A UNIFORM ARTIN-REES PROPERTY FOR SYZYGIES IN RINGS OF DIMENSION ONE AND TWO.

JANET STRIULI

ABSTRACT. Let $(R, \mathbf{m}, \mathsf{k})$ be a local Noetherian ring, let M be a finitely generated R-module and let $I \subset R$ be an $\mathbf{m}$-primary ideal. Let $\mathbf{F} = \{F_i, \partial_i\}$ be a free resolution of M. In this paper we study the question whether there exists an integer h such that $I^n F_i \cap \ker(\partial_i) \subset I^{n-h} \ker(\partial_i)$ holds for all i. We give a positive answer for rings of dimension at most two. We relate this property to the existence of an integer s such that I^s annihilates the modules $\operatorname{Tor}_i^R(M, R/I^n)$ for all $i > 0$ and all integers n.

1. Introduction

In this paper $(R, \mathbf{m}, \mathsf{k})$ denotes a local Noetherian ring, and all modules are finitely generated. As general reference we refer to [1, 4].

Let I be an ideal of R, let M be an R-module and N a submodule of M. The Artin-Rees lemma states that there exists an integer h depending on I, M and N such that for all $n \geq h$ one has

$$I^n M \cap N = I^{n-h}(I^h M \cap N). \tag{1.0.1}$$

A weaker property, which is often the one used in applications, is

$$I^n M \cap N \subset I^{n-h} N. \tag{1.0.2}$$

Much work has been done to determine whether h can be chosen uniformly, in the sense that (1.0.2) would be satisfied simultaneously for every ideal belonging to a given family; see [3, 6, 8–11]. We study another kind of uniformity.

(1.1) **Theorem.** *Let $(R, \mathbf{m}, \mathsf{k})$ be a local Noetherian ring with $\dim R \leq 2$. Let M a finitely generated R-module and $I \subset R$ an $\mathbf{m}$-primary ideal. There exists an integer h such that for every free resolution $\mathbf{F} = \{F_i, \partial_i^{\mathbf{F}}\}$ of M there are inclusions*

$$I^n F_{i-1} \cap \ker(\partial_i^{\mathbf{F}}) \subseteq I^{n-h} \ker(\partial_i^{\mathbf{F}}) \quad \text{for all } i \geq 1 \text{ and all } n > h. \tag{1.1.1}$$

The main motivation for this work is a theorem due to Eisenbud and Huneke [5, Theorem 3.1]): Let M be an R-module and let $\mathbf{F} = \{F_i, \partial_i^{\mathbf{F}}\}$ be a free resolution of M. If for every non-maximal prime ideal $\mathbf{p}$ of R the $R_{\mathbf{p}}$-module $M_{\mathbf{p}}$ has finite projective dimension and its rank is independent of $\mathbf{p}$, then there exists an integer h such that (1.1.1) holds.

To prove Theorem 1.1 we study the annihilators of the modules $\operatorname{Tor}_i^R(M, R/I^n)$; see also [5, Proposition 4.1].

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(1.2) **Theorem.** *Let $(R, \mathbf{m}, \mathsf{k})$ be a local Noetherian ring, let r be an integer and let $\mathcal{F}$ be a family of ideals. Assume that one of the following conditions holds:*

1. *$\dim R = 1$, $r = 2$ and $\mathcal{F}$ is the family of all $\mathbf{m}$-primary ideals;*
2. *$\dim R = 2$, $r = 3$ and $\mathcal{F}$ is the family of all parameter ideals.*

Then there exists an integer h such that

$$I^h \operatorname{Tor}_j^R(M, R/I^n) = 0$$

for every R-module M, every integer n, every $j \geq r$ and every $I \in \mathcal{F}$.

In the next section we define syzygetically Artin-Rees modules and study the case where the ring is Cohen-Macaulay. In Section Three we study uniform annihilators for certain Tor-modules. In Section Four we prove Theorems 1.2 and 1.1 (see Theorems 4.4 and 4.5) for rings of dimension one, and in Section Five we prove them (see Theorems 5.4 and 6.1) for rings of dimension two.

2. Syzygetically Artin-Rees modules

Given an R-module M and $\mathbf{F} = \{F_i, \partial_i^{\mathbf{F}}\}$ a minimal free resolution of M, we define $\Omega_i^R(M) := \ker(\partial_{i-1}^{\mathbf{F}})$.

(2.1) **Lemma.** *Let M be an R-module and let I be an ideal of R. Let h be an integer. The following conditions are equivalent:*

1. *for every free resolution $\mathbf{G} = \{G_i, \partial_i^{\mathbf{G}}\}$ one has*

(2.1.1) $$I^n G_i \cap \ker(\partial_i^{\mathbf{G}}) \subset I^{n-h} \ker(\partial_i^{\mathbf{G}}) \quad \text{for all } i \geq 1 \text{ and all } n > h;$$

2. *for some free resolution $\mathbf{G} = \{G_i, \partial_i^{\mathbf{G}}\}$ inclusion* (2.1.1) holds.

Proof. For every free resolution $\mathbf{G} = \{G_i, \partial_i^{\mathbf{G}}\}$, we can write $G_i = F_i \oplus C_i \oplus D_i$, where $\partial_i^{\mathbf{G}}|_{F_i} \subseteq \mathbf{m}F_{i-1}$, $\partial_i^{\mathbf{G}}(D_i) = 0$ and $\partial_i^{\mathbf{G}}(C_i) = C_{i-1}$. In particular, the inclusion $I^n G_i \cap \ker(\partial_i^{\mathbf{G}}) \subset I^{n-h} \ker(\partial_i^{\mathbf{G}})$ holds for all $i > 0$ and $n > h$ for a free resolution $\mathbf{G}$ of M if and only if it holds for the minimal free resolution $\mathbf{F}$ of M. □

(2.2) **Definition.** Let $(R, \mathbf{m}, \mathsf{k})$ be a local Noetherian ring. Let M be a finitely generated R-module, let I be an ideal of R and let h be an integer. An R-module M is *syzygetically Artin-Rees* of level h with respect to I if one of the equivalent conditions of Lemma 2.1 holds.

Let $\mathcal{F}$ be a family of ideals. If there exists an integer h such that (2.1.1) holds for every ideal $I \in \mathcal{F}$ then we say that M is *syzygetically Artin-Rees* with respect to $\mathcal{F}$, or simply *syzygetically Artin-Rees* if $\mathcal{F}$ is the family of all ideals.

(2.3) **Uniform Artin-Rees.** Let $(R, \mathbf{m}, \mathsf{k})$ be a local Noetherian ring. Given an R-module M and a submodule N, there exists an integer $h = h(M, N)$ such that $I^n M \cap N \subset I^{n-h} N$, for every ideal I of R and every $n > h$. See [6, Theorem 4.12].

(2.4) **Lemma.** *Let M be an R-module and let $\mathcal{F}$ be a family of ideals. Then the following hold*

1. *M is syzygetically Artin-Rees with respect to $\mathcal{F}$ if and only if $\Omega_i^R(M)$ is syzygetically Artin-Rees with respect to $\mathcal{F}$ for some integer $i > 0$.*
2. *Let $R \to S$ be a faithfully flat extension. If $M \otimes_R S$ is syzygetically Artin-Rees with respect to the family of ideals IS where $I \in \mathcal{F}$, then M is syzygetically Artin-Rees with respect to $\mathcal{F}$.*

Proof. For the first statement, assume that there exists integer $i > 0$ such that $\Omega_i^R(M)$ is syzygetically Artin-Rees with respect to $\mathcal{F}$ at level h. Let $\mathbf{F}$ be a minimal free resolution of M. Let h_1 the integer given in 2.3 for the R-modules $\bigoplus_{j=1}^{j=i} \Omega_j^R(M) \subset \bigoplus_{j=1}^{j=i-1} F_j$. If $s = \max\{h_0, h_1\}$, then M is syzygetically Artin-Rees with respect to $\mathcal{F}$ at level s.

For the second statement, notice that tensoring with a faithfully flat extension commutes with inclusions and intersections. □

The proof of the next theorem is due to D. Katz.

(2.5) **Theorem.** *Let $(R, \mathbf{m}, \mathsf{k})$ be a Cohen-Macaulay local ring and let M be an R-module. If I is an $\mathbf{m}$-primary ideal, then M is syzygetically Artin-Rees with respect to I.*

For the proof we need two lemmas.

(2.6) **Lemma.** *Let F be an R-module, K be a submodule of F and set $M = F/K$. Let $J = (a_1, \dots, a_l)$ be an ideal generated by an M-regular sequence. Then $J^n F \cap K = J^n K$ for all $n > 0$.*

Proof. Let $\xi \in J^n F \cap K$. Then there exists a homogeneous polynomial Φ in $F[x_1, \dots, x_l]$ of degree n such that $\Phi(a_1, \dots, a_l) = \xi$. By going modulo K, we have a homogeneous polynomial $\Phi_0 = \overline{\Phi}$ of degree n in $M[x_1, \dots, x_n]$ such that $\Phi_0(a_1, \dots, a_l) = 0$. We want to prove that Φ_0 is the zero polynomial, which implies that the coefficients of Φ are in K. Since $\Phi_0(a_1, \dots, a_l) = 0 \in J^{n+1}M$, the coefficients of Φ_0 are in JM, by [1, Theorem 1.1.7]. Therefore, there exists a homogeneous polynomial $\Phi_1 \in M[x_1, \dots, x_n]$ of degree $n + 1$ such that $\Phi_1(a_1, \dots, a_l) = \Phi_0(a_1, \dots, a_l) = 0$. By repeating this argument we can see that the coefficients of Φ_0 are in $J^n M$ for every n and therefore they are zero by the Krull Intersection Theorem. □

(2.7) **Lemma.** *Let $(R, \mathbf{m}, \mathsf{k})$ be a Cohen-Macaulay local ring with infinite residue field. Let I be an $\mathbf{m}$-primary ideal of R and let $J \subset I$ a minimal reduction with reduction number h. If M is a maximal Cohen-Macaulay R-module and*

$$0 \to K \to F \to M \to 0,$$

is an exact sequence of R-modules with F finitely generated, then

$$I^n F \cap K \subseteq I^{n-h} K, \quad \text{for every } n > h. \tag{2.7.1}$$

Proof. Let $J = (x_1, \dots, x_d)$. Since M is a maximal Cohen-Macaulay R-module, $x_1, \dots x_d$ is a regular sequence on M. For every $i > 0$ and for every $n > h$ we have

$$\begin{aligned} I^n F \cap K &= J^{n-h} I^h F \cap K \\ &\subseteq J^{n-h} F \cap K \\ &= J^{n-h} K, && \text{by Lemma 2.6} \\ &\subseteq I^{n-h} K. \end{aligned}$$ □

Now we are able to give the proof of Theorem 2.5.

Proof. By Lemma 2.4(2), we may assume that the residue field is infinite. Let $\mathbf{F}$ be a minimal free resolution of M. By Lemma 2.4(1) it is enough to show that $\Omega_d^R(M)$ is syzygetically Artin-Rees with respect to I. We can now use the inclusion (2.7.1) replacing K by $\Omega_i^R(M)$ and F by F_{i-1}, for every $i \geq d+1$ (see [1, Exercise 2.1.26]). □

3. Uniform annihilators of Tor modules

In this section we explore the relation between modules that are syzygetically Artin-Rees and the annihilators of a certain family of Tor modules.

(3.1) **Lemma.** *If M is a finitely generated R-module and if $\mathbf{F}$ is a minimal free resolution of M, then one has*

$$\operatorname{Tor}_j^R(M, R/I^n) \cong \frac{\Omega_j^R(M) \cap I^n F_{j-1}}{I^n \Omega_j^R(M)} \quad \textit{for every } j > 0.$$

Proof. Since $\operatorname{Tor}_1(\Omega_{j-1}^R(M), R/I^n) \cong \operatorname{Tor}_j^R(M, R/I^n)$ it is enough to consider the case $j = 1$. Tensor the exact sequence

$$0 \to \Omega_1^R(M) \to F_0 \to M \to 0$$

by R/I^n to obtain the exact sequence

$$0 \longrightarrow \operatorname{Tor}_1^R(M, R/I^n) \longrightarrow \Omega_1^R(M)/I^n\Omega_1^R(M) \longrightarrow F_0/I^nF_0.$$

The modules

$$\operatorname{Tor}_1^R(M, R/I^n) \quad \text{and} \quad \frac{\Omega_1^R(M) \cap I^n F_0}{I^n \Omega_1^R(M)}$$

are isomorphic as both are the kernel of the right-hand map. □

An immediate application of the previous lemma gives a stronger Artin-Rees property for the syzygies of the residue field.

(3.2) **Theorem.** *Let $(R, \mathbf{m}, \mathsf{k})$ be a local Noetherian ring. If $\mathbf{F} = \{F_i\}$ is the minimal free resolution of R-module k, then there exists an integer h such that*

$$\mathbf{m}^n F_{i-1} \cap \Omega_i^R(\mathsf{k}) = \mathbf{m}^{n-h}(\mathbf{m}^h F_{i-1} \cap \Omega_i^R(\mathsf{k})) \quad \textit{for all } n > h \textit{ and all } i > 0.$$

Proof. By [7, Corollary 3.16] there exists an integer h such that for $n \geq h$ and for all $j \geq 1$:

$$\operatorname{Tor}_j^R(\mathsf{k}, R/\mathbf{m}^n) \cong \frac{\mathbf{m}^{n-1}\Omega_j^R(\mathsf{k})}{\mathbf{m}^n \Omega_j^R(\mathsf{k})}.$$

Hence, for every $n \geq h$, we have:

$$\frac{\Omega_j^R(\mathsf{k}) \cap \mathbf{m}^n F_{j-1}}{\mathbf{m}^n \Omega_j^R(\mathsf{k})} \cong \operatorname{Tor}_j^R(\mathsf{k}, R/\mathbf{m}^n) \cong \frac{\mathbf{m}^{n-1}\Omega_j^R(\mathsf{k})}{\mathbf{m}^n \Omega_j^R(\mathsf{k})},$$

where the first isomorphism holds by Lemma 3.1. In particular the two modules

$$\frac{\mathbf{m}^{n-1}\Omega_j^R(\mathsf{k})}{\mathbf{m}^n \Omega_j^R(\mathsf{k})} \subseteq \frac{\Omega_j^R(\mathsf{k}) \cap \mathbf{m}^n F_{j-1}}{\mathbf{m}^n \Omega_j^R(\mathsf{k})}$$

have the same length and therefore they are equal. We have the following chain

$$\begin{aligned}\Omega_j^R(\mathsf{k}) \cap \mathbf{m}^n F_{j-1} &= \mathbf{m}^{n-1}\Omega_j^R(\mathsf{k}) \\ &= \mathbf{m}(\mathbf{m}^{n-2}\Omega_j^R(\mathsf{k})) \\ &\subseteq \mathbf{m}(\Omega_j^R(\mathsf{k}) \cap \mathbf{m}^{n-1}F_{j-1}) \\ &\subseteq \Omega_j^R(\mathsf{k}) \cap \mathbf{m}^n F_{j-1}.\end{aligned}$$ □

(3.3) **Definition.** Let $\mathcal{M}$ be a family of finitely generated R-modules, let $\mathcal{F}$ be a family of ideals of R and let h be an integer. We say that $\operatorname{Tor}_j(\mathcal{M}, \quad)$ is *uniformly $\mathcal{F}$-annihilated at level h* if

(3.3.1) $$I^h \operatorname{Tor}_j(M, R/I^n) = 0 \quad \text{for all } M \in \mathcal{M}, \text{ all } I \in \mathcal{F} \text{ and all } n \in \mathbb{Z}.$$

If (3.3.1) holds for every $j \geq 1$, then we say that $\operatorname{Tor}(\mathcal{M}, \quad)$ is *uniformly $\mathcal{F}$-annihilated at level h.*

Note that the phrase 'at level h' is dropped if h is not explicitly specified. When $\mathcal{M}$ consists of a single module M and $\mathcal{F}$ consists of a single ideal I, we say that $\operatorname{Tor}(M, \quad)$ is uniformly I-annihilated.

(3.4) **Lemma.** *Let M be an R-module, let I be an ideal of R and let j, h be an integers. The following hold.*

(1) *If $I^n F_{j-1} \cap \Omega_j^R(M) = I(I^{n-1}F_{j-1} \cap \Omega_j^R(M))$ for every $n > h$, then*
$$\operatorname{ann}_R(\operatorname{Tor}_j^R(M, R/I^n)) \subseteq \operatorname{ann}_R(\operatorname{Tor}_j^R(M, R/I^{n+1})) \quad \text{for every } n > h.$$
(2) *If $I^n F_{i-1} \cap \Omega_i^R(M) \subset I^{n-h}\Omega_i^R(M)$ for every $n \geq h$, then $\operatorname{Tor}_j(M, \quad)$ is uniformly I-annihilated at level h.*

Proof. For the first statement, let $x \in \operatorname{ann}_R(\operatorname{Tor}_j(M, R/I^n))$. Lemma 3.1 yields $x(I^n F_{j-1} \cap \Omega_j^R(M)) \subseteq I^n\Omega_j(M)$. For every $n > h$, one has

$$\begin{aligned} x(I^{n+1}F_{j-1} \cap \Omega_j^R(M)) &= xI(I^n F_{j-1} \cap \Omega_j^R(M)), \qquad \text{since } n > h, \\ &= Ix(I^n F_{j-1} \cap \Omega_j^R(M)) \\ &\subseteq II^n\Omega_j^R(M) = I^{n+1}\Omega_j^R(M). \end{aligned}$$

By Lemma 3.1 one has $x \in \operatorname{ann}_R(\operatorname{Tor}_j^R(M, R/I^{n+1}))$.

For the second statement, notice that $I^h \operatorname{Tor}_j^R(M, R/I^n) = 0$ for every $n \leq h$. Using Lemma 3.1 we have

$$\operatorname{Tor}_j^R(M, R/I^n) = (I^n F_{j-1} \cap \Omega_j^R(M))/I^n\Omega_j(M) \subseteq I^{n-h}\Omega_j(M)/I^n\Omega_j(M),$$

for every $n > h$, proving that $I^h \subset \operatorname{ann}_R(\operatorname{Tor}_j^R(M, R/I^n))$. □

An immediate consequence of Lemma 3.4(2) is the following

(3.5) **Proposition.** *If M is syzygetically Artin-Rees at level h with respect to I, then $\operatorname{Tor}(M, \quad)$ is uniformly I-annihilated at level h.*

From 2.5 one deduces

(3.6) **Corollary.** *Let $(R, \mathbf{m}, \mathsf{k})$ be a Cohen-Macaulay local ring and let $\mathcal{M}$ be the family of maximal Cohen-Macaulay R-modules. If I is an $\mathbf{m}$-primary ideal, then* $\mathrm{Tor}(\mathcal{M}, \quad)$ *is uniformly I-annihilated.*

It is natural to ask the following.

(3.7) **Question.** Let $(R, \mathbf{m}, \mathsf{k})$ be a local Noetherian ring. Let $I \subseteq R$ be an $\mathbf{m}$-primary ideal of R and let M be a finitely generated R-module. If $\mathrm{Tor}(M, \quad)$ is uniformly I-annihilated is M syzygetically Artin-Rees with respect to I?

In the next section we shall use the following.

(3.8) **Lemma.** *Let M be an R-module and let $\mathcal{F}$ be a family of ideals of R. If there exist integers h and q such that* $\mathrm{Tor}_i(M, \quad)$ *is uniformly $\mathcal{F}$-annihilated at level h for every $i \geq q$, then* $\mathrm{Tor}(M, \quad)$ *is uniformly $\mathcal{F}$-annihilated.*

Proof. Let s be an integer as in 2.3 for the R-modules $\bigoplus_{i=1}^{q+1} \Omega_i(M) \subset \bigoplus_{i=0}^{q} F_i$. Lemma 3.4(2) implies that $\mathrm{Tor}_i(M, \quad)$ are uniformly $\mathcal{F}$-annihilated at level s. If $l := \max\{s, h\}$, then $\mathrm{Tor}(M, \quad)$ is uniformly $\mathcal{F}$-annihilated at level l. □

4. Rings of dimension one

In this section we prove that every R-module over a one dimensional ring is syzygetically Artin-Rees.

(4.1) **Superficial elements.** Let $(R, \mathbf{m}, \mathsf{k})$ be a local Noetherian ring and let $I \subset R$ an ideal. An element $x \in I$ is said to be *superficial* in I if there exists an integer c such that

$$(I^n : x) \cap I^c = I^{n-1} \quad \text{for every } n > c.$$

Superficial elements always exist if k is infinite. See for example [12, Proposition 3.2, Chapter 1].

(4.2) **Lemma.** *Let $(R, \mathbf{m}, \mathsf{k})$ be a Noetherian ring.*

(1) *If I is an ideal and x is superficial in I for R, then there exists an integer c such that*

(4.2.1) $$(0 :_F x) \cap I^c F = 0 \quad \textit{for every free module } F.$$

Moreover, if (4.2.1) holds for x, then it does for every power of x.

(2) If $\dim R = 1$, then there exists an integer c such that equality (4.2.1) holds for all $\mathbf{m}$-primary ideals I and for all elements x superficial in I which are not in $\cup_{\mathbf{p} \in \mathrm{ass}_R(R) - \{\mathbf{m}\}} \mathbf{p}$.

Proof. For the first statement, if x is a superficial element, then there exists an integer c such that $(I^n : x) \cap I^c = I^{n-1}$ for all $n > c$. Therefore,

$$\begin{aligned} (0 :_F x) \cap I^c F &= (\cap_{n \geq c} I^n F :_F x) \cap I^c F = \cap_{n \geq c} (I^n F :_F x) \cap I^c F \\ &= \cap_{n \geq c} I^{n-1} F = 0. \end{aligned}$$

For the second statement, if $x \notin \cup_{\mathbf{p} \in \mathrm{ass}_R(R) - \{\mathbf{m}\}} \mathbf{p}$, then $(0 : x) \subset \mathrm{H}^0_{\mathbf{m}}(R)$. Let s be an integer as in 2.3 for the R-modules $\mathrm{H}^0_{\mathbf{m}}(R) \subset R$ and set the integer $t = \mathrm{length}(\mathrm{H}^0_{\mathbf{m}}(R))$, then

$$(0 :_F x) \cap I^{t+s} F \subset \mathrm{H}^0_{\mathbf{m}}(F) \cap I^{t+s} F \subset I^t \, \mathrm{H}^0_{\mathbf{m}}(F) \subset \mathbf{m}^t \, \mathrm{H}^0_{\mathbf{m}}(F) = 0.$$

Set $c = t + s$. □

(4.3) **Strong uniform Artin-Rees in one dimensional rings.** Let $(R, \mathbf{m}, \mathsf{k})$ be a one-dimensional Noetherian ring with infinite residue field.

(1) There exists an integer $r > 0$, depending only on the ring, such that for every $\mathbf{m}$-primary ideal I there exists a reduction $(x) \subset I$ such that

$$I^n = xI^{n-1} \quad \text{for all } n > r.$$

See [12, Lemma 2.6] and [14, Proposition 2.6]. Such x can be choosen in a non-empty Zariski-open subset of $I/\mathbf{m}I$. Since there exists a non-empty Zariski-open U subset of $I/\mathbf{m}I$ such that each element $r \in U$ is superficial for I, we may assume that x is superficial for I with respect to R, (see [15]).

(2) Let $N \subset M$ be two finitely generated R-modules and let $J \subset R$ be an ideal such that $JM \subseteq N$. Let h_0 be an integer such that $\mathbf{m}^{h_0}\,\mathrm{H}^0_{\mathbf{m}}(M/N) = 0$. If $\dim R/J = 0$ then $\mathrm{H}^0_{\mathbf{m}}(M/N) = M/N$, so for all ideals I and for all $n > h_0+1$ one has $I^nM \cap N = I(I^{n-1}M \cap N)$. If $\dim R/J = 1$, by [14, Proposition 2.10] there exists an integer h_1, depending on R/J, such that $I^n \cap J = I^{n-h_1}(I^{h_1} \cap J)$ for all ideals I. For I now an arbitrary $\mathbf{m}$-primary ideal, apply (1) for R/J to get an integer h_2 and set $h = \max\{h_1, h_0 + \max\{h_0, h_2\}\}$. Then

$$I^nM \cap N = I^{n-h}(I^hM \cap N),$$

for every ideals I and every $n > h$; for details see [14, Proposition 2.11].

(4.4) **Theorem.** *Let $(R, \mathbf{m}, \mathsf{k})$ be a one-dimensional local Noetherian ring. Let M be an R-module and let $\mathcal{N}$ be the family of all submodules of free R-modules. If $\mathcal{F}$ is the family of all $\mathbf{m}$-primary ideals, then*

(1) $\mathrm{Tor}(\mathcal{N}, \quad)$ *is uniformly $\mathcal{F}$-annihilated.*

(2) $\mathrm{Tor}(M, \quad)$ *is uniformly $\mathcal{F}$-annihilated.*

Proof. Without loss of generality we may assume that the residue field is infinite. Since any higher Tor module can be realized as Tor_1, for (1) it is enough to prove that $\mathrm{Tor}_1(\mathcal{N}, \quad)$ is uniformly $\mathcal{F}$-annihilated.

Let $N \in \mathcal{N}$, let h_1 be a positive integer such that $\mathbf{m}^{h_1}\,\mathrm{H}^0_{\mathbf{m}}(R) = 0$. Since N is a first syzygy, we have $\mathrm{H}^0_{\mathbf{m}}(N) \subseteq \mathrm{H}^0_{\mathbf{m}}(F)$ where F is some free module. Therefore

$$I^{h_1}\,\mathrm{H}^0_{\mathbf{m}}(N) \subseteq \mathbf{m}^{h_1}\,\mathrm{H}^0_{\mathbf{m}}(N) = 0$$

and $I^{h_1}\,\mathrm{Tor}^R_1(R/I^n, \mathrm{H}^0_{\mathbf{m}}(N)) = 0$, for every $n > 0$. Consider the exact sequence

$$0 \to \mathrm{H}^0_{\mathbf{m}}(N) \to N \to N/\,\mathrm{H}^0_{\mathbf{m}}(N) \to 0.$$

After tensoring with R/I^n we obtain the exact sequence

$$\mathrm{Tor}^R_1(R/I^n, \mathrm{H}^0_{\mathbf{m}}(N)) \to \mathrm{Tor}^R_1(R/I^n, N) \to \mathrm{Tor}^R_1(R/I^n, N/\,\mathrm{H}^0_{\mathbf{m}}(N)).$$

The R-module $N/\,\mathrm{H}^0_{\mathbf{m}}(N)$ is a maximal Cohen-Macaulay. Let $F \to N/\,\mathrm{H}^0_{\mathbf{m}}(N)$ be a surjective homomorphism where F is a finitely generated free R-module and let K be the kernel. Let h_2 be an integer as in 4.3(1). By Lemma 2.7 one has $I^{h_2}(I^nF \cap K) \subset I^nK$, for every $I \in \mathcal{F}$. Lemma 3.1 yields the equality $I^{h_2}\,\mathrm{Tor}^R_1(N/\,\mathrm{H}^0_{\mathbf{m}}(N), R/I^n) = 0$, for every $I \in \mathcal{F}$ and every integer n. If $h = h_1 + h_2$, then $\mathrm{Tor}^R_1(N, \quad)$ is uniformly $\mathcal{F}$-annihilated at level h.

For the second part apply Lemma 3.8. □

The previous theorem contains Theorem 1.2(1), while the following is a stronger version of Theorem 1.1 for one-dimensional rings.

(4.5) **Theorem.** *Let $(R, \mathbf{m}, \mathsf{k})$ be a Noetherian local ring of dimension one. Then each finitely generated R-module M is syzygetically Artin-Rees.*

Proof. Without loss of generality we may assume that the residue field is infinite. By Proposition [6, 2.3] it is enough to show that M is syzygetically Artin-Rees with respect to the family of all $\mathbf{m}$-primary ideals. Let $\mathbf{F}$ be a free resolution of M. For every $\mathbf{m}$-primary ideal I, choose a reduction $(x) \subset I$ and h_1 as in 4.3(1). Let c an integer as in 4.2(2) and let h_2 be an integer as in Theorem 4.4(2). Let $h = \max\{h_1, h_2\}$. Then for every $j > 0$ and every $n > 2h + c$ we have

$$x^h(I^n F_{j-1} \cap \Omega_j^R(M)) \subseteq I^n \Omega_j^R(M) = x^{n-h} I^h \Omega_j^R(M).$$

Let $u \in I^n F_{j-1} \cap \Omega_j^R(M)$. Then $x^h u = x^{n-h} v$, for some $v \in I^h \Omega_j^R(M)$. Since $n - 2h > c$ we have $u - x^{n-2h} v \in I^c F_{j-1} \cap (0 :_{F_{j-1}} x^h)$. Since x is superficial, we get $(0 :_{F_{j-1}} x^h) \cap I^c F_{j-1} = 0$ from 4.2(a), and therefore

$$u = x^{n-2h} v \in I^{n-h} \Omega_j^R(M) \subseteq I^{n-(2h+c)} \Omega_j^R(M). \qquad \square$$

5. Uniform annihilators in dimension two

In this section we prove Theorem 1.2(2). We need two lemmas. The first one can be found in [16, 2.2.6]. We include the proof for completeness.

(5.1) **Lemma.** *Let $(R, \mathbf{m}, \mathsf{k})$ be a two-dimensional local ring and let M be a finitely generated R-module. Let a, b be a system of parameters with b a non-zero-divisor on M and let $A = \mathbb{Z}[a, b]$. Let $x \in R$ be an element satisfying $x(b^l M :_M a^s) \subseteq b^l M$, for every $l, s \geq 0$. Then for any integer n one has*

$$x \operatorname{Tor}_1^R(R/(a,b)^n, M) = x \operatorname{Tor}_1^A(A/(a,b)^n, M) = 0, \tag{5.1.1}$$

Proof. The equality $x \operatorname{Tor}_1^R(R/(a,b)^n, M) = 0$ follows from the second equality in (5.1.1), as the ring homomorphism $A \to R$ induces a surjective homomorphism $\operatorname{Tor}_1^A(A/(a,b)^n, M) \to \operatorname{Tor}_1^R(R/(a,b)^n, M)$.

Since the elements a, b form a regular sequence on A, we can compute the A-module $\operatorname{Tor}_1^A(A/(a,b)^i, M)$ as the homology of the complex

$$M^i \xrightarrow{\phi_1} M^{i+1} \xrightarrow{\phi_2} M,$$

where ϕ_1 is given by

$$\begin{pmatrix} b & 0 & 0 & \cdots & 0 & 0 & 0 \\ -a & b & 0 & \cdots & 0 & 0 & 0 \\ 0 & -a & b & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & b & 0 & 0 \\ 0 & 0 & 0 & \dots & -a & b & 0 \\ 0 & 0 & 0 & \dots & 0 & -a & b \end{pmatrix}$$

and ϕ_2 is given by $\begin{pmatrix} a^i & a^{i-1}b & \cdots & b^i \end{pmatrix}$, see for example [4, Exercise 17.11]. If $\overline{m} = (m_1, \ldots, m_{i+1})$ is in $\ker(\phi_2)$, then in the localized module M_b one has $\overline{m} = \phi_1(\overline{n})$, where $\overline{n} = (n_1, \ldots, n_i)$ and

$$n_j = \frac{a^{j-1}m_1 + \cdots + b^{j-1}m_j}{b^j},$$

for every j, $1 \le j \le i$. The containment

$$a^{j-1}m_1 + \cdots + b^{j-1}m_j \in (b^j M : a^{i-j+1})$$

resulting from $\overline{m}$ being in $\ker(\phi_2)$ yields $xn_j \in M$, and $xm \in \operatorname{Image} \phi_1$. □

(5.2) **Generalized Cohen-Macaulay modules.** Let M be a finitely generated R-module and set $d = \dim_R(M)$. The R-module M is called *generalized Cohen-Macaulay* if the R-modules $\mathrm{H}^i_{\mathbf{m}}(M)$ are finitely generated for $i \le d-1$. Recall that the R-module $\mathrm{H}^i_{\mathbf{m}}(M)$ vanishes for $i < \operatorname{depth} M$.

Since the modules $\mathrm{H}^i_{\mathbf{m}}(M)$ are artinian, if M is generalized Cohen-Macaulay then there exists an ideal J such that $J\,\mathrm{H}^i_{\mathbf{m}}(M) = 0$ for all $i \le d-1$. Then

$$J((a_1, \ldots, a_i)M : a_{i+1}) \subseteq (a_1, \ldots, a_i)M, \tag{5.2.1}$$

for every part of system of parameters $a_1, \ldots, a_{i+1}$; for a proof see [13, Satz 2.4.2, page 44].

(5.3) **Lemma.** *Let $(R, \mathbf{m}, \mathsf{k})$ be a two-dimensional local ring and let $I = (a, b)$ be a parameter ideal. Let M a two-dimensional generalized Cohen-Macaulay module R-module and let h_0, and h_1 two integers such that*

$$\mathbf{m}^{h_0}\,\mathrm{H}^0_{\mathbf{m}}(M) = 0 \quad \textit{and} \quad \mathbf{m}^{h_1}\,\mathrm{H}^1_{\mathbf{m}}(M) = 0.$$

One then has $\mathbf{m}^{2h_0+h_1}\operatorname{Tor}^R_1(M, R/I^n) = 0$ for every n.

Proof. From the following exact sequence

$$0 \to \mathrm{H}^0_{\mathbf{m}}(M) \to M \to M/H^0_{\mathbf{m}}(M) \to 0$$

we obtain

$$\operatorname{Tor}^R_1(\mathrm{H}^0_{\mathbf{m}}(M), R/I^n) \to \operatorname{Tor}^R_1(M, R/I^n) \to \operatorname{Tor}^R_1(M/\mathrm{H}^0_{\mathbf{m}}(M), R/I^n). \tag{5.3.1}$$

The assumption on h_0 gives

$$\mathbf{m}^{h_0}\operatorname{Tor}^R_1(\mathrm{H}^0_{\mathbf{m}}(M), R/I^n) = 0 \quad \text{for every } n > 0.$$

Notice that the module $M/\mathrm{H}^0_{\mathbf{m}}(M)$ has positive depth and

$$\mathrm{H}^1_{\mathbf{m}}(M) = \mathrm{H}^1_{\mathbf{m}}(M/H^0_{\mathbf{m}}(M)).$$

By prime avoidance we may assume that b is a non-zero-divisor on $M/\mathrm{H}^0_{\mathbf{m}}(M)$. Let a be an element such that $I = (a, b)$. By 5.2 one has $\mathbf{m}^{h_0+h_1}(a^iM : b^j) \subseteq a^iM$ while Lemma 5.1 gives $\mathbf{m}^{h_0+h_1}\operatorname{Tor}^R_1(R/I^n, M/\mathrm{H}^0_{\mathbf{m}}(M)) = 0$ for every $n > 0$. The exact sequence (5.3.1) concludes the proof. □

This lemma is used in the proof of the next theorem, (see Theorem 1.2(2)). We shall find bounds for h_0 and h_1 that do not depend on the module M. In that way we obtain a power of the maximal ideal that annihilates the Tor_1 module and does not depend on M.

(5.4) **Theorem.** *Let $(R, \mathbf{m}, \mathsf{k})$ be a two-dimensional complete local ring. If $\mathcal{M}$ is the family of R-modules which are second syzygies and $\mathcal{F}$ is the family of parameter ideals, then* $\mathrm{Tor}(\mathcal{M},\quad)$ *is uniformly $\mathcal{F}$-annihilated.*

Proof. We may assume that k is infinite. Since any higher Tor can be realize as Tor_1 it is enough to show that $\mathrm{Tor}_1(\mathcal{M},\quad)$ is uniformly $\mathcal{F}$-annihilated. Let $0 = \mathbf{q}_1 \cap \cdots \cap \mathbf{q}_s \cap \mathbf{Q}_1 \cap \cdots \cap \mathbf{Q}_n$ be a primary decomposition of the zero ideal, such that $\dim(R/\mathbf{q}_i) = 2$ for $i \le s$, and $\dim(R/\mathbf{Q}_i) \le 1$ for $i \le n$. Let $\mathbf{q} = \mathbf{q}_1 \cap \cdots \cap \mathbf{q}_s$.

For any second syzygy M we may choose an exact sequence

$$0 \to M \to F_1 \to F_0$$

We define submodules of M as follows:

$$M_0 = \mathbf{q}F_1 \cap M \quad \text{and} \quad M_i = M_0 \cap (\mathbf{Q}_1 \cdots \cap \mathbf{Q}_i)F_1.$$

By induction on i we will prove that there exists an integer l_i such that

$$I^{l_i}\, \mathrm{Tor}_1^R(M/M_i, R/I^n) = 0$$

for every parameter ideal I and every $n > 0$. The case $i = n$ will prove the proposition since $M = M/M_n$.

Let us prove the claim for $i = 0$. The R-module $R/\mathbf{q}$ is generalized Cohen-Macaulay. For it, notice that the associated primes of $R/\mathbf{q}$ are the minimal primes of R. In particular $R/\mathbf{q}$ is an equidimensional local ring and it is Cohen-Macaulay after localizing at a prime different from the maximal ideal. In particular there exist integers k_0, k_1

$$\mathbf{m}^{k_0}\, \mathrm{H}^0_{\mathbf{m}}(R/\mathbf{q}) = 0,$$
$$\mathbf{m}^{k_1}\, \mathrm{H}^1_{\mathbf{m}}(R/\mathbf{q}) = 0,$$

and k_2 such that $\mathbf{m}^{k_2}\, \mathrm{H}^0_{\mathbf{m}}(R) = 0$. Set $h_0 = k_0$ and $h_1 = k_1 + k_2$. From

$$0 \to F_1/M \to F_0. \tag{5.4.1}$$

we obtain $\mathrm{H}^0_{\mathbf{m}}(F_1/M) \subseteq \mathrm{H}^0_{\mathbf{m}}(F_0)$ and $\mathbf{m}^{k_2}\, \mathrm{H}^0_{\mathbf{m}}(F_1/M) = 0$. From

$$0 \to M/M_0 \to F_1/\mathbf{q}F_1 \to F_1/M \to 0 \tag{5.4.2}$$

we obtain

$$\mathrm{H}^0_{\mathbf{m}}(M/M_0) \subseteq \mathrm{H}^0_{\mathbf{m}}(F_1/\mathbf{q}F_1)$$

and

$$\mathrm{H}^0_{\mathbf{m}}(F_1/M) \to \mathrm{H}^1_{\mathbf{m}}(M/M_0) \to \mathrm{H}^1_{\mathbf{m}}(F_1/\mathbf{q}F_1).$$

In particular M/M_0 is a generalized Cohen-Macaulay module and

$$\mathbf{m}^{h_0}\, \mathrm{H}^0_{\mathbf{m}}(M/M_0) = 0 \quad \text{and} \quad \mathbf{m}^{h_1}\, \mathrm{H}^1_{\mathbf{m}}(M/M_0) = 0.$$

Now apply Lemma 5.3 to finish the case $i = 0$.

Assume that the claim holds for some $i \ge 0$. Consider the exact sequences

$$0 \to M_i/M_{i+1} \to M/M_{i+1} \to M/M_i \to 0. \tag{5.4.3}$$

From the corresponding long exact sequence of Tor we need to find an h such that $\mathbf{m}^h\, \mathrm{Tor}_1(M_i/M_{i+1}, R/I^n) = 0$ for all $n > 0$, for all parameter ideals and for all second syzygies M.

Consider the following short exact sequence

$$0 \to K \to G \to M_i/M_{i+1} \to 0$$

where G is a free R-module. By Lemma 3.4(2) it is enough to find an integer h_i not depending on the parameter ideal I and on the module M such that $I^nG \cap K \subseteq I^{n-h_i}K$ for all $n > h_i$. If

$$J = \operatorname{ann}_R(\mathbf{q} \cap \mathbf{Q}_1 \cap \cdots \cap \mathbf{Q}_i / \mathbf{q} \cap \mathbf{Q}_1 \cap \ldots \mathbf{Q}_{i+1})$$

then, $\dim(R/J) \leq 1$. The injective homomorphism

$$(5.4.4) \qquad M_i/M_{i+1} \to (\mathbf{q} \cap \mathbf{Q}_1 \cap \cdots \cap \mathbf{Q}_i)F/(\mathbf{q} \cap \mathbf{Q}_1 \cap \ldots \mathbf{Q}_{i+1})F$$

yields $\dim_R(M_i/M_{i+1}) \leq \dim(R/J) \leq 1$. Since $JG \subseteq K$, by 4.3(2), there exist an integer-valued function h increasing in the power of the maximal ideal annihilating the local cohomology $\mathrm{H}^0_{\mathbf{m}}(M_i/M_{i+1})$, such that

$$I^nG \cap K \subseteq I^{n-h}K, \quad \text{for } n > h.$$

By the inclusion (5.4.4) we can bound above such power by the a power of the maximal ideal annihilating $\mathrm{H}^0_{\mathbf{m}}(\mathbf{q} \cap \mathbf{Q}_1 \cdots \cap \mathbf{Q}_i)F/(\mathbf{q} \cap \mathbf{Q}_1 \cap \cdots \cap \mathbf{Q}_{i+1})F)$, which does not depend on the module M. □

An application of Lemma 3.8 gives the following:

(5.5) **Corollary.** *Let $(R, \mathbf{m}, \mathsf{k})$ be a two-dimensional complete local ring. If M is a finitely generated R-module and $\mathcal{F}$ is the family of all parameter ideals, then* $\mathrm{Tor}(M, \quad)$ *is uniformly $\mathcal{F}$-annihilated.*

6. Syzygetycally Artin-Rees modules in dimension two

In this section we prove Theorem 1.1 for two-dimensional rings.

(6.1) **Theorem.** *Let $(R, \mathbf{m})$ be a two-dimensional local ring. Let $I \subset R$ be an $\mathbf{m}$-primary ideal. Then every finitely generated R-module M is syzygetically Artin-Rees with respect to I.*

Proof. Without loss of generality we may assume that the ring is complete and the residue field is infinite. Let $J \subseteq I$ be a reduction of I. By countable prime avoidance (see [2, Lemma 3]) we can choose a system of parameters x, y such that y is a non-zero-divisor on all the modules $\Omega_i^R(M)/\mathrm{H}^0_{\mathbf{m}}(\Omega_i^R(M))$, x is a superficial element in I for R and $J = (x, y)$.

By 4.2(1) there exists an integer h_0 such that

$$(0 :_R x) \cap I^n = 0,$$

for all $n > h_0$.

By Corollary 5.5 there exists an h_1 such that

$$K^{h_1} \operatorname{Tor}_i^R(R/K^n, M) = 0,$$

for every $i > 0$ and for every $n > 0$ and for every ideal K generated by a system of parameters.

Let h_2 the least integer such that

$$I^n = J^{n-h_2} I^{h_2} \qquad \text{for all } n \geq h_2.$$

Let h_3 be the least integer such that

$$(y)^n \cap (x^{h_1}) \subseteq x^{h_1} y^{n-h_3}, \qquad \text{for every } n > h_3.$$

Finally, an application of the Artin-Rees Lemma and the fact that $H^0_{\mathbf{m}}(R)$ is a finite length module gives an integer h_4 such that

$$I^{h_4} \cap H^0_{\mathbf{m}}(R) = 0.$$

We claim that for $n > h_0 + h_1 + h_2 + h_3 + h_4$ and for every $i \geq 2$ we have

$$I^n F_{i-1} \cap \Omega_i^R(M) \subseteq I^{n-h_0-h_1-h_2-h_3-h_4} \Omega_i^R(M).$$

By the choice of J and h_2 we have for such n

$$\begin{array}{rcl} I^n F_{i-1} \cap \Omega_i^R(M) & = & J^{n-h_2} I^{h_2} F_{i-1} \cap \Omega_i^R(M) \\ & \subseteq & J^{n-h_2} F_{i-1} \cap \Omega_i^R(M). \end{array}$$

In particular,

$$x^{h_1}(I^n F_{i-1} \cap \Omega_i^R(M)) \subseteq x^{h_1}(J^{n-h_2} F_{i-1} \cap \Omega_i^R(M)) \subseteq J^{n-h_2} \Omega_i^R(M),$$

where the last inclusion is given by the choice of h_1,x and Lemma 3.1. So

$$I^n F_{i-1} \cap \Omega_i^R(M) \subseteq J^{n-h_2} \Omega_i^R(M) : x^{h_1}.$$

Let $r \in J^{n-h_2} \Omega_i^R(M) : x^{h_1}$, then

$$r x^{h_1} = \sum_{j=0}^{h_1-1} m_j x^j y^{n-h_2-j} + \sum_{j=h_1}^{n-h_2} m_j x^j y^{n-h_2-j},$$

for $m_j \in \Omega_i^R(M)$. In particular,

$$x^{h_1}(r - \sum_{j=h_1}^{n-h_2} m_j x^{j-h_1} y^{n-h_2-j}) \in y^{n-h_1-h_2} \Omega_i^R(M).$$

So

$$\begin{aligned} I^n F_{i-1} \cap \Omega_i^R(M) &\subseteq J^{n-h_2} \Omega_i^R(M) : x^{h_1} \\ &\subseteq J^{n-h_2-h_1} \Omega_i^R(M) + (y^{n-h_2-h_1} \Omega_i^R(M) : x^{h_1}) \\ &\subseteq J^{n-h_2-h_1} \Omega_i^R(M) + (y^{n-h_2-h_1} F_{i-1} : x^{h_1}). \end{aligned}$$

By intersecting the last term of the inclusions by $I^{n-h_1-h_2} F_{i-1}$ we can write

$$\begin{aligned} I^n F_{i-1} \cap \Omega_i^R(M) &\subseteq (J^{n-h_2-h_1} \Omega_i^R(M) + (y^{n-h_2-h_1} F_{i-1} : x^{h_1})) \cap I^{n-h_1-h_2} F_{i-1} \\ &\subseteq J^{n-h_2-h_1} \Omega_i^R(M) + (y^{n-h_2-h_1} F_{i-1} : x^{h_1})) \cap I^{n-h_1-h_2} F_{i-1}, \end{aligned}$$

where the last inclusion holds since $J^{n-h_2-h_1} \Omega_i^R(M) \subseteq I^{n-h_1-h_2} F_{i-1}$. In particular, each $a \in I^n F_{i-1} \cap \Omega_i^R(M)$ can be written as $a = b + s$, where

$$b \in J^{n-h_1-h_2} \Omega_i^R(M) \quad \text{and} \quad s \in (y^{n-h_2-h_1} F_{i-1} : x^{h_1}) \cap I^{n-h_2-h_1} F_{i-1}.$$

By the choice of h_3 we have

$$\begin{array}{rcl} x^{h_1} s \in (x^{h_1}) F_{i-1} \cap y^{n-h_2-h_1} F_{i-1} & = & ((x^{h_1}) \cap (y^{n-h_2-h_1})) F_{i-1} \\ & \subseteq & x^{h_1} y^{n-h_1-h_2-h_3} F_{i-1}. \end{array}$$

Therefore we can write $x^{h_1} s = x^{h_1} y^{n-h_1-h_2-h_3} v$ with $v \in F_{i-1}$ and

$$s - y^{n-h_1-h_2-h_3} v \in (0 : x^{h_1}) \cap I^{n-h_1-h_2-h_3} F_{i-1}.$$

Since x is a superficial element and $n - h_1 - h_2 - h_3 > h_0$, we have

$$(0 :_{F_{i-1}} x^{h_1}) \cap I^{n-h_1-h_2-h_3} F_{i-1} = 0.$$

In particular $s = y^{n-h_1-h_2-h_3}v$.

Since $s = a - b \in \Omega_i^R(M)$, we obtain that

$$\partial_{i-1}(s) = y^{n-h_1-h_2-h_3}\partial_{i-1}(v) = 0$$

in $\Omega_{i-1}(M)$. But y is a non-zero-divisor on $\Omega_{i-1}(M)/\operatorname{H}^0_{\mathbf{m}}(\Omega_{i-1}(M))$, so that $\partial_{i-1}(v) \in H^0_{\mathbf{m}}(\Omega_{i-1}(M))$.

By the choice of h_4 we have

$$I^{h_4}F_{i-2} \cap \operatorname{H}^0_{\mathbf{m}}(F_{i-2}) = 0,$$

and because $\operatorname{H}^0_{\mathbf{m}}(\Omega_{i-1}(M)) \subseteq \operatorname{H}^0_{\mathbf{m}}(F_{i-2})$ we obtain

$$I^{h_4}\operatorname{H}^0_{\mathbf{m}}(\Omega_{i-1}(M)) = 0.$$

In particular $y^{h_4}\partial_{i-1}(v) = 0 \in \Omega_{i-1}(M)$, hence $y^{h_4}v \in \Omega_i(M)$.

Therefore,

$$s = y^{n-h_2-h_3-h_1}v = y^{n-h_1-h_2-h_3-h_4}y^{h_4}v$$

is an element of $y^{n-h_1-h_2-h_3-h_4}\Omega_i^R(M) \subseteq I^{n-h_2-h_3-h_1-h_4}\Omega_i^R(M)$, and $a = b + s$ is an element of

$$J^{n-h_1-h_2}\Omega_i^R(M) + I^{n-h_2-h_3-h_1-h_4}\Omega_i^R(M) \subseteq I^{n-h_2-h_3-h_1-h_4}\Omega_i^R(M). \qquad \square$$

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References

1. Winfried Bruns and Jürgen Herzog, *Cohen-Macaulay rings*, Cambridge Studies in Advanced Mathematics, vol. 39, Cambridge University Press, Cambridge, 1993. MR **1251956 (95h:**13020)
2. Lindsay Burch, *Codimension and analytic spread*, Proc. Cambridge Philos. Soc. **72** (1972), 369–373. MR MR0304377 (46 #3512)
3. A. J. Duncan and L. O'Carroll, *A full uniform Artin-Rees theorem*, J. Reine Angew. Math. **394** (1989), 203–207. MR **MR977443 (90c:**13011)
4. David Eisenbud, *Commutative algebra with a view toward algebraic geometry*, Graduate Texts in Mathematics, vol. 150, Springer-Verlag, New York, 1995. MR **97a:**13001
5. David Eisenbud and Craig Huneke, *A finiteness property of infinite resolutions*, J. Pure Appl. Algebra **201** (2005), no. 1-3, 284–294. MR 2158760
6. Craig Huneke, *Uniform bounds in Noetherian rings*, Invent. Math. **107** (1992), no. 1, 203–223. MR **93b:**13027
7. Gerson Levin, *Local rings and Golod homomorphisms*, J. Algebra **37** (1975), no. 2, 266–289. MR 55 #2878
8. L. O'Carroll, *Addendum to: "A note on Artin-Rees numbers"*, Bull. London Math. Soc. **23** (1991), no. 6, 555–556. MR **MR1135185 (92i:**13001b)
9. ———, *A note on Artin-Rees numbers*, Bull. London Math. Soc. **23** (1991), no. 3, 209–212. MR **92i:**13001a
10. Liam O'Carroll, *A uniform Artin-Rees theorem and Zariski's main lemma on holomorphic functions*, Invent. Math. **90** (1987), no. 3, 647–652. MR **914854 (89h:**13013)
11. Francesc Planas-Vilanova, *The strong uniform Artin-Rees property in codimension one*, J. Reine Angew. Math. **527** (2000), 185–201. MR **2001g:**13051
12. Judith D. Sally, *Numbers of generators of ideals in local rings*, Marcel Dekker Inc., New York, 1978. MR 0485852 (58 #5654)
13. Peter Schenzel, *Dualisierende Komplexe in der lokalen Algebra und Buchsbaum-Ringe*, Lecture Notes in Mathematics, vol. 907, Springer-Verlag, Berlin, 1982, With an English summary. MR **654151 (83i:**13013)

14. Janet Striuli, *Strong artin-rees property in rings of dimension one and two*, preprint (2006), available from http://arxiv.org/abs/math.AC/0607177.
15. Irena Swanson and Craig Huneke, *Integral Closure of Ideals, Rings, and Modules*, London Mathematical Society Lecture Note Series, vol. 336, Cambridge University Press, Cambridge, 2006.
16. Hsin-Ju Wang, *Jacobian ideals, resolutions, relation types of parameters*, Ph.D. thesis, Purdue University, 1994.

Department of Mathematics, University of Nebraska, Lincoln, NE 68588-0130, U.S.A.
E-mail address: jstriuli2@math.unl.edu