artin-Rees number. □

Let $I = (f_1, \ldots, f_n)$ be an ideal in R minimally generated by the polynomials f_i. Map the polynomial ring, with the standard grading, $R[x_1, \ldots, x_n]$, onto the Rees algebra $R[It]$ by sending x_i to $f_i t$. Let L be the kernel of this map. Then L is an homogeneous ideal and the $\mathrm{rt}(I)$ is defined to be the minimum integer s such that the ideal L can be genenerated by elements of degree less than or equal to s. This number does not depend on the choice of the generators of the ideal I; see for example [19]. The following lemma had been basically proved by Wang in [22].

Lemma 2.9. *Let $(R, \mathbf{m})$ be a Noetherian local ring and J be an ideal of R; denote $\bar{R} = R/J$. Let $I = (x_1, \ldots, x_m)$ be an ideal of R and suppose that $\mathrm{rt}(I\bar{R}) \leq h$, for some $h > 0$. Then for any $n \geq h$,*

$$I^n \cap J = I^{n-h}(I^h \cap J).$$

Proof. Let $n \geq h$ and let $x \in I^n \cap J$. Then there exists $F(X_1, \dots, X_m) \in R[X_1, \dots, X_m]$, a homogeneous polynomial of degree n, such that $F(x_1, \dots, x_m) = x$. Going modulo J, $\bar{F}$ is a relation on the $\bar{x}_i$'s, so by hypothesis there are polynomials G_i of degree h, and H_i, of degree $n-h$, such that $\bar{F} = \sum \bar{G}_i \bar{H}_i$ in $\bar{R}[X_1, \dots, X_m]$ and $\bar{G}_i$ are relations on the $\bar{x}_i$. Therefore $F = \sum G_i H_i + K$ for some $K \in R[X_1, \dots, X_m]$ of degree n and coefficients in J. Since:

$$\begin{aligned} K(x_1, \dots, x_m) &\in JI^n \subset I^{n-h}(I^h \cap J), \\ G_i(x_1, \dots, x_m) &\in I^h \cap J, \quad \text{and} \\ H_i(x_1, \dots, x_m) &\in I^{n-h}, \end{aligned}$$

$x = F(x_1, \dots, x_m) = \sum G_i(x_1, \dots, x_m) H_i(x_1, \dots, x_m) \in I^{n-h}(I^h \cap J)$. □

Remark 2.10. See [19], Lemma 6.3. If $(R, \mathbf{m})$ is a one dimensional, Cohen-Macaulay local ring and I is an m-primary ideal, then $\operatorname{rt}(I) \leq e$, where e is the multiplicity of R.

Lemma 2.11. *Let $(R, \mathbf{m})$ a Noetherian local ring. Suppose J is an ideal of R such that $\dim(R/J) \leq 1$. Then there exists an integer k such that for any $n > k$ and for any $\mathbf{m}$-primary ideal I, $I^n \cap J = I(I^{n-1} \cap J)$.*

Proof. If $\dim(R/J) = 0$ then there exists a power of the maximal ideal $\mathbf{m}^h \subset J$. Therefore for $n \geq h+1$ and for any ideal I we have the following:

$$I^n \cap J = I^n = II^{n-1} = I(I^{n-1} \cap J).$$

Suppose that $\dim(R/J) = 1$. If R/J is Cohen-Macaulay then the conclusion holds by the previous remark and by Lemma 2.9.

Suppose R/J has dimension one and it is not Cohen-Macaulay. Let $J \subset J'$ such that R/J' is Cohen-Macaulay and let l such that $\mathbf{m}^l J' \subset J$, so that for every ideal $I \subset R$ we have $I^l J' \subset J$. By the Cohen-Macaulay case there exists an Artin-Rees number $s = s(J' \subset R)$. Let $h = \max\{s, l\}$ and let $k = h + l$. Then, with an argument we already used, for any $n > k$ we have:

$$\begin{aligned} I^n \cap J &= I^n \cap J' \cap J \\ &= I^{n-h}(I^h \cap J') \cap J \\ &= I^{n-h}(I^h \cap J') \\ &= I^{n-h-l}(I^{h+l} \cap J') \\ &= I^{n-k}(I^k \cap J' \cap J) \\ &= I^{n-k}(I^k \cap J). \end{aligned}$$

This concludes the proof of the theorem. □

Using techniques from [6], we can prove the following more general proposition.

Proposition 2.12. *Suppose $N \subseteq M$ are finitely generated modules over a Noetherian ring. Let $J = \mathrm{Ann}(M/N)$. Suppose there exist a $k(J \subset R)$, such that $I^n \cap J = I^{n-k}(I^k \cap J)$, for any ideal $I \subset R$ and for any $n > k$. Assume that there is a strong uniform Artin-Rees number $s(N/JM \subset M/JM)$. Then $I^n M \cap N = I^{n-s}(I^s M \cap N)$, for any ideal I of R and for any $n > k$.*

Proof. Let $\phi : R^m \to M$, a surjection of a free module onto M. Denote by $K = \ker(\phi)$ and by $L = \phi^{-1}(N)$, the pre-image of the submodule $N \subset M$. Then, as shown in [6], it is enough to show that there exists a k such that for any $n > k$ and for any ideal $I \subset R$, we have $I^n R^m \cap L = I^{n-k}(I^k R^m \cap L)$. Indeed,

$$\begin{aligned}
\phi^{-1}(I^n M \cap N) &= \phi^{-1}(I^n M) \cap \phi^{-1}(N) \\
&= (I^n R^m + K) \cap L \\
&= K + (I^n R^m \cap L) \\
&= K + I^{n-k}(I^k R^m \cap L);
\end{aligned}$$

by applying ϕ across the equality between the first and the last term we obtain the claim. Therefore without loss of generality we may assume M is a free module. By hypothesis, for any $n > s$ and for any ideal I, we have

$$I^n M \cap N \subset I^{n-s}(I^s M \cap N) + JM.$$

Therefore

$$I^n M \cap N \subset I^{n-s}(I^s M \cap N) + JM \cap I^n M = I^{n-s}(I^s M \cap N) + (I^n \cap J)M,$$

where the last equality holds since M is a free module. If $n > k$, we have $I^n \cap J = I^{n-s}(I^s \cap J)$. Hence,

$$\begin{aligned}
I^n M \cap N &= I^{n-s}(I^s M \cap N) + I^{n-s}(I^s \cap J)M \\
&= I^{n-s}(I^s M \cap N) + I^{n-s}(I^s M \cap JM) \\
&\subset I^{n-s}(I^s M \cap N), \quad \text{since} \quad JM \subseteq N.
\end{aligned}$$

$\square$

Corollary 2.13. *Let* $(R, \mathbf{m})$ *a local Noetherian ring. Given two finitely generated* R*-modules* $N \subset M$ *such that* $\dim(M/N) \leq 1$*, there exists an Artin-Rees number* $s(N \subset M)$*. In other words, there exist an integer* s *such that for any* $n > s$ *and for any ideal* $I \subset R$ *we have* $I^n M \cap N = I(I^{n-1}M \cap N)$*.*

Proof. Apply Lemma 2.12 and Lemma 2.11. □

The proof of the following corollary can be found in [19]. We give it here for easy reference.

Corollary 2.14. *Let* R *be an excellent ring. Let* $N \subset M$ *be finitely generated* R*-modules such that* $\dim(M/N) \leq 1$*. Then there exists a strong uniform Artin-Rees number* $s(N \subset M)$*.*

Proof. Let $J = \mathrm{Ann}(M/N)$. Notice that $s(N \subset M) = s(N \subset M, \mathcal{W})$, where $\mathcal{W}$ is the family of maximal ideals in $V(J)$. Let $\{P_1, \ldots, P_n\} = \min(R/J)$ define the following subset of $V(J)$:

$$\begin{aligned}
A_1 &= \mathrm{Ass}(R/J) \\
A_2 &= \mathrm{Ass}(M/N) \\
\Sigma_1 &= \cup_{j>1, j<n, i_1<\ldots,<i_j} V(P_{i_1} + \cdots + P_{i_j}) \\
\Sigma_1 &= \emptyset \quad \text{if} \quad n = 1 \\
\Sigma_2 &= \cup_{i=1}^{n} \mathrm{Sing}(A/P_i).
\end{aligned}$$

All the sets above are finite sets of maximal ideals of R/J. For each such maximal ideal $\mathbf{m}$ there exists a strong Artin-Rees number for $N_{\mathbf{m}} \subset M_{\mathbf{m}}$. Let s be the maximum of such $s_{\mathbf{m}}$.

Denote R/J by $\overline{R}$. Let $\mathbf{m} \notin S := A_1 \cup A_2 \cup \Sigma_1 \cup \Sigma_2$, then $(M/N)_{\mathbf{m}}$ is a

Cohen-Macaulay module over $(\overline{R})_{\mathbf{m}}$, which is a Cohen-Macaulay local ring with a unique minimal prime P such that $(\overline{R}/P)_{\mathbf{m}}$ is a DVR with uniform parameter t. Notice that P is the image in $\overline{R}$ of P_i for some i. Since $(M/N)_{\mathbf{m}}$ is a Cohen-Macaulay module over a Cohen-Macaulay ring it is enough to show that there a uniform bound for $e(\overline{R})_{\mathbf{m}}$ for any $\mathbf{m} \notin S$, see the proofs of Proposition 2.10, Proposition 2.11 and Proposition 2.12.

Let

$$n = \max_i \{ \sum_{j=0}^{k} \mu(P_i^j) \quad | \quad i = 1, \ldots, n \},$$

where k be a minimum integer such that the k-th power of the nilradical of $\overline{R}$ is zero. Then

$$\begin{aligned} e(\overline{R}_{\mathbf{m}}) &= \mu(\mathbf{m}^i \overline{R}_{\mathbf{m}}), \quad \text{for} \quad i >> 0, \\ &= \mu((P+t)^i \overline{R}_{\mathbf{m}}), \quad \text{since } (\overline{R}/P\overline{R})_{\mathbf{m}} \text{ is a DVR}, \\ &= \sum_{j=0}^{k} \mu(P^j \overline{R}_{\mathbf{m}}), \\ &= \sum_{j=0}^{k} \mu(P^j) \leq n, \quad \text{by the definition of n.} \end{aligned}$$

□

2.2 Two dimensional rings

The following example was inspired by the example in [22], it shows that the uniform Artin-Rees property does not hold for two dimensional rings.

Example 2.15. Let $R = \mathsf{k}[x, y, z]/(z^2)$. Consider the following family of ideals:

$$I_n = < x^n, y^n, x^{n-1}y + z >,$$

for any $n \in \mathbb{N}$. Let J the ideal generated by z.
We claim that $I_n(I_n^{n-1} \cap J) \neq I_n^n \cap J$, for any n. In particular we will show that

$$x^{(n-1)^2}y^{n-1}z \in I_n^n \cap J \quad \text{but} \quad x^{(n-1)^2}y^{n-1}z \notin I_n(I_n^{n-1} \cap J).$$

Denote $x^{(n-1)^2}y^{n-1}z$ by ξ. Notice that I_n is a homogeneous ideal if we assign degree one to x and y and degree n to z. With such assigment, the degree of ξ is $(n-1)^2+n-1+n = n^2$. Since $x^{(n-1)^2}y^{n-1}z = (x^{n-1}y+z)^n-(x^n)^{n-1}y^n \in I_n^n$ the first claim holds.
Suppose that $x^{(n-1)^2}y^{n-1}z \in I_n(I_n^{n-1} \cap J)$, this remains true going modulo the elements $x^{(n-1)^2+1}, y^n$. Notice that $I_n^{n-1} \mod (x^{(n-1)^2+1}, y^n)R$ is generated by

$$< x^{n(n-1-i)}(x^{n-1}y+z)^i \quad | \quad i = 0, 1, \ldots, n-1 > .$$

Moreover,

$$\begin{aligned} x^{n(n-1-i)}(x^{n-1}y+z)^i &= x^{n(n-1-i)}(x^{(n-1)i}y^i + x^{(n-1)(i-1)}y^{i-1}z) \\ &= x^{n^2-n-i}y^i + x^{n^2-2n-i+1}y^{i-1}z. \end{aligned}$$

But $n^2-n-i \geq (n-1)^2+1$ for $i \leq n-2$. Therefore $I_{n-1}^n \mod (x^{(n-1)^2+1}, y^n)$ is generated by

$$< x^{(n-1)^2}y^{n-1} + x^{(n-1)(n-2)}y^{(n-2)}z, \quad x^{n^2-2n-i+1}y^{i-1}z \quad | \quad i = 1, \ldots, n-2 > .$$

Denote by

$$f = x^{(n-1)^2}y^{n-1} + x^{(n-1)(n-2)}y^{(n-2)}z$$

and by

$$g_i = x^{n^2-2n-i+1}y^{i-1}z.$$

Notice that $\xi \notin I_nG$, where G is the ideal generated by the g_i's. Indeed, the degree of x in the g_i is $n^2-2n-i+1$ and $(n^2-2n-i+1)+(n-1) > n^2-2n+1$ for $i \leq n-2$.

Let $hf + \sum h_ig_i$ be a homogeneous element of $I_n^{n-1} \cap J$ that appear in the expression on ξ as element of $I_n(I_n^{n-1} \cap J)$. By the previous observation, we can assume $h \neq 0$. Let m be a homogeneous monomial of h. If z does not divide m, then

$$mf = m'(x,y)x^{(n-1)(n-2)+1}y^{(n-2)}z$$

or

$$mf = m'(x,y)x^{(n-1)(n-2)}y^{(n-2)+1}z;$$

if z does divide m then $mf = m'(x,y)x^{(n-1)^2}y^{n-1}z$, with m' possibly a unit. By a degree counting we can see that $\deg(hf) \geq n^2-n+1$. Therefore, for any element $i \in I_{n-1}$ we have $\deg(ihf) > n^2 = \deg(\xi)$. This shows a contradiction.

The following example shows that the Artin-Rees property fails in a two dimensional ring, even if the ring is reduced. Notice that, by Corollary 2.13, R cannot be a two dimensional domain.

Example 2.16. Let $R = \mathsf{k}[x,y,z]/(xz)$. Consider the following family of ideals:

$$I_n = (x^n, y^n, x^{n-1}y + z^n),$$

for any $n \in \mathbb{N}$. Let $J = (z)$. Again, we claim that for any positive integer n, $I_n(I_n^{n-1} \cap J) \neq I_n^n \cap J$. More in particular we claim that

$$z^{n^2} \in I_n^n \cap J \quad \text{but} \quad z^{n^2} \notin I_n(I_n^{n-1} \cap J).$$

Indeed, $z^{n^2} = (x^{n-1}y + z^n)^n - (x^n)^{n-1}y^n \in I_n^n$ and trivially $z^{n^2} \in J$. On the other hand I_n^{n-1} is generated by:

$$< x^{n(n-1)}, x^{(n-1)^2}y^{n-1} + z^{n(n-1)}, y^n L, x^{(n-1)^2+i}y^{n-1-i} \quad | \quad i = 1, \ldots n-1 >,$$

for some ideal L in R. Notice that if $z^{n^2} \in I_n(I_n^{n-1} \cap J)$ then this holds going modulo y^n. Moreover, if a homogeneous element of $I_n^{n-1} \cap (z)$

$$f(x,y)x^{n(n-1)} + g(x,y,z)(x^{(n-1)^2}y^{n-1} + z^{n(n-1)}) + \sum h_i(x,y)x^{(n-1)^2+i}y^{n-1-i}$$

is in J then f, g, h_i must be in the maximal homogeneous ideal (they cannot be units since $n^2-n, (n-1)^2+i > (n-1)^2$). Writing $g(x,y,z) = g''(x,y)+zg'$. Since the expression above has to belong to the ideal generated by z, we see that

$$f(x,y)x^{n(n-1)} + g''(x,y)x^{(n-1)^2}y^{n-1} + \sum h_i(x,y)x^{(n-1)^2+i}y^{n-1-i} = 0.$$

But if this is the case, since $xz = 0$ in R, we have

$$fx^{n(n-1)} + g(x^{(n-1)^2}y^{n-1} + z^{n(n-1)}) + \sum h_i x^{(n-1)^2+i} = zg'z^{n(n-1)}.$$

But $zg'z^{n(n-1)}$ is an homogeneous element of degree at least $n^2 - n + 1$ and the multiplication by any element in I_n increases the degree by n. Therefore any element in $I_n(I_n^n \cap J)$ has degree at least $n^2 + 1$ while z^{n^2} has degree strictly smaller.

Chapter 3

Artin-Rees bounds on syzygies

In this chapter we study a uniform property for syzygies.

Definition 3.1. Uniform Artin-Rees for syzygies. Let $(R, \mathbf{m}, \mathsf{k})$ be a local Noetherian ring and let M be a finitely generated R-module. Let $\mathbb{F} = \{F_i\}$ be a free resolution of M. Denote by $M_i \subset F_{i-1}$ the i-th syzygy. We say that M has the uniform Artin-Rees property for syzygies with respect to an ideal I if there exists an integer h such that

$$M_i \cap I^n F_{i-1} \subset I^{n-h} M_i, \tag{3.0.1}$$

for all $n > h$ and for all $i > 0$. We denote the minimum integer h that satisfies 3.0.1 by $\mathrm{ar}_{\mathrm{syz}}(I, M)$. If 3.0.1 holds for every ideal $I \subset R$ then we say that M has the uniform Artin-Rees property for syzygies.

We can make a stronger definition:

Definition 3.2. Strong uniform Artin-Rees for syzygies. In the same setting of the above, we say that M has the strong uniform Artin-Rees property for syzygies with respect to an ideal I if there an integer h such that

$$M_i \cap I^n F_{i-1} = I(M_i \cap I^{n-1} F_{i-1}), \tag{3.0.2}$$

for all $n > h$ and for all $i > 0$. We denote the minimum integer h that satisfies 3.0.2 by $\mathrm{AR}_{\mathrm{syz}}(I, M)$. If 3.0.2 holds for every ideal $I \subset R$ then we say that M has the strong uniform Artin-Rees property for syzygies.

Eisenbud and Huneke [9] showed that there M has the uniform Artin-Rees property for syzygies when the module M has finite projective dimension on the punctured spectrum.

Throughout this chapter $(R, \mathbf{m}, \mathsf{k})$ will be a Noetherian local ring and we will denote by M_i the i-th syzygy of a minimal free resolution of a finitely generated R-module M while $\Omega_i(\mathsf{k})$ is the i-th syzygy of the residue field k. In this chapter we show that the residue field has the strong uniform Artin-Rees property for syzygies with respect to the maximal ideal $\mathbf{m}$. We also study uniform Artin-Rees property for syzygies and how the property behaves in short exact sequence. We show that the property with respect to an ideal $J \subset R$ relates to having a uniform annihilator for the family of modules $\mathcal{T}_{i,n} = \mathrm{Tor}_i^R(M, R/J^n)$. By an argument due to Katz, we can show that over Cohen-Macaulay rings, any R module has the uniform Artin-Rees property with respect to a $\mathbf{m}$-primary ideal.

It is easy to see that a module has the (strong) uniform Artin-Rees property for syzygies if and only if one of its syzygies has it.

3.1 Strong uniform Artin-Rees bounds for syzygies and the residue field

The residue field k has the strong uniform Artin-Rees property for syzygies. This fact is a consequence of the content of Corollary 3.16, in [13].

Theorem 3.3 (Levin). *Let* $(R, \mathbf{m}, \mathsf{k})$ *be a local Noetherian ring and let* $\mathbb{F} = \{F_i, d_i\}$ *be a minimal free resolution of* k. *Denote by* $\Omega_i(\mathsf{k}) = \ker d_{i-1}$. *Then there exists an integer* r_0 *such that for* $n \geq n_0$ *and for all* $i \geq 1$*:*

$$\operatorname{Tor}_i^R(\mathsf{k}, R/\mathbf{m}^n) \cong \frac{\mathbf{m}^{n-1}\Omega_i(\mathsf{k})}{\mathbf{m}^n\Omega_i(\mathsf{k})}.$$

Remark 3.4. If M is a finitely generated R-module we have

$$\frac{M_i \cap \mathbf{m}^n F_{i-1}}{\mathbf{m}^n M_i} = \operatorname{Tor}_i^R(M, R/\mathbf{m}^n).$$

Indeed, from the short exact sequence $0 \to M_i \to F_{i-1} \to M_{i-1} \to 0$, by tensoring with $R/\mathbf{m}^n$ we have

$$0 \to \operatorname{Tor}_1^R(M_{i-1}, R/\mathbf{m}^n) \to M_i/\mathbf{m}^n M_i \to F_{i-1}/\mathbf{m}^n F_{i-1} \to M_{i-1}/\mathbf{m}^n M_{i-1} \to 0.$$

From this last exact sequence we can see that the modules

$$\operatorname{Tor}_1^R(M_{i-1}, R/\mathbf{m}^n) = \operatorname{Tor}_i^R(M, R/\mathbf{m}^n)$$

and $\frac{M_i \cap \mathbf{m}^n F_{i-1}}{\mathbf{m}^n M_i}$ are the same module being the kernel of the middle map.

Corollary 3.5. *Let* $(R, \mathbf{m}, \mathsf{k})$ *be a local Noetherian ring. Then* k *has the strong uniform Artin-Rees property for syzygies.*

Proof. Let n_0 as in the previous theorem, then for any $n \geq n_0$, we have:

$$\frac{\Omega_i(\mathsf{k}) \cap \mathbf{m}^n F_{i-1}}{\mathbf{m}^n\Omega_i(\mathsf{k})} = \operatorname{Tor}_i^R(\mathsf{k}, R/\mathbf{m}^n) \cong \frac{\mathbf{m}^{n-1}\Omega_i(\mathsf{k})}{\mathbf{m}^n\Omega_i(\mathsf{k})},$$

where the first equality holds by Remark 3.4 and the second isomorphism is given by the previous Theorem. In particular the two modules

$$\frac{\mathbf{m}^{n-1}\Omega_i(\mathsf{k})}{\mathbf{m}^n\Omega_i(\mathsf{k})} \subseteq \frac{\Omega_i(\mathsf{k}) \cap \mathbf{m}^n F_{i-1}}{\mathbf{m}^n\Omega_i(\mathsf{k})}$$

have the same length and therefore they are equal. We have the following chain of inclusions,

$$\begin{aligned}\Omega_i(\mathsf{k}) \cap \mathbf{m}^n F_{i-1} &= \mathbf{m}^{n-1}\Omega_i(\mathsf{k}) \\ &= \mathbf{m}(\mathbf{m}^{n-2}\Omega_i(\mathsf{k})) \\ &\subset \mathbf{m}(\Omega_i(\mathsf{k}) \cap \mathbf{m}^{n-1}F_{i-1}).\end{aligned}$$

□

3.2 Uniform Artin-Rees bounds for syzygies

3.2.1 Uniform Artin-Rees bounds and short exact sequences

Recall from the previous chapter that $\operatorname{ar}(J, N \subset M)$ is the minimal integer such that for any $n \geq \operatorname{ar}(J, N \subseteq M)$ we have $J^n M \cap N \subset J^{n-h}N$. The proof of the following proposition is in [11].

Proposition 3.6. *Let $M_0 \subseteq M_1 \subseteq \cdots \subseteq M_n$ be finitely generated R-modules over a Noetherian ring and let J to be an ideal of R. Then*

(1) $\operatorname{ar}(J, M_0 \subset M_n) \leq \sum_{i=1}^n \operatorname{ar}(J, M_{i-1} \subset M_i)$*;*

(2) *If furthermore $M_i/M_{i-1} \cong R/I_i$ are cyclic modules for $i = 1, \ldots, n$, then* $\operatorname{ar}(J, M_0 \subset M_n) \leq \sum_{i=1}^n \operatorname{ar}(J, I_i \subset R)$*.*

The following definition and remark are due to Yongwei Yao, see [23].

Definition 3.7. For any ideal $J \subset R$ and any finitely generated R-module, define

$$\operatorname{ar}(J, M) = \max\{\operatorname{ar}(J, M' \subset M'') \quad | \quad M''/M' \cong M\}.$$

Notice that the maximum in the definition is well defined by Proposition 3.6(2).

Remark 3.8. (1) Let $\alpha : 0 \to N \to X \to M \to 0$ be a short exact sequence. Then $\mathrm{ar}(J, X) \leq \mathrm{ar}(J, N) + \mathrm{ar}(J, M)$. Indeed, let $X_1 \subset X_2$ two finitely generated R-modules such that $X_2/X_1 \cong X$ and let L_1 a finitely generated R-module such that $X_1 \subset L_1 \subset X_2$ and $L_1/X_1 \cong L$. Then

$$X_2/L_1 \cong (X_2/X_1)/(L_1/X_1) \cong X/L \cong M.$$

By definition and by Proposition 3.6(1), we have

$$\mathrm{ar}(J, X_1 \subset X_2) \leq \mathrm{ar}(J, X_1 \subset L_1) + \mathrm{ar}(J, L_1 \subset X_2) \leq \mathrm{ar}(J, L) + \mathrm{ar}(J, M).$$

By taking the maximum over finitely generated R-moduls $X_1 \subset X_2$ such that $X_1/X_2 \cong X$ on the left side we get the desired result.

(2) $\mathrm{ar}(J, M) = \mathrm{ar}(J, N \subset F)$ where F is a free R-module. Indeed, let $M_1 \subset M_2$ two finitely generated R-modules such that $M_2/M_1 \cong M$. It is enough to show that for every $n > \mathrm{ar}(J, N \subset F) = k$, $\mathbf{m}^n M_2 \cap M_1 \subset \mathbf{m}^{n-k} M_1$. Since F is a free module we have the following commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & N & \longrightarrow & F & \xrightarrow{f} & M & \longrightarrow & 0 \\ & & \downarrow & & \downarrow{\scriptstyle h} & & \| & & \\ 0 & \longrightarrow & M_1 & \longrightarrow & M_2 & \xrightarrow{g} & M & \longrightarrow & 0. \end{array}$$

Notice that $h(F) + M_1 = M_2$ and $h(F) \cap M_1 = h(N)$. Therefore, for every $n > k$

$$\begin{aligned} J^n M_2 \cap M_1 &= J^n(h(F) + M_1) \cap M_1 \\ &= J^n M_1 + J^n h(F) \cap M_1 \\ &\subseteq J^n M_1 + h(J^n F) \cap h(N) \\ &\subseteq J^n M_1 + J^{n-k} h(N) \\ &\subseteq J^{n-k} M_1. \end{aligned}$$

A priori, the property of having a uniform Artin-Rees bound for syzygies depends on the free resolution chosen. It turns out that it is enough to look at one particular free resolution.

Remark 3.9. Let M be a finitely generated R-module, and let $\mathcal{G}$ and $\mathcal{F}$ be a free and a minimal free resolution of M. Denote by M_i' and M_i the i-th syzygies respectively of $\mathcal{G}$ and $\mathcal{F}$. Then there exists an isomorphism of complex $\phi_{i-1} : G_{i-1} \to F_{i-1} \oplus H_i$, where $\mathcal{H} = \{H_i\}$ is the trivial complex, see [7] page 495. Therefore,

$$\begin{aligned} J^n G_{i-1} \cap M_i' \subset J^{n-h} M_i' &\quad \text{if and only if} \\ \phi_{i-1}(J^n G_{i-1} \cap M_i') \subset \phi_{i-1}(J^{n-h} M_i') &\quad \text{if and only if} \\ J^n \phi_i(G_{i-1}) \cap \phi_i(M_i') \subset J^{n-h} \phi_i(M_i') &\quad \text{if and only if} \\ J^n(F_{i-1} \oplus H_i) \cap (M_i \oplus H_i) \subset J^{n-h}(M_i \oplus H_i) &\quad \text{if and only if} \\ J^n F_{i-1} \cap M_i \subset J^{n-h} M_i. & \end{aligned}$$

Proposition 3.10. *Let $\alpha : 0 \to N \to X \to M \to 0$ be a short exact sequence. Assume that N and M have the uniform Artin-Rees property with respect to an ideal $J \subset R$. Then X has the uniform Artin-Rees property on* *syzygies with respect to J and*

$$\mathrm{ar}_{\mathrm{syz}}(J, X) \leq \mathrm{ar}_{\mathrm{syz}}(J, M) + \mathrm{ar}_{\mathrm{syz}}(J, N).$$

Proof. Let $\mathcal{F} = \{F_i\}$ and $\mathcal{G} = \{G_i\}$ be minimal free resolutions of M and N. Then X has a free resolution, not necessarily minimal, where the i-th free module is the direct sum of F_i and G_i. By Remark 3.8(2), we have that

$$\mathrm{ar}(J, M_i) = \mathrm{ar}(J, M_{i+1} \subset F_{i+1}) \leq \mathrm{ar}_{\mathrm{syz}}(M)$$

and

$$\mathrm{ar}(J, N_i) = \mathrm{ar}(J, N_{i+1} \subset F_{i+1}) \leq \mathrm{ar}_{\mathrm{syz}}(J, N),$$

for any $i > 0$. Therefore,

$$\begin{aligned}
\mathrm{ar}(J, X_{i+1} \subset F_{i+1} \oplus G_{i+1}) &= \mathrm{ar}(J, X_i), && \text{by Remark 3.8(2),}\\
&\leq \mathrm{ar}(J, M_i)\\
&+ \mathrm{ar}(J, N_i), && \text{by Remark 3.8(1),}\\
&= \mathrm{ar}(J, M_{i+1} \subset F_{i+1})\\
&+ \mathrm{ar}(J, N_{i+1} \subset G_{i+1}), && \text{by Remark 3.8(2),}\\
&\leq \mathrm{ar}_{\mathrm{syz}}(J, N) + \mathrm{ar}_{\mathrm{syz}}(J, M), && \text{by hypothesis.}
\end{aligned}$$

and hence

$$\mathrm{ar}_{\mathrm{syz}}(J, X) = \sup_{i \in \mathbb{N}} \{\mathrm{ar}(J, X_i \subseteq F_i \oplus G_i)\} \leq \mathrm{ar}_{\mathrm{syz}}(J, N) + \mathrm{ar}_{\mathrm{syz}}(J, M).$$

□

More is true for short exact sequences.

Proposition 3.11. *Let $\alpha : 0 \to N \to X \to M \to 0$ be a short exact sequence. If two modules have the uniform Artin-Rees property for syzygies with respect to an ideal J then also the third does.*

Proof. If M and N have the uniform Artin-Rees property for syzygies, then so does X, by Proposition 3.10.

Assume N and X have the uniform Artin-Rees property for syzygies with respect to an ideal J. Let F a free module surjecting on X. We have the following diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & N & \longrightarrow & X & \longrightarrow & M & \longrightarrow & 0 \\
 & & \uparrow & & \twoheaduparrow & & \twoheaduparrow & & \\
0 & \longrightarrow & 0 & \longrightarrow & F & = & F & \longrightarrow & 0 \\
 & & & & \hookuparrow & & \hookuparrow & & \\
 & & & & L & \longrightarrow & P & &
\end{array}$$

and hence, by Snake Lemma, we have the following short exact sequence:

$$0 \longrightarrow L \longrightarrow P \longrightarrow N \longrightarrow 0.$$

By hypothesis, L and N have the uniform Artin-Rees property on syzygies with respect to J. By Proposition 3.10, P and hence M have the uniform Artin-Rees property on syzygies with respect to J.

Assume that M and X have the uniform Artin-Rees property on syzygies with respect to J. Consider the following diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & N & \longrightarrow & X & \longrightarrow & M & \longrightarrow & 0 \\
 & & \twoheaduparrow & & \twoheaduparrow & & \twoheaduparrow & & \\
0 & \longrightarrow & F & \longrightarrow & H & \longrightarrow & G & \longrightarrow & 0, \\
 & & \hookuparrow & & \hookuparrow & & \hookuparrow & & \\
0 & \longrightarrow & N' & \longrightarrow & X' & \longrightarrow & M' & \longrightarrow & 0.
\end{array}$$

where each row and column is exact, and where F, G and H are free modules. From the above diagram we have the following:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & N & \longrightarrow & X & \longrightarrow & M & \longrightarrow & 0 \\
 & & \| & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & N & \longrightarrow & H/N' & \longrightarrow & G & \longrightarrow & 0, \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & 0 & \longrightarrow & X'/N' & \longrightarrow & M' & \longrightarrow & 0.
\end{array}
$$

Since M has the uniform property, $X'/N' \cong M'$ has it. By Proposition 3.10, H/N' has the uniform property with respect to J. Since the middle row is split exact, N has the uniform property with respect to J. □

It seems hard to prove the uniform Artin-Rees property for syzygies by using a prime filtration.

3.2.2 Uniform Artin-Rees property for syzygies and annihilators of the Tor*s modules*

Fix an integer i; the classical Artin-Rees Lemma, applied to the i-th syzygy, gives uniform annihilators for the family of modules $\mathcal{T}_n = \operatorname{Tor}_i^R(M, R/J^n)$.

Proposition 3.12. *Let $(R, \mathbf{m}, \mathsf{k})$ be a local Noetherian ring. Let M be a finitely generated R-module, J an ideal of R and let i be a fixed integer. Then there exists an integer h such that for every $n > 0$*

$$J^h \operatorname{Tor}_i^R(M, R/J^n) = 0.$$

Moreover there exists an integer k such that

$$\operatorname{Ann}(\operatorname{Tor}_i^R(M, R/J^{k+i})) = \operatorname{Ann}(\operatorname{Tor}_i^R(M, R/J^k)),$$

for every $i > 0$.

Proof. By the usual Artin-Rees Lemma, there exists a strong Artin-Rees number $\mathrm{AR}(J, M_i \subset F_{i-1}) = h$. For any $n > h$, we have $J^n F_{i-1} \cap M_i = J(J^{n-1}F_{i-1} \cap M_i)$. We claim that for any $n > h$ the following inclusion holds

$$\mathrm{Ann}(\mathrm{Tor}_i(M, R/J^n)) \subset \mathrm{Ann}(\mathrm{Tor}_i(M, R/J^{n+1}))$$

To prove the claim, let $c \in \mathrm{Ann}(\mathrm{Tor}_i(M, R/J^n))$ then, by 3.4

$$c(J^n F_{i-1} \cap M_i)) \subset J^n M_i.$$

Therefore,

$$c(J^{n+1}F_{i-1} \cap M_i) \subset cJ(J^n F_{i-1} \cap M_i) \subset J^{n+1}M_i,$$

proving that $c \in \mathrm{Ann}(\mathrm{Tor}_i^R(M, R/J^{n+1}))$. Because the ring R is Noetherian, these annihilators are eventually stable. Assume that h is the integer such that $\mathrm{Ann}(\mathrm{Tor}_i^R(M, R/J^{h+i})) \subset \mathrm{Ann}(\mathrm{Tor}_i^R(M, R/J^h))$, for every $i > 0$. Then $J^h \subset \mathrm{Ann}(\mathrm{Tor}_i^R(M, R/J^n))$, for every $n > 0$. □

In particular the annihilator of the R-module

$$L_i = \oplus_n \mathrm{Tor}_i^R(M, R/\mathbf{m}^n)$$

is $\mathbf{m}$-primary. If the uniform Artin-Rees Lemma for syzygies holds, we have a uniform annihilator for the family of modules $\mathcal{T}_{i,n} = \mathrm{Tor}_i^R(M, R/J^n)$. In fact, the following Lemma holds.

Lemma 3.13. *Let* $(R, \mathbf{m})$ *be local ring. If* M *has the uniform Artin-Rees property for syzygies then there exists an integer* h *such that for any* $i > 0$ *and for any* $n > 0$, $J^h \subset \mathrm{Ann}(\mathrm{Tor}_i^R(M, R/J^n))$.

Proof. Notice that $J^h \operatorname{Tor}_i^R(M, R/J^n) = 0$ for every $i > 0$ and for every $n \leq h$. Let $\mathbb{F}_\bullet = \{F_i\}$ be a minimal free resolution of M. If M has the uniform Artin-Rees property for syzygies, then there exists an integer h such that $J^n F_{i-1} \cap M_i \subset J^{n-h} M_i$, for every $n > h$ and for every $i > 0$. By Remark 3.4 we have

$$\operatorname{Tor}_i^R(M, R/J^n) = (J^n F_{i-1} \cap M_i)/J^n M_i \subset J^{n-h} M_i / J^n M_i,$$

for every $n > h$ and for every $i > 0$, showing that

$$J^h \operatorname{Tor}_i^R(M, R/J^n) = 0.$$

□

The existence of an element which uniformly annihilates the family of modules $\mathcal{T}_{i,n} = \operatorname{Tor}_i^R(M, R/\mathbf{m}^n)$ is sometimes equivalent to the uniform Artin-Rees property for syzygies with respect to the maximal ideal $\mathbf{m}$.

Proposition 3.14. *Let $(R, \mathbf{m})$ be a local Noetherian ring and let M be a finitely generated maximal Cohen-Macaulay R-module. Assume that there exists an element $c \in R$ which is regular on the module M and on R. Then M has the uniform Artin-Rees property on syzygies with respect to the maximal ideal $\mathbf{m}$ if and only if there exists a non-zerodivisor c such that*

$$c \operatorname{Tor}_i^R(M, R/\mathbf{m}^n) = 0$$

for every $n \gg 0$ and for every $i > 0$, and M/cM satisfies the uniform Artin-Rees property for syzygies with respect to the maximal ideal $\mathbf{m}$.

Proof. By Lemma 3.13, if M has the Artin-Rees property on syzygies with respect too the maximal ideal $\mathbf{m}$, then there exists an integer h such that

$\mathbf{m}^h \subset \mathrm{Ann}(\mathrm{Tor}_i^R(M, R/\mathbf{m}^n))$, for any $i > 0$ and for any $n > h$. Since R is Cohen-Macaulay of positive dimension, there exists a non-zero divisor as in the proposition.

For the other direction, let $(\mathbb{F}, \delta)$ be a minimal free resolution of M and notice that, since M is maximal Cohen-Macaulay and c a non-zerodivisor on R, $\mathbb{F} \otimes R/(c)$ is a minimal free resolution for M/cM. Denote by $\overline{}$ the operation of going modulo c. By assumption, $\overline{M}$ has the uniform Artin-Rees property on syzygies with respect to the maximal ideal $\mathbf{m}$, therefore there exists a h such that

$$\mathbf{m}^n \overline{F}_i \cap \overline{M}_i \subset \mathbf{m}^{n-h} \overline{M}_i, \tag{3.2.1}$$

for every $i > 0$ and for every $n > h$.

Let $u \in \mathbf{m}^n F_{i-1} \cap M_i$. By taking a larger h, we can assume that 3.2.1 holds and $c \in \bigcap_{i>0, n>h} \mathrm{Ann}\,\mathrm{Tor}_i^R(M, R/\mathbf{m}^n)$. For such values of n, since

$$\mathrm{Tor}_i^R(M, R/\mathbf{m}^n) = (\mathbf{m}^n F_{i-1} \cap M_i)/\mathbf{m}^n M_i,$$

we have that $cu \in \mathbf{m}^n M_i$. This means that there exist $y \in F_{i+1}$, $u_j \in F_{i+1}$ and $m_j \in \mathbf{m}^n$, such that

$$cu = c\delta(y) = \sum m_j \delta(u_j).$$

This implies that $cy - \sum m_j u_j \in M_{i+1}$ and therefore

$$\overline{\sum m_j u_j} \in \overline{M}_{i+1} \cap \mathbf{m}^n \overline{F}_{i+1} \subset \mathbf{m}^{n-h} \overline{M}_{i+1}.$$

This shows that $\sum m_j u_j \in \mathbf{m}^{n-h} F_{i+1} + cF_{i+1}$ so that we can write $\sum m_j u_j = z + cy'$. Then, $cy' \in \mathbf{m}^{n-h} F_{i+1} \cap cF_{i+1}$. By the usual Artin-Rees Lemma, there exists an l such that $(c) \cap \mathbf{m}^n \subset \mathbf{m}^{n-l}(c)$, for any $n > l$. Therefore,

$$cy' \in c\mathbf{m}^{n-h-l} F_{i+1}, \quad \text{for every} \quad n > h + l,$$

which implies that $y' \in \mathbf{m}^{n-h-l}F_{i+1}$ being c a non-zerodivisor. If $u' = \delta(y')$ then $u' \in \mathbf{m}^{n-h-l}M_i$.

We claim that $u \in \mathbf{m}^{n-h-l}M_i$, proving the uniform Artin-Rees property for syzygies with respect to the maximal ideal $\mathbf{m}$, with bound $h+l$. For,

$$cu = \delta(cy) = \delta(\sum m_j u_j) = \delta(z + cy') = c\delta(y') = cu'$$

But c is a non zero divisor and therefore $u = u' \in \mathbf{m}^{n-l-h}M_i$, as desired. □

3.3 Uniform Artin-Rees property for syzygies for modules over Cohen-Macaulay rings

We show that any module over a Cohen-Macaulay ring has the uniform Artin-Rees property for syzygies with respect to the maximal ideal $\mathbf{m}$. Recall the following

Definition 3.15. Let R be a Noetherian ring and $I \subset R$ an ideal. $J \subset I$ is a reduction of I if there exists an integer h such that for any $n > h$ we have $I^n = J^{n-h}I^h$. The integer h is call a reduction number.

For easy reference we recall here the following theorem due to Rees

Theorem 3.16. *Let R be a ring and M an R-module. Let $a_1, \ldots, a_n$ be a sequence regular on M and $I = (a_1, \ldots, a_n)$. Let $x_1, \ldots, x_n$ be indeterminates over R. If $g \in M[x_1, \ldots, x_n]$ is homogeneous of degree d and $g(a_1, \ldots, a_d) \in I^{d+1}M$ then g has the coefficients in IM.*

The proof can be found in [5], page 6.

Lemma 3.17. *Let $(R, \mathbf{m})$ be a Noetherian local ring. Let K be a submodule of a free module F and let $M = F/K$. Let $J = (a_1, \ldots, a_l)$ be an ideal generated by a sequence which is regular on M. Then $J^n F \cap K = J^n K$ for any $n > 0$.*

Proof. Let $m \in J^n F \cap K$ then there exists a homogeneous polynomial g of degree n in l variable and with coefficients in F such that $g(a_1, \ldots, a_l) = m$. By going modulo K, we have a homogeneous polynomial $\overline{g}$ of degree n in $M[x_1, \ldots, x_n]$ such that $\overline{g}(a_1, \ldots, a_l) = 0$. We want to prove that $\overline{g}$ is the zero polynomial. By applying Theorem 3.16, since $\overline{g}(a_1, \ldots, a_l) = 0 \in I^{n+1}M$, we have that the coefficients of $\overline{g}$ are in IM. Therefore there exists a homogeneous polynomial g_1 of degree $n+1$ in $M[x_1, \ldots, x_n]$ such that $g_1(a_1, \ldots, a_l) = \overline{g}(a_1, \ldots, a_l) = 0$. By repeating this argument we can see that the coefficients of $\overline{g}$ are in $I^n M$ for every n and therefore, by Krull Intersection Theorem, they are zero.

□

Theorem 3.18. *Let $(R, \mathbf{m}, \mathsf{k})$ be a local Noetherian ring of dimension d, let M be a finitely generated R-module which is maximal Cohen-Macaulay and let I be a $\mathbf{m}$-primary ideal. Let F be a free module and $\phi : F \to M$ a surjection. Assume that the residue field is infinite. Then there exists an integer h, not depending on the module M, such that for every $n > h$ we have*

$$I^n F \cap K \subset I^{n-h} K.$$

Assume further that R is Cohen-Macaulay then any module (not necessarly maximal Cohen-Macaulay) M has the uniform Artin-Rees property for syzygies with respect to any $\mathbf{m}$-primary ideal.

Proof. Let $J = (x_1, \ldots, x_d) \subset I$ be a minimal reduction such that $x_1, \ldots x_d$ is a regular sequence on M. Let h be the reduction number. Let $K = \ker(\phi)$. For every $i > 0$ and for every $n > h$ we have

$$\begin{aligned} I^n F \cap K &= J^{n-h} I^h F \cap K && (3.3.1) \\ &\subset J^{n-h} F \cap K && (3.3.2) \\ &= J^{n-h} K \subset J^{n-h} K, && (3.3.3) \end{aligned}$$

where the last equality is given by the previous lemma with $d = l$, which we can apply since $F/K \cong M$ is a maximal Cohen-Macaulay module.
Assume now that R is Cohen-Macaulay and let M be any finitely generated R-module. By replacing M with its d-th syzygy we can assume that M is maximal Cohen-Macaulay. Recall that M_i is the i-th syzygy of M. Notice that we can use equation 3.3.1 replacing K by M_i and F by F_{i-1}. □

Lemma 3.13 gives the following

Corollary 3.19. *Let $(R, \mathbf{m}, \mathsf{k})$ be a Cohen-Macaulay local ring with infinite residue field and let M be a maximal Cohen-Macaulay module. Given a* **m***-primary ideal, there exists an integer h such that for any $i > 0$ and for any $n > 0$*

$$J^h \subset \operatorname{Ann}(\operatorname{Tor}_i^R(M, R/J^n))$$

.

3.4 Uniform Artin-Rees property for syzygies in rings of dimension one

By applying Theorem 3.18 with $d = 1$ we obtain

Proposition 3.20. *Let* $(R, \mathbf{m}, \mathsf{k})$ *be a local Noetherian ring of dimension one with infinite residue field. There exists an integer* h *such that for every finitely generated* R*-module* M *of* depth 1 *and for every* $n > h$

$$\mathbf{m}^h \operatorname{Tor}_1^R(M, R/\mathbf{m}^n) = 0.$$

Note that in the previous corollary the integer h does not depend on the module.

Proposition 3.21. *Let* $(R, \mathbf{m}, \mathsf{k})$ *be a local Noetherian ring of dimension one with infinite residue field. Let* M *be any finitely generated* R*-module which is a first syzygy. There exists an integer* l*, not depending on the module, such that for every* $n > 0$

$$\mathbf{m}^l \operatorname{Tor}_1^R(M, R/\mathbf{m}^n) = 0.$$

Proof. Without loss of generality we may assume that $\operatorname{depth}(M) = 0$ and M is not a finite length module. Consider the following short exact sequence

$$0 \to H^0_{\mathbf{m}}(M) \to M \to M/H^0_{\mathbf{m}}(M) \to 0.$$

After tensoring by $R/\mathbf{m}^n$ we obtain the following exact sequence

$$\operatorname{Tor}_1^R(R/\mathbf{m}^n, H^0_{\mathbf{m}}(M)) \to \operatorname{Tor}_1^R(R/\mathbf{m}^n, M) \to \operatorname{Tor}_1^R(R/\mathbf{m}^n, M/H^0_{\mathbf{m}}(M)).$$

Chose a positive integer h_1 such that $\mathbf{m}^{h_1} H^0_{\mathbf{m}}(R) = 0$. Since M is a first syzygy, we have $H^0_{\mathbf{m}}(M) \subset H^0_{\mathbf{m}}(F)$ where F is some free module. Therefore $\mathbf{m}^{h_1} H^0_{\mathbf{m}}(M) = 0$ and $\mathbf{m}^{h_1} \operatorname{Tor}_1^R(R/\mathbf{m}^n, H^0_{\mathbf{m}}(M)) = 0$, for every $n > 0$. Since $M/H^0_{\mathbf{m}}(M)$ has positive depth, by the previous proposition there exists an integer h_2, not depending on the module $M/H^0_{\mathbf{m}}(M)$, such that $\mathbf{m}^{h_2} \operatorname{Tor}_1^R(M/H^0_{\mathbf{m}}(M), R/\mathbf{m}^n) = 0$, for every $n > h_2$. Let $l = h_1 h_2$. For every $n > h_2$ and therefore for every $n > l$, we have $\mathbf{m}^l \operatorname{Tor}_1^R(M, R/\mathbf{m}^n) = 0$. On the other hand, $\mathbf{m}^l \subset \operatorname{Ann}(\operatorname{Tor}_1^R(M, R/\mathbf{m}^n))$ for every $n \leq l$. □

Corollary 3.22. *Let $(R, \mathbf{m}, \mathsf{k})$ be a local Noetherian ring of dimension one with infinite residue field. Let M be any finitely generated R-module. There exists an integer l, depending on the module, such that for every $n > 0$ and for every $i > 0$*

$$\mathbf{m}^l \operatorname{Tor}_i^R(M, R/\mathbf{m}^n) = 0.$$

Proof. By Proposition 3.21 there exists an integer q such that for every $n > 0$ and for every $i > 1$ we have

$$\mathbf{m}^q \operatorname{Tor}_i^R(M, R/\mathbf{m}^n) = \mathbf{m}^q \operatorname{Tor}_1^R(M_{i-1}, R/\mathbf{m}^n) = 0.$$

By taking $i = 1$ in Remark 3.12, there exists an integer p such that

$$\mathbf{m}^p \operatorname{Tor}_1^R(M, R/\mathbf{m}^n) = 0,$$

for all integer $n > 0$. Let l be the maximum between p and q. □

Recall the following definition

Definition 3.23. Let $(R, \mathbf{m})$ be a local Noetherian ring. an element $x \in \mathbf{m}$ is said to be superficial if there exists an integer c such that

$$(\mathbf{m}^n : x) \cap \mathbf{m}^c = \mathbf{m}^{n-1},$$

for all $n > c$.

Superficial elements always exist if the residue field k is infinite, see [20] page 7. If x is superficial for R then for every free module F we have

$$(0 :_F x) \cap \mathbf{m}^c F \subset (\mathbf{m}^n :_F x) \cap \mathbf{m}^c F = \mathbf{m}^{n-1} F,$$

for all $n > c$. Therefore $(0 :_F x) \cap \mathbf{m}^c F = 0$ by Krull Intersection Theorem. Finally we have the following

Theorem 3.24. *Let $(R, \mathbf{m}, \mathsf{k})$ be a Noetherian local ring of dimension one with infinite residue field. Then any finitely generated R-module M has the uniform Artin-Rees property for syzygies with respect to the maximal ideal.*

Proof. Let $x \in \mathbf{m}$ be an element which is a reduction of $\mathbf{m}$ and a superficial element for R. Since x is superficial we have $(0 :_{F_{i-1}} x) \cap \mathbf{m}^c F_{i-1} = 0$, for every $n > c$. Let h_1 be an integer as in Corollary 3.22. Let h_2 be a reduction number of $(x) \subset \mathbf{m}$. We may assume that both h_1 and h_2 are bigger then c. Let $h = \max\{h_1, h_2\}$. Then for every $i > 0$ we have

$$x^h(\mathbf{m}^n F_{i-1} \cap M_i) \subset \mathbf{m}^n M_i = x^{n-h}\mathbf{m}^h M_i.$$

Let $u \in \mathbf{m}^n F_{i-1} \cap M_i$ then $x^h u = x^{n-h} v$, where $v \in \mathbf{m}^h M_i$. Then for every $n > 2h$ we have $x^h(u - x^{n-2h}v) = 0$. Since $h > c$ we that $u - x^{n-2h}v \in \mathbf{m}^c F_{i-1}$. Therefore, since x is superficial by applying the equation $(0 : x) \cap \mathbf{m}^c = 0$ we have $u = x^{n-2h}v \in \mathbf{m}^{n-2h} M_i$. □

Chapter 4

Conclusion

In this thesis we study problems in commutative algebra. The modules we study are finitely generated R-modules over Noetherian rings.
In the first chapter of this thesis we study closely the correspondence between short exact sequences and elements of the Ext module. We prove an extension of Miyata's theorem. On one hand the theorem it is interesting on its own, on the other hand the many applications of this theorem and of its partial converse, show that this extension is actually a nice tool for dealing with short exact sequences. We can give simpler proofs of previous results, we can improve bounds, we can descrive minimal generators of the Ext module and we can give, in certain cases, new information about the structure of the Ext modules.

In the second and third chapter we study uniform versions of the classical Artin-Rees Lemma. In the second chapter we study the strong uniform Artin-Rees property, where an unique integer makes the classical Artin-Rees Lemma working for an infinite family of ideals. We see that the strong uniform Artin-Rees property holds for one dimensional excellent rings. We give examples of two dimensional rings in which the property fails.

In the third chapter we study the uniform property for syzygies. The ring we work with is always a local Noetherian ring. In this case the ideal is fixed and we search for an integer that makes the weaker version of the Artin-Rees Lemma hold for all the syzygies inside the free modules of a free resolution. We investigate also the strong uniform Artin-Rees property for syzygies. We show that the residue field k has the strong uniform Artin-Rees property for syzygies with respect to the maximal ideal and that any module over a local Noetherian ring of dimension one has the uniform Artin-Rees property for syzygies with respect to any $\mathbf{m}$-primary ideal. The fact that the uniform Artin-Rees Lemma for syzygies with respect to any $\mathbf{m}$-primary ideals holds in Cohen-Macaulay rings of any dimensions gives an indication that the the property might hold in other higher dimensional rings. On the other hand it would be interesting to know if any finitely generated R-module of finite length has the strong uniform Artin-Rees property for syzygies with respect to the maximal ideal.

References

[1] M. Auslander, *Representation theory of finite-dimensional algebras*, Algebraists' homage: papers in ring theory and related topics (New Haven, Conn., 1981), Contemp. Math., vol. 13, Amer. Math. Soc., Providence, R.I., 1982, pp. 27–39. MR 84b:16031

[2] Maurice Auslander, *Isolated singularities and existence of almost split sequences*, Representation theory, II (Ottawa, Ont., 1984), Lecture Notes in Math., vol. 1178, Springer, Berlin, 1986, pp. 194–242. MR 87j:13029

[3] Maurice Auslander and Idun Reiten, *Modules determined by their composition factors*, Illinois J. Math. **29** (1985), no. 2, 280–301. MR 86i:16032

[4] Klaus Bongartz, *A generalization of a theorem of M. Auslander*, Bull. London Math. Soc. **21** (1989), no. 3, 255–256. MR 90b:16031

[5] Winfried Bruns and Jürgen Herzog, *Cohen-Macaulay rings*, Cambridge Studies in Advanced Mathematics, vol. 39, Cambridge University Press, Cambridge, 1993. MR MR1251956 (95h:13020)

[6] A. J. Duncan and L. O'Carroll, *A full uniform Artin-Rees theorem*, J. Reine Angew. Math. **394** (1989), 203–207. MR MR977443 (90c:13011)

[7] David Eisenbud, *Commutative algebra*, Graduate Texts in Mathematics, vol. 150, Springer-Verlag, New York, 1995, With a view toward algebraic geometry. MR 97a:13001

[8] David Eisenbud and Melvin Hochster, *A Nullstellensatz with nilpotents and Zariski's main lemma on holomorphic functions*, J. Algebra **58** (1979), no. 1, 157–161. MR MR535850 (80g:14002)

[9] David Eisenbud and Craig Huneke, *A finiteness property of infinite resolutions*, Journal of Algebra (to appear).

[10] Robert M. Guralnick, *Lifting homomorphisms of modules*, Illinois J. Math. **29** (1985), no. 1, 153–156. MR 86g:13006

[11] Craig Huneke, *Uniform bounds in Noetherian rings*, Invent. Math. **107** (1992), no. 1, 203–223. MR 93b:13027

[12] Craig Huneke and Graham J. Leuschke, *Two theorems about maximal Cohen-Macaulay modules*, Math. Ann. **324** (2002), no. 2, 391–404. MR 2003j:13011

[13] Gerson Levin, *Local rings and Golod homomorphisms*, J. Algebra **37** (1975), no. 2, 266–289. MR 55 #2878

[14] Saunders Mac Lane, *Homology*, Classics in Mathematics, Springer-Verlag, Berlin, 1995, Reprint of the 1975 edition. MR 96d:18001

[15] Hideyuki Matsumura, *Commutative algebra*, second ed., Mathematics Lecture Note Series, vol. 56, Benjamin/Cummings Publishing Co., Inc., Reading, Mass., 1980. MR MR575344 (82i:13003)

[16] Takehiko Miyata, *Note on direct summands of modules*, J. Math. Kyoto Univ. **7** (1967), 65–69. MR 35 #5434

[17] L. O'Carroll, *A note on Artin-Rees numbers*, Bull. London Math. Soc. **23** (1991), no. 3, 209–212. MR 92i:13001a

[18] Liam O'Carroll and Dorin Popescu, *Free resolutions for deformations of maximal Cohen-Macaulay modules*, Comm. Algebra **28** (2000), no. 11, 5329–5352. MR MR1785504 (2001i:13015)

[19] Francesc Planas-Vilanova, *The strong uniform Artin-Rees property in codimension one*, J. Reine Angew. Math. **527** (2000), 185–201. MR 2001g:13051

[20] Judith D. Sally, *Numbers of generators of ideals in local rings*, Marcel Dekker Inc., New York, 1978. MR MR0485852 (58 #5654)

[21] Hsin-Ju Wang, *On the Fitting ideals in free resolutions*, Michigan Math. J. **41** (1994), no. 3, 587–608. MR 96b:13013

[22] ______, *Some uniform properties of* 2*-dimensional local rings*, J. Algebra **188** (1997), no. 1, 1–15. MR 98e:13006

[23] Yongwei Yao, *Linear growth property of Tor and Ext*, preliminary version.

[24] Yuji Yoshino, *Cohen-Macaulay modules over Cohen-Macaulay rings*, London Mathematical Society Lecture Note Series, vol. 146, Cambridge University Press, Cambridge, 1990. MR 92b:13016

[25] ______, *On degenerations of Cohen-Macaulay modules*, J. Algebra **248** (2002), no. 1, 272–290. MR 2002m:16016