On Extensions of Modules

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Abstract

In this paper we study closely Yoneda's correspondence between short exact sequences and the Ext^1 group. We prove a main theorem which gives conditions on the splitting of a short exact sequence after taking the tensor product with R/I, for any ideal I of R. As an application we prove a generalization of Miyata's Theorem on the splitting of short exact sequences and we improve a proposition of Yoshino about efficient systems of parameters. We introduce the notion of sparse module and we show that $\mathrm{Ext}^1_R(M, N)$ is a sparse module provided that there are finitely many isomorphism classes of maximal Cohen-Macaulay modules having multiplicity the sum of the multiplicities of M and N. We prove that sparse modules are Artinian. We also give some information on the structure of certain Ext^1 modules.

Key words: Extensions of modules. Rings of finite Cohen-Macaulay type.

1 Introduction

Let R a Noetherian ring and M and N finitely generated R-modules. If $I \subset R$ is an ideal and $\alpha : 0 \to N \to X \to M \to 0$ is a short exact sequence, we denote by $\alpha \otimes R/I$ the sequence $0 \to N/IN \to X/IX \to M/IM \to 0$.
In the first part of this paper we prove the following:

Theorem 1.1. *Let R be a Noetherian ring. Suppose $\alpha \in I\,\mathrm{Ext}^1_R(M, N)$ is a short exact sequence of finitely generated modules. Then $\alpha \otimes R/I$ is a split exact sequence.*

An immediate corollary is an extension of Miyata's theorem [1], which gives a necessary and sufficient condition on the splitting of short exact sequences:

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Theorem 1.2. *Let R be a Noetherian ring. Let $\alpha : 0 \to N \to X_\alpha \to M \to 0$ and $\beta : 0 \to N \to X_\beta \to M \to 0$ be two short exact sequences. If X_α is isomorphic to X_β and $\beta \in I \operatorname{Ext}^1_R(M, N)$ for some ideal $I \subset R$, then $\alpha \otimes R/I$ is split exact.*

In the second part of the paper, we give some applications of Theorem 1.1 to efficient systems of parameters and, more in particular, to the structure of $\operatorname{Ext}^1_R(M, N)$ for rings of finite Cohen-Macaulay type.

Recall that a finitely generated R-module is said to be maximal Cohen-Macaulay if the depth$(M) = \dim(R)$. We also say that R has finite Cohen-Macaulay type if there are a finite number of isomorphism classes of indecomposable maximal Cohen-Macaulay modules. Auslander [2] proved that if R is a complete local ring of finite Cohen-Macaulay type then the length of $\operatorname{Ext}^1_R(M, N)$ is finite, where M and N are maximal Cohen-Macaulay modules. Recently, Huneke and Leuschke [3] generalized this theorem to the non complete case. A different proof is given as an application of Theorem 1.1.
In section 4 we introduce the notion of sparse modules. More specifically, suppose that depth $R \geq 1$; then we say that a module is sparse if there are a finite number of submodules of the form xM where x is a non-zero-divisor on R. we prove several properties for sparse modules. In particular, we show that sparse modules are Artinian and that $\operatorname{Ext}^1_R(M, N)$ is sparse if M and N are maximal Cohen-Macaulay modules over a ring of finite Cohen-Macaulay type.
In the last section, we are able to give more information about the structure of $\operatorname{Ext}^1_R(M, N)$. In particular, we give an explicit bound for the power of the maximal ideal which kills the Ext module, depending on the number of isomorphism classes of maximal Cohen-Macaulay modules of multiplicity the sum of the multiplicities of M and N. Our bound improves the one given in [3]. Finally, we use the developed techniques to show that $\operatorname{Ext}^1_R(M, N)$ is a cyclic module, under certain conditions.

2 Main theorem and Miyata's theorem

We recall the theorem due to Miyata [1] on the splitting of short exact sequences.

Theorem 2.1 (Miyata). *Let $(R, \mathbf{m})$ be a local Noetherian ring and let*

$$\alpha : 0 \longrightarrow N \xrightarrow{i} X_\alpha \xrightarrow{\pi} M \longrightarrow 0$$

be a short exact sequence of finitely generated R-modules. Then, α is a split exact sequence if and only if X_α and $M \oplus N$ are isomorphic.

We can further relax the assumptions on α and prove the following:

Lemma 2.2. *Let $(R, \mathbf{m})$ be a local Noetherian ring and let α be the following exact complex of finitely generated R-modules:*

$$N \xrightarrow{i} X_\alpha \xrightarrow{\pi} M \longrightarrow 0$$

If X_α is isomorphic to $M \oplus N$ then α is split exact.

Proof. By Theorem 2.1, it suffices to prove that i is an injective map. By way of contradiction, suppose that there exists an element $c \neq 0$ such that $c \in \ker i$. Choose n such that $c \notin \mathbf{m}^n N$. Then we have the exact sequence

$$0 \longrightarrow C \longrightarrow N/\mathbf{m}^n N \xrightarrow{\bar{i}} X_\alpha/\mathbf{m}^n X_\alpha \xrightarrow{\pi} M/\mathbf{m}^n M \longrightarrow 0,$$

where C is the kernel of $i \otimes R/\mathbf{m}^n$. By the assumption on n, C is not the zero module because the equivalence class of c belongs to it. If λ denotes length, we have the following contradiction:

$$\begin{aligned} \lambda(X_\alpha/\mathbf{m}^n X_\alpha) &< \lambda(X_\alpha/\mathbf{m}^n X_\alpha) + \lambda(C) \\ &= \lambda(N/\mathbf{m}^n N) + \lambda(M/\mathbf{m}^n M), \quad && \text{from the exact sequence } \xi, \\ &= \lambda(X_\alpha/\mathbf{m}^n X_\alpha), && \text{from } X_\alpha \cong M \oplus N. \end{aligned}$$

□

Lemma 2.3. *Let R be a Noetherian ring and let α be the following short exact sequence:*

$$\alpha : \quad 0 \longrightarrow N \xrightarrow{f} X_\alpha \xrightarrow{g} M \longrightarrow 0.$$

Denote by C the image of the connecting homomorphism δ, obtained by applying the functor $\mathrm{Hom}_R(\quad, N)$ to α:

$$\cdots \longrightarrow \mathrm{Hom}_R(X_\alpha, N) \xrightarrow{f^*} \mathrm{Hom}_R(N, N) \xrightarrow{\delta} \mathrm{Ext}^1_R(M, N) \xrightarrow{g^*} \cdots$$

Then, α is a split exact sequence if and only if $C = 0$.

Proof. Suppose $C = 0$. Since $\alpha = \delta(\mathrm{id}_N)$, $\alpha = 0$ in $\mathrm{Ext}^1_R(M, N)$ and hence α is a split exact sequence.

On the other hand, if α is a split exact sequence then there exists an R-homomorphism $f' : X_\alpha \to N$, such that $f'f = \mathrm{id}_N$. To prove the lemma, it is enough to show that the map $f^* : \mathrm{Hom}(X_\alpha, N) \to \mathrm{Hom}(N, N)$ is a surjective map. For, if $l \in \mathrm{Hom}(N, N)$ we have $l = f^*(lf')$. □

Recall the following result (see [4], [5], [6],[7],[8]):

Proposition 2.4. *Let $(R, \mathbf{m})$ be a local Noetherian ring and let M and N two finitely generated R-modules. Denote by $\lambda(-)$ the length of a R-module. Then, the following are equivalent*

(1) $M \cong N$;
(2) $\lambda(\operatorname{Hom}_R(M, L)) = \lambda(\operatorname{Hom}_R(N, L))$, *for all modules L of finite length;*
(3) $\lambda(M \otimes_R L) = \lambda(N \otimes_R L)$, *for all modules L of finite length.*

We are now ready to give the proof of Theorem 1.1.

Proof. (Theorem 1.1). We first show that it is enough to prove the local statement. Assume that the local version of the theorem holds. If

$$\alpha : 0 \longrightarrow N \xrightarrow{f} X_\alpha \xrightarrow{g} M \longrightarrow 0$$

is a short exact sequence in $I \operatorname{Ext}^1_R(M, N)$ then $\alpha/1 \in \operatorname{Ext}^1_{R_\mathbf{m}}(M_\mathbf{m}, N_\mathbf{m})$ is given by the sequence

$$\alpha/1 : 0 \longrightarrow N_\mathbf{m} \xrightarrow{f/1} X_{\alpha\mathbf{m}} \xrightarrow{g/1} M_\mathbf{m} \longrightarrow 0,$$

for any maximal ideal $\mathbf{m}$. Let C be the kernel of $f \otimes \operatorname{id}_{R/I}$, then $\operatorname{Supp}(C) \subset V(I)$. Let $\mathbf{m} \in V(I)$. $\alpha \in I \operatorname{Ext}^1_R(M, N)$ implies $\alpha/1 \in IR_\mathbf{m} \operatorname{Ext}^1_{R_\mathbf{m}}(M_\mathbf{m}, N_\mathbf{m})$ and therefore $\alpha/1 \otimes R_\mathbf{m}/IR_\mathbf{m}$ is a split exact sequence. In particular,

$$C_\mathbf{m} = \ker(f \otimes \operatorname{id}_{R/I})_\mathbf{m} = \ker(f/1 \otimes \operatorname{id}_{R_\mathbf{m}/IR_\mathbf{m}}) = 0,$$

proving that $\alpha \otimes R/I$ is a short exact sequence. We need to show that $\alpha \otimes R/I$ is actually a split exact sequence. By Lemma 2.3, it is enough to prove that $\operatorname{Image}(\delta) = 0$, where δ is the connecting homomorphism:

$$\operatorname{Hom}_R(N/IN, N/IN) \xrightarrow{\delta} \operatorname{Ext}^1_R(M/IM, N/IN).$$

Call $C = \operatorname{Image}(\delta)$, again $\operatorname{Supp}(C) \subset V(I)$. Let $\mathbf{m}$ any maximal ideal in $V(I)$, then $C_\mathbf{m} = \operatorname{Image}(\delta/1)$, where $\delta/1$ is the connecting homomorphism:

$$\operatorname{Hom}_{R_\mathbf{m}}(N_\mathbf{m}/IN_\mathbf{m}, N_\mathbf{m}/IN_\mathbf{m}) \xrightarrow{\delta/1} \operatorname{Ext}^1_{R_\mathbf{m}}(M_\mathbf{m}/IM_\mathbf{m}, N_\mathbf{m}/IN_\mathbf{m}).$$

But, by Lemma 2.3, $\operatorname{Image}(\delta/1) = 0$ since $\alpha/1 \otimes R_\mathbf{m}/IR_\mathbf{m}$ is a split exact sequence. Therefore, $C_\mathbf{m} = 0$ for any $\mathbf{m} \in V(I)$, which implies that $C = 0$.

We may assume $(R, \mathbf{m})$ is a local Noetherian ring. By Lemma 2.2 it is enough to show that $X_\alpha/IX_\alpha \simeq M/IM \oplus N/IN$.
Let

$$F_2 \xrightarrow{d_2} F_1 \xrightarrow{d_1} F_0 \xrightarrow{d_0} M \longrightarrow 0$$

be part of a minimal resolution of M over R, and let $L \subset F_0$ be the kernel of d_0. Following Yoneda's correspondence (see [9], page 652-654), there exists a

$\Psi \in \ker d_2^* \subset \operatorname{Hom}_R(F_1, N)$ which induces a map ψ from $L = \ker d_0$ to N in such a way that α is the pushout of the following diagram:

$$\begin{array}{ccccccccc} \xi : 0 & \longrightarrow & L & \xrightarrow{i} & F_0 & \xrightarrow{d_0} & M & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ \alpha : 0 & \longrightarrow & N & \xrightarrow{f} & X_\alpha & \xrightarrow{g} & M & \longrightarrow & 0, \end{array}$$

where i is the inclusion. Since α is an element of $I \operatorname{Ext}_R^1(M, N)$, we can choose $\Psi \in I \operatorname{Hom}_R(F_1, N)$, which implies that $\psi(L) \subset IN$. Denote by ν_α the following exact sequence:

$$\nu_\alpha : 0 \longrightarrow L \xrightarrow{(i,-\psi)} F_0 \oplus N \xrightarrow{\pi} X_\alpha \longrightarrow 0.$$

Let Ω be a finitely generated module of finite length such that $I\Omega = 0$. Tensor both the sequences ξ and ν_α with Ω and set

$$\operatorname{Image}_1 = (i \otimes_R \operatorname{id})(L \otimes_R \Omega),$$

$$\operatorname{Image}_2 = (i \otimes_R \operatorname{id}, -\psi \otimes_R \operatorname{id})(L \otimes_R \Omega).$$

Since $\operatorname{Image}(\psi) \subset IN$, it follows that $\operatorname{Image}_1 = \operatorname{Image}_2 \subset F_0 \otimes_R \Omega$.

If $\lambda_R(M)$ denotes the length of an R-module M, we have:

$$\begin{aligned} \lambda_R(X_\alpha \otimes_R \Omega) &= \lambda_R(F_0 \otimes_R \Omega) + \lambda_R(N \otimes_R \Omega) - \lambda_R(\operatorname{Image}_2) \\ &= \lambda_R(M \otimes_R \Omega) + \lambda_R(\operatorname{Image}_1) + \lambda_R(N \otimes_R \Omega) - \lambda_R(\operatorname{Image}_2) \\ &= \lambda_R(M \otimes_R \Omega) + \lambda_R(N \otimes_R \Omega). \end{aligned}$$

Notice that if Y is any R-module, which is also an R/I-module, then we have $\lambda_R(Y) = \lambda_{R/I}(Y)$. Therefore, the equality above says:

$$\lambda_{R/I}(M/IM \otimes_R \Omega) + \lambda_{R/I}(N/IN \otimes_R \Omega) = \lambda_{R/I}(X_\alpha/IX_\alpha \otimes_R \Omega).$$

Since in this equality we can choose any R/I module of finite length Ω, by Proposition 2.4, we have :

$$M/IM \oplus N/IN \cong X_\alpha/IX_\alpha.$$

$\square$

As a corollary of the theorem we can prove 1.2:

Proof. (Theorem 1.2). Since $\beta \in I\,\mathrm{Ext}^1_R(M,N)$, Theorem 1.1 implies that $X_\beta/IX_\beta \simeq M/IM \oplus N/IN$ and hence $X_\alpha/IX_\alpha \simeq M/IM \oplus N/IN$. Applying Lemma 2.2, we have that $\alpha \otimes R/I$ is split exact. □

If the ideal I is the zero ideal Theorem 2.1 is recovered.

Another corollary of Theorem 1.1 is the following proposition, which we will use later.

Proposition 2.5. *Let $(R, \mathbf{m})$ be a local Noetherian ring. Consider the short exact sequence $\alpha : 0 \to N \to X_\alpha \to M \to 0$, and let M_1 the first syzygy of M in a minimal free resolution. Assume that $\mathrm{Ext}^1_R(X_\alpha, M_1) = 0$. If α is not the zero element in $\mathrm{Ext}^1_R(M,N)$, then α is a minimal generator of $\mathrm{Ext}^1_R(M,N)$.*

Proof. By way of contradiction, assume $\alpha \in \mathbf{m}\,\mathrm{Ext}^1_R(M,N)$. By Theorem 1.1, $\alpha \otimes R/\mathbf{m}$ is a split exact sequence and therefore

$$\mu(M) + \mu(N) = \mu(X_\alpha).$$

On the other hand, if $0 \to M_1 \to F \to M \to 0$ is the beginning of a minimal free resolution for M, there exists an R-homomorphism $\phi : M_1 \to N$ such that X_α is the cokernel as in the following short exact sequence:

$$\beta : 0 \longrightarrow M_1 \xrightarrow{(i,-\phi)} F \oplus N \longrightarrow X_\alpha \longrightarrow 0,$$

where i is the inclusion. Since $\mathrm{Ext}^1_R(X_\alpha, M_1) = 0$, β is a split exact sequence and $X_\alpha \oplus M_1 \cong F \oplus N$. Therefore we have the following contradiction:

$$\mu(M) + \mu(N) = \mu(X_\alpha) < \mu(X_\alpha) + \mu(M_1) = \mu(F) + \mu(N) = \mu(M) + \mu(N).$$

□

Remark 2.6. The same conclusion of Proposition 2.5 holds if we assume $\mathrm{Ext}^1_R(X_\alpha, N) = 0$ instead of $\mathrm{Ext}^1_R(X_\alpha, M_1) = 0$.

The converse of Theorem 1.1 does not hold as the following example shows.

Example 2.7. Let $R = k[|x^2, x^3|]$. Every non-zero element $\alpha \in \mathrm{Ext}^1_R(k,R)$ is a minimal generator and hence is not in $\mathbf{m}\,\mathrm{Ext}^1_R(k,R)$.

Let α be the short exact sequence as in the following diagram:

$$\begin{array}{ccccccccc}
 & 0 & \longrightarrow & \mathbf{m} & \xrightarrow{i} & R & \longrightarrow & k & \longrightarrow & 0 \\
 & & & \downarrow{\scriptstyle\psi} & & \downarrow & & \| & & \\
\alpha : & 0 & \longrightarrow & R & \longrightarrow & P & \longrightarrow & k & \longrightarrow & 0,
\end{array}$$

where ψ is the R-homomorphism sending x^2 to x^3 and x^3 to x^4 and P is the pushout. The short exact sequence α is not split exact because there is no map from R that extends ψ, hence α is not in $\mathbf{m}\operatorname{Ext}^1(k, R)$. On the other hand, the minimal number of generators of P is 2 and hence $P/\mathbf{m}P \simeq k \oplus k$.

However we can prove a converse of Theorem 1.1 in the following sense:

Proposition 2.8. *Let M and N be finitely generated R-modules, let $y \in R$ be a non-zero-divisor on R, M, N and let α be the short exact sequence*

$$0 \longrightarrow N \longrightarrow X_\alpha \longrightarrow M \longrightarrow 0.$$

Suppose that $X_\alpha/yX_\alpha \simeq M/yM \oplus N/yN$. Then $\alpha \in y\operatorname{Ext}^1_R(M, N)$.

Proof. Since y is a non-zero-divisor on N we have the following exact sequences:

$$0 \longrightarrow N \xrightarrow{y} N \xrightarrow{\pi} N/yN \longrightarrow 0,$$

and, by applying the functor $\operatorname{Hom}_R(M, \quad)$,

$$\cdots \longrightarrow \operatorname{Ext}^1_R(M, N) \xrightarrow{y} \operatorname{Ext}^1_R(M, N) \xrightarrow{\pi^*} \operatorname{Ext}^1_R(M, N/yN) \longrightarrow \cdots.$$

By exactness, to show that $\alpha \in y\operatorname{Ext}^1_R(M, N)$, it is enough to show that $\pi^*(\alpha) = 0$.

Denote by $\phi : \operatorname{Ext}^1_R(M, N/yN) \to \operatorname{Ext}^1_R(M/yM, N/yN)$ the isomorphism that takes a short exact sequence $\beta : 0 \to N/yN \to Y \to M \to 0$ to the short exact sequence $\beta \otimes R/yR : 0 \to N/yN \to Y/yY \to M/yM \to 0$ (which is exact since y is a non-zero-divisor on M). Since $\pi^*(\alpha)$ is $0 \to N/yN \to X_\alpha/yf(N) \to M \to 0$, $\phi\pi^*(\alpha)$ is the short exact sequence:

$$\alpha \otimes R/yR : 0 \longrightarrow N/yN \longrightarrow X_\alpha/yX_\alpha \longrightarrow M/yM \longrightarrow 0,$$

which is split exact. Hence, $\pi^*(\alpha) = 0$. ∎

Question 2.9. Let $(R, \mathbf{m})$ a Cohen-Macaulay local ring. Let M and N be finitely generated maximal Cohen-Macaulay modules over R and let $x_1, \ldots, x_n$ be a regular sequence on R. Let $\alpha \in \operatorname{Ext}^1_R(M, N)$ be a short exact sequence. Is it true that

$$\alpha \otimes R/(x_1, ..., x_n) \quad \text{is split exact}$$

if and only if

$$\alpha \in (x_1, \ldots, x_n)\operatorname{Ext}^1_R(M, N)?$$

In the remainder of this section we present another proposition on the annihilator of short exact sequences, which we will use later.

Let α and β will be the following two short exact sequences:

$$\alpha: \ 0 \longrightarrow N \xrightarrow{\alpha_1} X_\alpha \xrightarrow{\alpha_2} M \longrightarrow 0,$$

$$\beta: \ 0 \longrightarrow N \xrightarrow{\beta_1} X_\beta \xrightarrow{\beta_2} M \longrightarrow 0.$$

Assume $X_\alpha \cong X_\beta$. It follows from Miyata's theorem that $\mathrm{Rad}(\mathrm{Ann}(\alpha)) = \mathrm{Rad}(\mathrm{Ann}(\beta))$. The following example, due to Giulio Caviglia, shows that $\mathrm{Ann}\,\alpha$ and $\mathrm{Ann}\,\beta$ do not have the same integral closure (see exercise A3.29, page 656, [9]).

Example 2.10. Let α and β be the following sequences:

$$\alpha, \beta: \ 0 \longrightarrow \mathbb{Z}^2 \oplus \mathbb{Z}_2 \xrightarrow[\beta_1]{\alpha_1} \mathbb{Z}^2 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_2 \xrightarrow[\beta_2]{\alpha_2} \mathbb{Z}_4 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \longrightarrow 0,$$

where $\alpha_1(x, y, z) = (4x, y, 2z, 0)$ and $\beta_1(x, y, z) = (2x, 2y, 0, z)$. Notice that $\beta = \beta_1 \oplus \beta_2 \oplus \beta_2$, where β_1 is the split exact sequence $0 \to \mathbb{Z}_2 \to \mathbb{Z}_2 \oplus \mathbb{Z}_4 \to \mathbb{Z}_4 \to 0$ and β_2 is the generator of $\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}_2, \mathbb{Z})$. The annihilator of β is therefore $2\mathbb{Z}$. On the other hand α is the direct sum of the split exact sequence $0 \to \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z}_2 \to \mathbb{Z}_2 \to 0$ and the generators of $\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}_4, \mathbb{Z})$ and $\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}_2, \mathbb{Z}_2)$. Therefore $\mathrm{Ann}(\alpha) = 4\mathbb{Z}$.

Proposition 2.11. *Let α and β be two short exact sequences in $\mathrm{Ext}^1_R(M, N)$ with middle modules isomorphic to X. If $\mathrm{Ext}^1_R(X, N) = 0$ then $\mathrm{Ann}(\alpha) = \mathrm{Ann}(\beta)$.*

Proof. Applying the functor $\mathrm{Hom}_R(\ \ , N)$ to the short exact sequence α, we get the exact sequence $\mathrm{Hom}_R(N, N) \to \mathrm{Ext}^1_R(M, N) \to 0$. Therefore there exists a homomorphism $\psi \in \mathrm{Hom}_R(N, N)$ such that $\beta \in \mathrm{Ext}^1_R(M, N)$ is the pushout of ψ as in the following diagram:

$$\begin{array}{ccccccccc}
\alpha: 0 & \longrightarrow & N & \xrightarrow{\alpha_1} & X & \longrightarrow & M & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
\beta: 0 & \longrightarrow & N & \xrightarrow{\beta_1} & X & \longrightarrow & M & \longrightarrow & 0,
\end{array}$$

ϕ induces a homomorphism $\psi^* : \mathrm{Ext}^1_R(M, N) \to \mathrm{Ext}^1_R(M, N)$ taking α to β, which implies that $\mathrm{Ann}(\alpha) \subset \mathrm{Ann}(\beta)$. Since we can switch the role of α and β, we have the thesis. $\square$

3 Applications

3.1 Efficient systems of parameters

Recall the following definition:

Definition 3.1. A system of parameters $x_1, \dots, x_n$ is an efficient system of parameters if for any $i = 1, \dots, n$ there is a regular subring T_i of R over which R is finite and such that x_i belongs to the Noetherian different $\mathcal{N}_{T_i}^R$.

The proofs above can be used to give an improvement of the following proposition 6.17 in [10]:

Proposition 3.2. *Let $\mathbf{x} = \{x_1, \dots, x_n\}$ be an efficient system of parameters. Let $\alpha : 0 \to N \to X \to M \to 0$ be a short exact sequence of maximal Cohen-Macaulay modules. Denote by $\mathbf{x}^2$ the ideal generated by $x_1^2, \dots, x_n^2$ and let $\bar{\alpha}$ be the short exact sequence $0 \to N/\mathbf{x}^2N \to X/\mathbf{x}^2X \to M/\mathbf{x}^2M \to 0$. If $\bar{\alpha}$ is split exact then α is split exact.*

We can substitute $\mathbf{x}^2$ simply by $\mathbf{x}$ and prove the following Proposition:

Proposition 3.3. *Let $\mathbf{x} = \{x_1, \dots, x_n\}$ be an efficient system of parameters. Let $\alpha : 0 \to N \to X \to M \to 0$ be a short exact sequence of maximal Cohen-Macaulay modules and let $\bar{\alpha}$ be $0 \to N/\mathbf{x}N \to X/\mathbf{x}X \to M/\mathbf{x}M \to 0$. If $\bar{\alpha}$ is split exact then α is split exact.*

Proof. Denote by $\mathbf{x}_i = \{x_1, \dots, x_i\}$ and by α_i the following short exact sequence:

$$0 \longrightarrow N/\mathbf{x}_iN \longrightarrow X/\mathbf{x}_iX \longrightarrow M/\mathbf{x}_iM \longrightarrow 0.$$

We will show by descending induction on i that α_i is split exact. The case $i = 0$ will give the thesis of the improved Proposition 3.3. The assumptions says that α_n is a split exact sequence, which is the case $i = n$. Suppose that the case $i = k$ is true. Then, by Proposition 2.8,

$$\alpha_{k-1} \in x_k \operatorname{Ext}^1_{R/\mathbf{x}_{k-1}R}(M/\mathbf{x}_{k-1}M, N/\mathbf{x}_{k-1}N) \simeq x_k \operatorname{Ext}^1_R(M, N/\mathbf{x}_{k-1}N).$$

By definition of efficient systems of parameters, there exists a regular subring S_k such that the extension $S_k \subset R$ is finite and x_k is in $\mathcal{N}_{S_k}^R$, the Noetherian different. By a result proved by Wang ([11], Proposition 5.9), we have that $\mathcal{N}_{S_k}^R \operatorname{Ann}(\operatorname{Ext}^1_{S_k}(M, N/\mathbf{x}_{k-1}N)) \subset \operatorname{Ann}(\operatorname{Ext}^1_R(M, N/\mathbf{x}_{k-1}N))$. But M is a free S_k-module so that $\operatorname{Ann}(\operatorname{Ext}^1_{S_k}(M, N/\mathbf{x}_{k-1}N))$ is the unit ideal of S_k and hence $x_k \in \operatorname{Ann}(\operatorname{Ext}^1_R(M, N/\mathbf{x}_{k-1}N))$, which implies that α_{k-1} is a split exact sequence. □

Let us study the structure of the Ext modules and then prove some applications for rings of finite Cohen-Macaulay type. Assume that $(R, \mathbf{m})$ is a local Noetherian ring. First of all we need some notation. For every short exact sequence $\alpha \in \operatorname{Ext}^1_R(M,N)$ denote by X_α the middle module. For every finitely generated R-module X, denote by $[X]$ the isomorphism class of X and define the following set

$$E_{[X]} := \{\alpha \in \operatorname{Ext}^1_R(M,N) \quad | \quad X_\alpha \cong X\}. \tag{1}$$

For every $y \in \mathbf{m}$, define

$$\mathcal{S}_y := \{[X] \quad | \quad \exists \alpha \in y \operatorname{Ext}^1_R(M,N) \quad \text{and} \quad X \cong X_\alpha\}. \tag{2}$$

Lemma 3.4. *Let $(R, \mathbf{m})$ be a local Noetherian ring. Let y be a non-zero-divisor on M, N and R. Then, with the above notation,*

$$y \operatorname{Ext}^1_R(M,N) = \bigcup_{[X]\in\mathcal{S}_y} E_{[X]}.$$

Proof. The fact that $y \operatorname{Ext}^1_R(M,N) \subset \bigcup_{[X]\in\mathcal{S}_y} E_{[X]}$, follows immediately from the definition of the sets $\mathcal{S}_y$. For the other direction, all it is to prove is that if $E_{[X]} \cap y \operatorname{Ext}^1_R(M,N) \neq$ then $E_{[X]} \subset y \operatorname{Ext}^1_R(M,N)$. For it, let $\alpha \in E_{[X]} \cap y \operatorname{Ext}^1_R(M,N)$ and $\beta \in E_{[X]}$. By Theorem 1.2, $\beta \otimes R/(y)$ is a split exact sequence and, by Proposition 2.8, $\beta \in y \operatorname{Ext}^1_R(M,N)$, proving the lemma. □

Theorem 3.5. *Let $(R, \mathbf{m})$ be a Cohen-Macaulay local ring. Assume $\dim R \geq 1$ and let M and N be two finitely generated maximal Cohen-Macaulay modules over R. Assume that there are only finitely many isomorphism classes of maximal Cohen-Macaulay modules of multiplicity the sum of the multiplicities of M and N. If h is the number of such isomorphism classes, then there are at most 2^h submodules of the form $x \operatorname{Ext}^1_R(M,N)$, where x is a non-zero-divisor on R.*

Proof. Let $X_1, \ldots, X_h$ be representatives for the isomorphism classes of all the possible modules that can fit in the middle of a short exact sequence in $\operatorname{Ext}^1_R(M,N)$. Let $y \in \mathbf{m}$ but not in the union of the minimal primes of R. Consider the module $x \operatorname{Ext}^1_R(M,N)$. By Lemma 3.4, we have

$$y \operatorname{Ext}^1_R(M,N) = \bigcup_{[X]\in\mathcal{S}_y} E_{[X]}.$$

Since there are at most h modules that can fit in the middle of a short exact sequence in $\operatorname{Ext}^1_R(M,N)$, there at most 2^h different subsets of $\{X_1, \ldots, X_h\}$ and hence at most 2^h different sets $\mathcal{S}_y$, proving the theorem. □

We can apply this theorem to the theory of rings of finite Cohen-Macaulay type. Recall that a Noetherian local ring $(R, \mathbf{m})$ is said of finite Cohen-Macaulay type if there are only a finite number of classes of non-isomorphic maximal Cohen-Macaulay modules. As a corollary of 3.5 we can give a different proof of the following theorem due, in this generality, to [3]:

Theorem 3.6. *Let $(R, \mathbf{m})$ be a Noetherian of finite Cohen-Macaulay type and let M, N two maximal Cohen-Macaulay modules. Then there exists a t such that $\mathbf{m}^t \operatorname{Ext}^1_R(M, N) = 0$.*

Proof. If the dimension of the ring R is zero, then there is nothing to prove. Suppose that $\dim(R) \geq 1$. Assume, by way of contradiction, that the maximal ideal $\mathbf{m}$ is not the radical ideal of $\operatorname{Ann}(\operatorname{Ext}^1_R(M, N))$. By prime avoidance, we can pick an element x in the maximal ideal but not in the union

$$\operatorname{Ann}(\operatorname{Ext}^1_R(M, N)) \cup \bigcup_{P \in \operatorname{Ass}(R)} P.$$

Let h be the number of isomorphism classes of modules that can fit in the middle of a short exact sequence in $\operatorname{Ext}^1_R(M, N)$. By Theorem 3.5, there exist $i < j < 2^h + 1$ such that $x^i \operatorname{Ext}^1_R(M, N) = x^j \operatorname{Ext}^1_R(M, N)$. By Nakayama Lemma, $x^i \operatorname{Ext}^1_R(M, N) = 0$, showing a contradiction. □

We can actually give a bound for t in terms of the number of possible maximal Cohen-Macaulay modules that can fit in the middle of a short exact sequence in $\operatorname{Ext}^1_R(M, N)$, improving the bound of [3]. For this, see the last section.

4 Sparse Modules

Theorem 3.5 motivates us to make the following definition:

Definition 4.1. Let $(R, \mathbf{m})$ be a Noetherian local ring of depth at least 1. Let M a finitely generated R-module. M is said to be sparse if there is only a finite number of submodules xM, where $x \in R$ is a non-zero-divisor of R.

Remark 4.2. Notice that sparseness is closed under taking direct sums and moreover the quotient of a sparse module is sparse. On the other hand, a submodule of a sparse module does not need to be sparse.

Example 4.3. Let $R = k[|x, y|]$, with k infinite. Let M be the cokernel of the following map:

$$R^4 \xrightarrow{\beta} R^2 \longrightarrow 0$$

where β is given by the following matrix:

$$\begin{pmatrix} 0 & x & 0 & y^2 \\ x & y & y^2 & 0 \end{pmatrix}.$$

Notice that $\mathbf{m}^2 M = 0$. The only submodules of the form lM with $l \in \mathbf{m}$ are xM and $(x+y)M$. Let $N = Rm_1$ be the generator of M corresponding to $(1,0)$. Notice that $\mathbf{m}N$ is a two dimensional vector space and therefore it cannot be sparse.

Proposition 4.4. *Let $(R, \mathbf{m})$ be a local Noetherian ring of positive depth and let M be a finitely generated R- module. If M is sparse then M is Artinian.*

Proof. Choose $x_1, \dots, x_n$ to be the generators of the maximal ideal $\mathbf{m}$ in such a way that x_i is a non-zero-divisor of R for any $i = 1, \dots, n$. For each i, consider the modules $x_i^h M$. Since x_i^h is a non-zero-divisor for any h and since M is sparse, $x_i^n M = x_i^m M$ for some $m > n$. By Nakayama's lemma, this implies that $x_i^n M = 0$. It follows that M is Artinian. □

Let us suppose that the ring contains the residue field k, which we assume to be infinite.

Proposition 4.5. *Let $(R, \mathbf{m}, k)$ be a local Noetherian ring of positive depth and with infinite residue field. Suppose M is a finitely generated R-module. If M is sparse then there exists an element $l \in \mathbf{m}$ such that $\mathbf{m}M = lM$.*

Proof. Since M is sparse, we can list the all possible submodules of the form xM, where x is a non-zero-divisor. Let the list be $l_1 M, \dots, l_h M$. Let $x_1, \dots, x_n$ be a set of minimal generators of the maximal ideal. Choose n vectors with entries in k which are linearly independent. Say $s^i =< s^i_1, ..., s^i_n >$, for $i = 1, ..., n$. For any $i > n$ choose a vector s^i which is linearly independent with respect to any subset of $n-1$ vectors s^i, $i \in \{1, ..., i-1\}$. For $i = nh+1$, there exists a j, $1 \le j \le h$, and a sequence of n linearly independent vectors, $s^{i_1}, ..., s^{i_n}$, such that

$$(\sum_{h=1}^{n} s_k^{i_g} x_h)M = l_j M,$$

for any $g = 1, \dots, n$. Without loss of generality we may assume that $j = 1$ and $\{s^{i_1}, ..., s^{i_n}\}$=$\{s^1, \dots, s^n\}$. Since $s^1, ..., s^n$ are linearly independent, $u_j = \sum s_i^j x_i$ for $j = 1, ..., n$, are a minimal system of generators for the maximal ideal. We claim that $\mathbf{m}M = lM$. For, it is enough to show that $uM \subset lM$, for any $u \in \mathbf{m}$. If $u = f_1 u_1 + ... + f_n u_n \in \mathbf{m}$ then $uM \subset u_1 M + ... + u_n M = lM$. □

Corollary 4.6. *Let $(R, \mathbf{m}, k)$ be a local Noetherian ring of positive depth and with infinite residue field. Let M a finitely generated sparse R-module. Denote* *by μ the minimal number of generators. Then $\mu(\mathbf{m}^k M) \leq \mu(M)$, for any $k \geq 0$.*

Proof. By Proposition 4.5, for any integer h, we have that $\mathbf{m}^h M = l^h M$, from which the proposition follows. □

Suppose that R is a standard graded ring over k. Assume k is infinite and R has depth at least one. Let M be a finitely generated graded R-module. Then, we can formulate the following:

Definition 4.7. M is said to be homogeneously sparse if there are a finite number of submodules xM where $x \in \mathbf{m}$ is a homogeneous non-zero-divisor of R.

Question 4.8. In the case of graded modules over graded ring, being sparse implies being homogeneously sparse. Is the converse true?

Theorem 4.9. *Let R be a standard graded ring over an algebraically closed field k and assume that R_1 is generated by two elements. Let M be a standard graded module over R. Then, M is homogeneously sparse if and only if M is Artinian and there exists a linear form $l \in \mathbf{m}$ such that $lM = \mathbf{m}M$, where $\mathbf{m}$ is the homogeneous maximal ideal.*

Proof. The proof for the "only if" direction is the graded version of Proposition 4.4 and Proposition 4.5. For the other direction, suppose that M is Artinian and that there exists an $l \in \mathbf{m}$ such that $lM = \mathbf{m}M$. Since M is Artinian, it is enough to show that there are finitely many submodules of the form fM, where f is a homogeneous polynomial of degree less than d, for some $d > 0$. Since k is algebraically closed, any homogeneous polynomial in two variables can be factored into linear terms; hence it is enough to show that there are finitely many submodules of the form tM, where t is a linear form. Moreover, since $M = \oplus M_i$, it is enough to show that there are finitely many tM_i, where t is a linear form and M_i is the i-th degree component of the module M. It is enough to show that there are finitely many k vector spaces of the form tM_0, for t a linear form. Indeed, if we want to show that there are finitely many tM_i for $i > 0$, replace the module M by the module generated by M_i. Without loss of generality, let R_1 be generated by l and s. Since $lM = \mathbf{m}M$, $\mu(\mathbf{m}M) \leq \mu(M)$. Suppose $m_1, ..., m_h$ are minimal generators in degree zero, such that $lm_1, ..., lm_h$ generate $\mathbf{m}M$. Complete $m_1, ..., m_h$ to a minimal system of generators of M, $m_1, ..., m_n$ say. Let $\alpha l + \beta s$ be a general linear form. Let A to be the matrix where the columns are the images in M_1 of $(\alpha l + \beta s)m_i$ in terms of the lm_i, $i = 1, \ldots, h$:

$$\begin{pmatrix} \alpha+\beta a_{11} & \beta a_{12} & \ldots & \beta a_{1h} & \ldots & \beta a_{1n} \\ \beta a_{21} & \alpha+\beta a_{22} & \ldots & \beta a_{2h} & \ldots & \beta a_{1n} \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \beta a_{h1} & \ldots & \ldots & \alpha+\beta a_{hh} & \ldots & \beta a_{hn} \end{pmatrix}.$$

If there are only finitely many (α, β) such that the first $h-$columns have a zero determinant, then the module M is sparse. Since the determinant of the first h-columns is a homogeneous polynomial of degree h in two variables, it is enough to show that it is not identically zero. But the values $\alpha = 1$, $\beta = 0$ give a non zero determinant. □

5 Structure of $\operatorname{Ext}^1_R(M, N)$

Remark 5.1. By Theorem 3.5, $\operatorname{Ext}^1_R(M, N)$ is a sparse module when M and N are maximal Cohen-Macaulay modules over a ring of finite Cohen-Macaualy type.

We are ready to improve the bound given in [3] for the power of the maximal ideal that kills the module $\operatorname{Ext}^1_R(M, N)$ when R is a ring of finite Cohen-Macaulay type and M and N are maximal Cohen-Macaulay modules. More precisely:

Theorem 5.2. *Let $(R, \mathbf{m}, k)$ be a Cohen-Macaulay local ring of positive depth and with infinite residue field. Assume, as in Theorem 3.5, that there are only h isomorphism classes of maximal Cohen-Macaulay modules of multiplicity the sum of the multiplicities of M and N. Then $\mathbf{m}^{h-1} \operatorname{Ext}^1_R(M, N) = 0$.*

Proof. Let $X_1, \ldots, X_h \cong M \oplus N$ be a complete list of representative of isomorphism classes of modules that can fit in the middle of a short exact sequence in $\operatorname{Ext}^1_R(M, N)$ and let E_{X_j} the set defined in 1. Suppose by way of contradiction that $\mathbf{m}^{h-1} \operatorname{Ext}^1_R(M, N) \neq 0$. By Theorem 3.5, $\operatorname{Ext}^1_R(M, N)$ is a sparse module; hence there exists a non-zero-divisor $l \in \mathbf{m}$ such that $\mathbf{m} \operatorname{Ext}^1_R(M, N) = l \operatorname{Ext}^1_R(M, N)$, by Proposition 4.5. Let $S_1 = \{0, 1, \ldots, h-1\}$ and $S_2 = S_1 \backslash \{0\}$. Let ϕ a multivalued map from S_1 to S_2 such that $\phi(i) \subset S_2$, and $j \in \phi(i)$ if and only if there exists a minimal generator of $\mathbf{m}^i \operatorname{Ext}^1_R(M, N)$ in E_{X_j}. If we prove that $\phi(i) \cap \phi(j) = \emptyset$ for $i < j$, we will get a contradiction since the cardinality of S_2 is strictly smaller than the cardinality of S_1. Suppose that there exists a $t \in \phi(i) \cap \phi(j)$; then E_{X_t} contains a minimal generator α of $\mathbf{m}^i \operatorname{Ext}^1_R(M, N) = l^i \operatorname{Ext}^1_R(M, N)$ and a minimal generator β of

$\mathbf{m}^j \operatorname{Ext}^1_R(M,N) = l^j \operatorname{Ext}^1_R(M,N)$. By Theorem 1.1, $\beta \otimes R/l^jR$ is split exact and, by Theorem 1.2, $\alpha \otimes R/l^jR$ is split exact. By Proposition 2.8,

$$\alpha \in l^j \operatorname{Ext}^1_R(M,N) = l^{j-i} l^i \operatorname{Ext}^1_R(M,N) \subset \mathbf{m} l^i \operatorname{Ext}^1_R(M,N),$$

which contradicts α being a minimal generator of $l^i \operatorname{Ext}^1_R(M,N)$. □

Proposition 5.3. *Let $(R,\mathbf{m},k)$ be a Cohen-Macaulay local Noetherian ring of positive depth and with algebraically closed residue field. Let M be a finitely generated R-module which is maximal Cohen-Macauly and let M_1 be the first syzygy in a minimal free resolution of M. Assume that there are only h isomorphism classes of maximal Cohen-Macaulay modules of multiplicity the sum of the multiplicities of M and M_1 and $\mathbf{m}^{h-2} \operatorname{Ext}^1_R(M,M_1) \neq 0$. Then $\operatorname{Ext}^1_R(M,M_1)$ is the direct sum of cyclic modules.*

Proof. Let $X_1,\ldots,X_h \cong M \oplus N$ be a complete list of representative of isomorphism classes of modules that can fit in the middle of a short exact sequence in $\operatorname{Ext}^1_R(M,N)$ and let E_{X_j} the set defined in 1. Let , let $S_1 := \{0,1,\ldots,h-2\}$ and $S_2 := \{1,\ldots,h-2\}$. Moreover, let $\phi : S_1 \to S_2$ a multivalued map such that $\phi(i) \subset S_2$ and $j \in \phi(i)$ if and only if there exists a minimal generator of $\mathbf{m}^i \operatorname{Ext}^1_R(M,M_1)$ in E_{X_j}. Since the sets $\phi(i)$ are disjoint and $\mathbf{m}^{h-2} \operatorname{Ext}^1_R(M,M_1) \neq 0$, we have that the cardinality of $\phi(i)$ is one, for every $i \in \mathcal{S}_1$. In particular, all the short exact sequence, which are minimal generators for the R-modules $\mathbf{m}^i \operatorname{Ext}^1_R(M,M_1)$, have isomorphic modules in the middle. By Proposition 2.5, the initial part of the minimal free resolution of M, $\alpha : 0 \to M_1 \to F \to M \to 0$, is a minimal generator for the R-module $\operatorname{Ext}^1_R(M,M_1)$ and hence any other minimal generator of the R-module $\operatorname{Ext}^1_R(M,M_1)$ belongs to E_{X_1}, where we set X_1 to be the free module F.

By Proposition 4.5, there exists an element which is a non-zero-divisor on R, $l \in \mathbf{m}\backslash\mathbf{m}^2$, such that $l^{h-2} \operatorname{Ext}^1_R(M,M_1) = \mathbf{m}^{h-2} \operatorname{Ext}^1_R(M,M_1) \neq 0$; hence there exists a $\beta \in E_1$ such that $l^j\beta \neq 0$, for $j \in \mathcal{S}_2$. By Proposition 2.11, $l^j\gamma \neq 0$, for every $\gamma \in E_1$ and $j \in \mathcal{S}_2$.

Let $\alpha_1,\ldots,\alpha_m$ be a minimal set of generators for $\operatorname{Ext}^1_R(M,M_1)$. We first claim that for any $j \in \mathcal{S}_2$, $l^j\alpha_1,\ldots,l^j\alpha_m$ are a minimal system of generators for the modules $\mathbf{m}^j \operatorname{Ext}^1_R(M,M_1)$. For it, assume there exists a linear combination $\sum_{i=1}^m \lambda_i l^j \alpha_i = 0$, where not all the λ_i are zero, hence $l^j(\sum_{i=1}^m \lambda_i\alpha_i) = 0$. Since $\sum_{i=1}^m \lambda_i\alpha_i \in E_{X_1}$, applying Proposition 2.11 we get that $l^j\gamma = 0$ for every $\gamma \in E_{X_1}$, contradicting the last statement of the above paragraph.

The last step is to prove that we can complete l to a minimal system of generators for $\mathbf{m}$, say $l, l_1,\ldots,l_n$, such that $l_i \operatorname{Ext}^1_R(M,N) = 0$, for every $i = 1,\ldots,n$. For it, we claim that if $l, l_1,\ldots,l_n$ is a system of minimal generators for $\mathbf{m}$, then there exists a $\lambda_i \in k$ such that $(l_i - \lambda_i l)\gamma = 0$ for every $\gamma \in E_{X_1}$ and for every $i = 1, \ldots, n$. To prove the claim, notice that the multiplication by l_i

$$\frac{\operatorname{Ext}^1_R(M, M_1)}{\mathbf{m} \operatorname{Ext}^1_R(M, M_1)} \to \frac{\mathbf{m} \operatorname{Ext}^1_R(M, M_1)}{\mathbf{m}^2 \operatorname{Ext}^1_R(M, M_1)},$$

is a k-linear map between k^m. Since k is algebraically closed, there exists an eigenvalue λ_i and an eigenvector γ_i, in particular we can write $l_i\gamma_i = \lambda_i l \gamma_i$. This means that $(l_i - \lambda_i l)\gamma_i = 0$, since $\gamma_i \in E_{X_1}$, by Proposition 2.11, we have $(l_i - \lambda_i l)\gamma = 0$ for every $\gamma \in E_{X_1}$. By replacing l_i with $l_i - \lambda_i l$, we have the claim.

This shows that we can write $\operatorname{Ext}^1_R(M, M_1) = \oplus_{i=1}^m R\alpha_i$.

□

Remark 5.4. Call M_i the modules $R\alpha_i$. The above proof shows that the Hilbert function of M_i can have just 1 or 0 as possible values. Recall that a module is called uniserial if there exists a unique composition series. In the previous Proposition, if $R/(l_1, \ldots, l_m)$ is a DVR, then the modules M_i are uniserial.

Remark 5.5. The above proposition shows that if all the minimal generators of $\operatorname{Ext}^1_R(M, N)$ have a free module in the middle, then $\operatorname{Ext}^1_R(M, N)$ is direct sum of cyclic modules.

Acknowledgment

I would like to thank my advisor Craig Huneke for helpful conversations, which made this paper possible.

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