Some Homological Properties of Almost Gorenstein Rings

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ABSTRACT. We show that the residue field k is a direct summand of the second syzygy of the canonical module for some almost Gorenstein rings. This implies that over a Teter ring the only totally reflexive modules are the free ones. We provide an example of an almost Gorenstein ring which has infinitely many non-isomorphic totally reflexive modules.

Introduction

Let $(R, \mathfrak{m}, \mathsf{k})$ be a local Noetherian ring. For an R-module M denote by M^* the R-module $\mathrm{Hom}_R(M, R)$.

DEFINITION 0.1. An R-module M is totally reflexive if and only if $M^{**} \cong M$ and $\mathrm{Ext}^i_R(M, R) = 0 = \mathrm{Ext}^i_R(M^*, R)$, for all $i \gg 0$.

Free modules are totally reflexive and so is every maximal Cohen-Macauly module over a Gorenstein ring. Over a general ring, it is an open problem to determine conditions that are necessary and sufficient for the existence of a non-free totally reflexive module, see for example [**4**]. The starting point of our investigation was to consider the problem for some artinian rings, and in particular for the class of *Teter rings* for which we show that the only totally reflexive modules are the free ones, see Corollary 2.3.

Teter rings are a particular example of *almost Gorenstein* rings, as defined in [**8**]. Our investigation lead us to the study of the syzygies of the canonical module for a certain class of almost Gorenstein rings, for which we prove the following:

THEOREM 0.2 (Main Theorem). *Let $(R, \mathfrak{m}, \mathsf{k})$ be a local noetherian ring which is almost Gorenstein with canonical module ω_R. Assume that R is not Gorenstein and $R = S/J$, where S is an artinian Gorenstein ring. Let $c = \dim_{\mathsf{k}}(J :_S \mathfrak{m})/(\mathfrak{m}J :_S \mathfrak{m}) > 0$. Then the vector space k^c is a direct summand of the second syzygy of the canonical module ω_R.*

The proof of the Main Theorem is the content of Section 1. The connection between the Main Theorem and the problem above is given in Section 2, where we also give an example of an almost Gorenstein ring that has infinitely many totally reflexive modules. In section 3 we consider *strongly* almost Gorenstein rings that

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are quotients of a polynomial ring modulo a monomial ideal and we show that k is a direct summand of the first or the second syzygy of the canonical module. This implies that over such rings there are no non-free totally reflexive modules. We do not know whether the class of strongly almost Gorenstein rings is strictly contained in the class of almost Gorenstein rings.

1. The canonical module over almost Gorenstein rings

The following definition appears in [**8**].

Definition 1.1. An artinian local ring $(R,\mathfrak{m},\mathsf{k})$ is *almost Gorenstein* if the inclusion $0:_R(0:I)\subseteq(I:_R\mathfrak{m})$ holds for every ideal $I\subset R$.

For any artinian ring R, one may assume, by the Cohen structure theorem, that R is a quotient S/J where S is a Gorenstein artinian ring. If S is a Gorenstein ring, then $0:_S(0:I)=I$ for all ideals I in S. Therefore without loss of generality we may assume that $J=0:K$ for some ideal $K\subseteq S$. Assume that K is generated by $f_1,\dots f_n$. The following result is an adaptation of Proposition 4.1 in [**8**].

Lemma 1.2. *Let S be a Gorenstein artinian ring and let $K=(f_1,\dots,f_n)$ be an ideal such that the ring $R=S/(0:_S K)$ is almost Gorenstein, but not Gorenstein. Then the following equality holds:*

$$\mathfrak{m}_S=f_i:_S(f_1,\dots,\hat{f_i},\dots f_n)+(f_1,\dots,\hat{f_i},\dots f_n):_S f_i,$$

for all $i\in\{1,\dots,n\}$.

Proof. Without loss of generality we may assume that $i=1$. Let $I=(0:_S f_1)$ and denote by $J=(0:_S K)$. As $J\subseteq I$ and S/J is almost Gorenstein, one has the inclusion $J:_S(J:_S I)\subset I:_S\mathfrak{m}_S$. For the first term of the equality, we have the following equalities:

$$\begin{aligned} J:_S(J:_S I) &= (0:_S K):_S(J:_S I)\\ &= (0:_S K(J:_S I))\\ &= (0:_S K[(0:_S K):_S I)])\\ &= (0:_S K(0:_S KI))\\ &= (0:_S(0:_S KI)):_S K\\ &= KI:_S K \end{aligned}$$

For the right hand term of the equality:

$$I:_S\mathfrak{m}_S=(0:f_1):_S\mathfrak{m}_S=0:_S f_1\mathfrak{m}.$$

So that $KI:_S K\subseteq 0:_S f_1\mathfrak{m}$. Using that S is Gorenstein, we have $(f_1\mathfrak{m}_S)=0:_S(0:_S f_1\mathfrak{m}_S)\subseteq 0:_S(KI:_S K)$. But

$$\begin{aligned} 0:_S(KI:_S K) &= K(0:_S KI)\\ &= K[(0:_S I):_S K]\\ &= K(f_1:_S K) \end{aligned}$$

Putting all together we obtain that $f_1\mathfrak{m}\subseteq K(f_1:_S K)$. In particular, for every element $x\in\mathfrak{m}$ we can write $xf_1=\sum_{i=1}^n u_if_i$, with $u_i\in(f_1:_S K)$ and hence $(x-u_1)f_1=\sum_{i=2}^n u_1f_i$. This implies that $x-u_1\in(f_2,\dots,f_n):_S f_1$ and finally

$x \in (f_2, \dots, f_n) :_S f_1 + f_1 :_S K = (f_2, \dots, f_n) :_S (f_1) + (f_1) :_S (f_2, \dots, f_n)$. As x is a random element in the maximal ideal $\mathfrak{m}$, we have the thesis. □

The conditions in 1.2 are not sufficient for a ring to be almost Gorenstein. Take for example $S = \mathsf{k}[x, y, z, u]/(x^2, y^2, z^2, u^2)$, $f_1 = xz$, $f_2 = yz, f_3 = xu$, $f_4 = yu$. The conclusion of Lemma 1.2 holds, but $R = \mathsf{k}[x, y, z, u]/(x^2, y^2, z^2, u^2, xy, zu)$ is not almost Gorenstein.

Now we give the proof of the Main Theorem, Theorem 0.2. We will use $\Omega^i_R(M)$ to denote the ith syzygy of an R-module M.

Proof. In the following, denote by y' the image in R of the element $y \in S$. Since S is Gorenstein, we may assume that $J = (0 :_S K)$, for some ideal $K = (f_1, \dots, f_n)$. The canonical module ω_R is given by $\mathrm{Hom}_S(R, S) = \mathrm{Hom}_S(S/(0 :_S K), S)$ which is isomorphic to $(0 :_S (0 :_S K)) = K$. Let

$$\cdots \longrightarrow R^p \xrightarrow{\partial_2} R^m \xrightarrow{\partial_1} R^n \xrightarrow{\partial_0} K \longrightarrow 0$$

be a minimal presentation of the canonical module. By Lemma 1.2, we can choose a set of minimal generators $x_1, \dots, x_e$ of the maximal ideal $\mathfrak{m}_S$, such that $x_i \in f_1 :_S (f_2, \dots f_n)$ or $x_i \in (f_2, \dots, f_n) : f_1$ for every $i = 1, \dots, e$. In any case there is a relation $a_{1i}f_1 + a_{2i}f_2 + \cdots + a_{ni}f_n = 0$ in S, such that either $a_{1i} = x_i$ or $a_{2i} = x_i$. The column vectors $D'_i = (a'_{1i}, \dots, a'_{ni})$ are part of a minimal generating set for the first syzygy. After a choice of basis, $D'_1, \dots, D'_e$ are the first e columns of the matrix representing ∂_1, let $D'_{e+1}, \dots, D'_m$ be the other columns. Let u be an element of $(J : \mathfrak{m})/(\mathfrak{m}J : \mathfrak{m})$, we claim that $\mathbf{u}' = (u', u', 0, \dots, 0) \mid$ for every $j = 1, \dots, c\}$ is part of a minimal generating set for the second syzygy of ω_R. To prove the claim, denote by $B' = (b'_{ij})$ the matrix representing ∂_2. Assume that $\mathbf{u}'$ is not part of a minimal set of generators of the second syzygy. This implies that

$$\mathbf{u} = c_1 \begin{pmatrix} b_{11} \\ \vdots \\ b_{m1} \end{pmatrix} + \cdots + c_p \begin{pmatrix} b_{1p} \\ \vdots \\ b_{mp} \end{pmatrix} + JS^m$$

where c_i are elements of the maximal ideal $\mathfrak{m}_S$. Moreover we have

$$b_{1i}D_1 + \cdots + b_{mi}D_m \in JS^n$$

for all $i = 1, \dots, p$. This implies that $D\mathbf{u} = \sum c_j(b_{1j}D_1 + \cdots + b_{mj}D_m) \in \mathfrak{m}JS^n$. For every $u \in (J :_S \mathfrak{m})$ there exists $i = 1, \dots, e$ such that $ux_i \in J \setminus \mathfrak{m}J$. By looking at the component $(D\mathbf{u})_i$ one reaches a contradiction.

This shows that if $u_1, \dots, u_c$ are elements in $(J : \mathfrak{m})$ such that their representatives in $(J : \mathfrak{m})/(\mathfrak{m}J : \mathfrak{m})$ are a basis then $\mathbf{u}'_i$ is part of a minimal set of generators for the second syzygy of ω_R. Moreover for every linear combination $u = \alpha_1 u_1 + \cdots + \alpha_c u_c$ with coefficients in k, the element u' is still a basis element for $(J : \mathfrak{m})/(\mathfrak{m}J : \mathfrak{m})$ and therefore $\mathbf{u}'$ is a minimal generator for $\Omega^2_R(\omega_R)$ unless all the coefficients α_i are equal to zero. □

Remark 1.3. If $R = S/J$ with S artinian Gorenstein, and we write $J = 0 :_S K$, then we have $\mathfrak{m}J : \mathfrak{m} \neq J : \mathfrak{m}$ if and only if $\mathfrak{m}K : \mathfrak{m} \neq K : \mathfrak{m}$.

Indeed, one has $J : \mathfrak{m} = \mathfrak{m}J : \mathfrak{m}$ if and only if $0 : \mathfrak{m}K = \mathfrak{m}(0 : K) : \mathfrak{m}$ if and only if $\mathfrak{m}K = 0 : [\mathfrak{m}(0 : K) : \mathfrak{m}] = \mathfrak{m}[0 : \mathfrak{m}(0 : K)] = \mathfrak{m}[K : \mathfrak{m}]$ if and only if $K : \mathfrak{m} = \mathfrak{m}K : \mathfrak{m}$.

2. Almost Gorenstein rings and totally reflexive modules

The following lemma is well-known by the experts. We include the proof for easy reference.

LEMMA 2.1. *Let* $(R, \mathfrak{m}, \mathsf{k})$ *be a local ring with canonical module* ω_R. *If* k *is a direct summand of any syzygy of* ω_R *then there are no non-free totally reflexive modules.*

PROOF. Let X be a totally reflexive module. By definition, $\operatorname{Ext}_R^i(X, F) = 0$ for every free module F and for every $i > 0$. Applying the functor $\operatorname{Hom}_R(X, \quad)$ to the short exact sequence $0 \to \Omega_R^1(\omega_R) \to F \to \omega_R \to 0$, one obtains the equalities $\operatorname{Ext}_R^1(X, \omega_R) = \operatorname{Ext}_R^{i+1}(X, \Omega_R^i(\omega_R))$ for every R-module M. In particular, $\operatorname{Ext}_R^{i+1}(X, \mathsf{k}) = 0$ if k is a direct summand of $\Omega_R^i(\omega_R)$. This shows that X has finite projective dimension and therefore it is free, see by the Auslander-Bridger formula (see for example Theorem 1.4.8 [**6**]) and the Auslander-Buchsbaum formula (see for example Theorem 1.3.3 [**5**]). □

REMARK 2.2. In [**9**] Theorem 1.6 and Remark 1.8 (e) it is shown that if a local ring R can be written as a quotient S/J , where $(S, \mathfrak{m}_S)$ is a local ring such that $\dim_{\mathsf{k}}(J : \mathfrak{m}_S)/(\mathfrak{m}J : \mathfrak{m}_S) \geq 2$ then there are no non-free totally reflexive modules. Theorem 0.2 and Lemma 2.1 shows that for almost Gorenstein rings the conclusion holds even in the case when $\dim_{\mathsf{k}}(J : \mathfrak{m})/(\mathfrak{m}J : \mathfrak{m}) \geq 1$.

We say that an artinian ring R is a Teter ring if there exists a local artinian Gorenstein ring $(S, \mathfrak{m}_S)$ such that $R = S/(\delta)$, where $(\delta) = (0) :_S \mathfrak{m}_S$

THEOREM 2.3. *Let* R *be a Teter ring, then every totally reflexive module is free.*

PROOF. Write $R = S/\delta$ where S is a Gorenstein artinian ring with socle equal to δ. The condition $(\delta) :_S \mathfrak{m} \neq (0) :_S \mathfrak{m} = (\delta)$ holds, therefore one can apply Theorem 0.2 and Lemma 2.1. □

Teter rings are the ring of smallest Gorenstein colength, for a deinition see [**1**]. The following example shows that it is possible to have totally reflexive modules over rings of Gorenstein colength 2.

EXAMPLE 2.4. The ring $R = \mathsf{k}[|x, y, z|]/(x^2, y^2, z^2, yz)$ has totally reflexive modules which are not free. On the other hand, let $S = \mathsf{k}[|x, y, z|]/(x^2, y^2, z^2)$ and $J = (yz)S$ then $(J :_S \mathfrak{m}_S) = (\mathfrak{m}_S J; \mathfrak{m}_s)$. The ring R has Gorenstein colength 2.

The following proof is an adaptation from [**8**].

LEMMA 2.5. *Let* $(R, \mathfrak{m}, \mathsf{k})$ *be a Cohen-Macaulay ring such that* $\mathfrak{m}\operatorname{Ext}_R^1(M, R) = 0$ *for all maximal Cohen-Macaulay module* M. *Then* $R/\boldsymbol{x}$ *is an almost Gorenstein ring for all system of parameters* $\boldsymbol{x}$.

PROOF. Let I be any ideal of R containing the ideal generated by $\boldsymbol{x}$. We need to show that $\boldsymbol{x} :_R (\boldsymbol{x} :_R I) \subseteq I :_R \mathfrak{m}$. Assume that I is generated by $f_1, \ldots, f_n$ and consider the short exact sequence

$$0 \to \frac{R}{\boldsymbol{x} : I} \to \left(\frac{R}{\boldsymbol{x}}\right)^n \to N \to 0,$$

where the first map is given by $\overline{u} \to (\overline{f_1 u}, \dots, \overline{f_n u})$. Applying the functor $\mathrm{Hom}_R(\quad, R/\boldsymbol{x})$ to the short exact sequence we obtain:

$$0 \to \mathrm{Hom}_R(N, \frac{R}{\boldsymbol{x}}) \to \mathrm{Hom}_R(\frac{R}{I}, \frac{R}{\boldsymbol{x}})^n \to \mathrm{Hom}_R(\frac{R}{\boldsymbol{x}:I}, \frac{R}{\boldsymbol{x}}) \to \mathrm{Ext}^1_R(N, \frac{R}{\boldsymbol{x}}).$$

The cokernel of the middle map is the cokernel of:

$$\oplus \frac{\boldsymbol{x} :_R I}{\boldsymbol{x}} \to \frac{\boldsymbol{x} :_R (\boldsymbol{x} :_R I)}{\boldsymbol{x}}$$

given by $(\overline{u}_1, \dots, \overline{u}_n) \to \overline{f_1 u_1 + \cdots + f_n u_n}$. The cokernel is therefore isomorphic to $\frac{\boldsymbol{x}:_R(\boldsymbol{x}:_R I)}{I}$ and embeds in $\mathrm{Ext}^1_R(N, R/\boldsymbol{x})$. As $\boldsymbol{x} \subseteq \mathrm{ann}_R \mathrm{Ext}^1_R(N, \quad)$, we obtain the isomorphism $\mathrm{Ext}^1_R(N, R/\boldsymbol{x}) \cong \mathrm{Ext}^{d+1}_R(N, R)$ which is isomorphic to $\mathrm{Ext}^1_R(\Omega^d(N), R)$ and therefore annihilated by $\mathfrak{m}$. This implies that $\mathfrak{m}\frac{\boldsymbol{x}:_R(\boldsymbol{x}:_R I)}{I} = 0$ and therefore the thesis. □

Remark 2.6. In [**3**], Theorem 3.1, the authors prove that if $(R, \mathfrak{m})$ is a local ring and $\boldsymbol{y} = y_1, \dots, y_d$ is a regular sequence in $\mathfrak{m}^2$ then $R/\boldsymbol{y}$ has a totally reflexive module.

Remark 2.7. Let $(R, \mathfrak{m})$ be a local ring. Let M be a finitely generated R-module and $0 \to \Omega^1_R(M) \to F \to M \to 0$ be the beginning of a minimal free resolution of M. For every element x of the maximal ideal, denote by μ_x the multiplication by x. If for every x in a minimal set of generators of $\mathfrak{m}$ there exists a linear map ϕ_x such that the diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Omega^1_R(M) & \longrightarrow & F & \longrightarrow & M & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \mu_x} & \swarrow{\scriptstyle \phi_x} & & & & & \\ 0 & \longrightarrow & \Omega^1_R(M) & \longrightarrow & F & \longrightarrow & M & \longrightarrow & 0 \end{array}$$

commutes, then $\mathfrak{m}\,\mathrm{Ext}^1_R(M, N) = 0$ for all modules N.

Remark 2.8. Let $(R, \mathfrak{m})$ be a local Cohen-Macaulay ring with canonical module ω_R. For every R-module N, denote by $N^\vee$ the R-module $\mathrm{Hom}_R(N, \omega_R)$. Let M and L two maximal Cohen-Macaulay modules. There exists an isomorphism $\phi : \mathrm{Ext}^1_R(M, L) \to \mathrm{Ext}^1_R(L^\vee, M^\vee)$ such that $\phi(\xi_1) = \xi_2$ where

$$\xi_1 : 0 \to L \to X \to M \to 0$$

and

$$\xi_2 : 0 \to M^\vee \to X^\vee \to L^\vee \to 0.$$

Example 2.9. In this example we show that there exists an almost Gorenstein ring that admits a totally reflexive module which is not free, and in fact infinitely many by [**7**]. The ring $R = \mathbb{C}[[x, y, z, u, v]]/(xz - y^2, xv - yu, yv - zy)$ is of finite Cohen Macaulay type and its only indecomposable maximal Cohen-Macaulay modules are R, the ideals $\omega_R \cong \alpha = (x, y)$, $\alpha^2 = (x^2, y^2, xy)$, $\beta = (x, y, u)$ and the R-module $\Omega^1_R(\beta)$. For a proof of this see for example [**10**]. In the following we show that the maximal ideal $\mathfrak{m}$ annihilates all the R-modules $\mathrm{Ext}^1_R(M, R)$ for M maximal Cohen-Macaulay. We first show that $\mathfrak{m}\,\mathrm{Ext}^1_R(\alpha, \Omega^1(\alpha)) = 0$.

For the ideal α, the first syzygy $\Omega^1_R(\alpha)$ is generated by

$$[-v, u], [-z, y], [-y, x].$$

The following list gives the maps of Remark 2.7

$$\phi_x = \begin{pmatrix} y & z-y \\ -x & -y+x \end{pmatrix} \qquad \phi_y = \begin{pmatrix} -y & 0 \\ x & 0 \end{pmatrix}$$
$$\phi_z = \begin{pmatrix} z & 0 \\ -y & 0 \end{pmatrix} \qquad \phi_v = \begin{pmatrix} v-y & -z \\ -u+x & y \end{pmatrix}$$
$$\phi_u = \begin{pmatrix} y & 0 \\ -u & 0 \end{pmatrix}$$

In particular, by remark 2.7, we have that $\mathrm{Ext}^1_R(\alpha, N) = 0$ for every R-module N.

For every maximal Cohen-Macaulay module M, the following holds:

$$\mathrm{ann}_R(\mathrm{Ext}^1_R(M,R)) = \mathrm{ann}_R(\mathrm{Ext}^1_R(\omega_R, M^\vee) \cong \mathrm{Ext}^1_R(\alpha, M^\vee) = \mathfrak{m},$$

where the second equality follows from Remark 2.8.

3. The monomial case

The following version of the notion of almost Gorenstein ring is considered in [**8**] without being given a name. We will term it strongly almost Gorenstein, since it is shown in [**8**] that it implies the almost Gorenstein property. We do not know whether the two properties are equivalent.

Definition 3.1. An artinian local ring R is *strongly almost Gorenstein* if $\omega_R^*(\omega_R) \supseteq \mathfrak{m}$, where ω_R denotes the canonical module of R, and

$$\omega_R^*(\omega_R) = \{y \in R \,|\, y = f(x) \text{ for some } \ x \in \omega_R \text{ and some } f \in \omega_R^*\}.$$

The main result of this section deals with artinian strongly almost Gorenstein rings which are obtained as quotients of polynomial rings by monomial ideals.

Theorem 3.2. *Let $S = \mathsf{k}[x_1, \dots, x_d]/(x_1^{A_1}, \dots, x_d^{A_d})$, and let $f_1, \dots, f_n$ be monomials in S such that $R = S/0 :_S (f_1, \dots, f_n)$ is strongly almost Gorenstein. Then R does not admit non-free totally reflexive modules.*

The proof of Theorem 3.2 will be given after we prove the following:

Theorem 3.3. *Let $S = \mathsf{k}[x_1, \dots, x_d]/(x_1^{A_1}, \dots, x_d^{A_d})$, and let $f_1, \dots, f_n$ be monomials in S such that*

(1) *f_i does not divide f_j for every $i \neq j$;*
(2) *x_i divides f_j for all $i \in \{1, \dots, d\}$ and for all $j \in \{1, \dots n\}$;*
(3) *$(x_1, \dots, x_u) \subseteq \sum_{i=1}^n f_i :_S (f_1, \dots, f_n)$.*

then, one of the following conclusions holds:

(A) *There exists an $i \in \{1, \dots, n\}$ and a $j \in \{1, \dots, u\}$ such that*
$$\frac{f_i}{x_j} \in (f_1, \dots, f_n) : (x_1, \dots, x_u);$$
(B) *There exist mutually disjoint sets $S_1, \dots, S_n \subset \{1, \dots, u\}$ such that for all $i \in \{1, \dots, n\}$ and all $j \in \{1, \dots, u\}$, $x_j f_i \neq 0 \Leftrightarrow j \in S_i$.*

Proof. Before we proceed with the proof, we establish some claims that we will use later. Write each $f_j = \Pi_{i=1}^n x_i^{N_{ji}}$, with $N_{ji} < A_i$.

Claim 1: If $x_i f_j \in (f_k)$, for some integers i, j, k then one of the following cases hold:

$$\text{(i)} \quad \begin{cases} N_{ji} = N_{ki} - 1 \\ N_{jl} \geq N_{kl}, \quad \text{for every} \quad l \neq i \end{cases}$$

(ii) $N_{ji} = A_i - 1$

Moreover, for fixed j, k, the first case can hold for at most one i.

Proof of Claim 1: Note that (ii) is equivalent to $x_i f_j = 0$ in S. If $0 \neq x_i f_j \in (f_k)$, then (i) is obtained by comparing the exponents of each variable for $x_i f_j$ and f_k. The fact that $N_{ji} = N_{ki} - 1$ is due to the assumption that f_k does not divide f_j. For the last statement, assume that there are two indeces i_1 and i_2 such that

$$\begin{cases} N_{ji_1} = N_{ki_1} - 1 \\ N_{jl} \geq N_{kl}, \quad \text{for every} \quad l \neq i_1 \end{cases}$$

and

$$\begin{cases} N_{ji_2} = N_{ki_2} - 1 \\ N_{jl} \geq N_{kl}, \quad \text{for every} \quad l \neq i_2 \end{cases}$$

then, $N_{ki_2} - 1 = N_{ji_2} \geq N_{ki_2}$ which is a contradiction.

Claim 2: If conclusion B holds but A does not hold, then we have the following:

(i) each set S_i has cardinality at least 2;

(ii) for every $k \in S_i$ we have $x_k \in (f_i) : (f_1, \dots, f_n)$, and $x_k \notin (f_j) : (f_1, \dots, f_n)$ for all $j \neq i$.

Proof of Claim 2: Assume that there exist indeces i and k such that $S_i = \{x_k\}$. Since $x_k f_j = 0$ for all $j \neq i$, it follows that case (A) holds, as

$$\frac{f_i}{x_k} \in (f_i) : (f_1, \dots, f_n).$$

For (ii), let $k \in S_i$. Assume that $x_k \in (f_j) : (f_1, \dots, f_n)$ for some $j \neq i$. Then $0 \neq x_k f_i \in (f_j)$. As we may assume (i), there exists an $l \in S_i$ such that $l \neq k$. By Claim 1, we have $N_{il} \geq N_{jl}$. As $l \notin S_j$, we have $x_l f_j = 0$, and thus $N_{jl} = A_l - 1$. This contradicts the fact that $N_{il} < A_l - 1$.

The proof of the theorem goes by induction on the number of variables d, the case $d = 1$ being obvious. Assume that the theorem holds for $d - 1$ variables. We now induct on the number n of polynomials. Assume that the theorem holds in the case of $n - 1$ polynomials.

Claim 3: If there exists $k \in \{1, \dots, u\}$ such that $x_k f_i = 0$ for all $i \in \{1, \dots, n\}$, then we are done by induction on the number of variables. In particular, whenever conclusion B holds for a subset of $\{f_1, \dots, f_n\}$ with respect to a subset $\{x_1, \dots, x_s\}$ of $\{x_1, \dots, x_u\}$, we may assume that the sets $S_1, S_2, \dots$, asserted in Conclusion B form a partition of $\{1, \dots, s\}$.

Indeed, we can write $f_i = x_k^{A_k - 1} f_i'$, with $f_i' \in \mathsf{k}[x_1, \dots, \hat{x_k}, \dots, x_d]$. Assumptions (1), (2), (3) hold for $\{f_1', \dots, f_n'\}$ viewed as monomials in $d - 1$ variables. If conclusion (A) holds for $\{f_1', \dots, f_n'\}$, then it also holds for $\{f_1, \dots, f_n\}$. Similarly, if conclusion (B) holds for $\{f_1', \dots, f_n'\}$, then it also holds for $\{f_1, \dots, f_n\}$ (with the same choice of the sets S_i).

Claim 4: Assume that conclusion B holds for $\{f_1, \dots, f_{n-1}\}$ with respect to a set of variables $\{x_1, \dots, x_s\}$, with $s \leq u$. Let $S_1', \dots, S_{n-1}' \subset \{1, \dots, s\}$ be the sets asserted in conclusion B. Let $k \in \{1, \dots, s\}$, and let $i \in \{1, \dots, n-1\}$ be such that $k \in S_i'$.

Then we have either $x_k \in (f_i) : (f_1, \dots, f_n)$, or $x_k \in (f_n) : (f_1, \dots, f_n)$. Among the k's for which the first situation occurs, we can have $x_k f_n \neq 0$ for at most one such k.

Proof of Claim 4: By Claim 2 (ii), we cannot have $x_k \in (f_j) : (f_1, \ldots, f_{n-1})$ for any $i \neq j \leqslant n-1$. Thus, we have either $x_k \in (f_i) : (f_1, \ldots, f_n)$, or $x_k \in (f_n) : (f_1, \ldots, f_n)$. For the last part of the claim, assume that $x_{k_1} f_n \in (f_{i_1})$, and $x_{k_2} f_n \in (f_{i_2})$, with $k_1 \in S'_{i_1}$, and $k_2 \in S'_{i_2}$. We need to show that one of $x_{k_1} f_n$ or $x_{k_2} f_n$ is zero. If $i_1 = i_2$, this follows from Claim 1. Assume that $i_1 \neq i_2$ and $x_{k_1} f_n \neq 0$. Then $N_{nk_1} = N_{i_1 k_1} - 1$, $N_{nl} \geqslant N_{i_1 l}$ for all $l \neq k_1$. In particular, $N_{nk_2} \geqslant N_{i_1 k_2}$. Since $k_2 \notin S'_{i_1}$, we have $x_{k_2} f_{i_1} = 0$, and thus $x_{k_2} f_n = 0$.

Claim 5: If

$$(x_1, \ldots, x_u) \subseteq \sum_{l \neq i} (f_l) : (f_1, \ldots, f_n).$$

for some $i \in \{1, \ldots, n\}$, then conclusion A holds.

Proof of Claim 5: Assume that $(x_1, \ldots, x_u) \subseteq \sum_{i=1}^{n-1} (f_i) : (f_1, \ldots, f_n)$. The assumptions (1),(2),and (3) in the theorem are satisfied for $\{f_1, \ldots, f_{n-1}\}$ with respect to the variables $\{x_1, \ldots, x_u\}$, and by the induction hypothesis either A of B holds. If (A) holds for $\{f_1, \ldots, f_{n-1}\}$ then it also holds for $\{f_1, \ldots, f_n\}$, and we are done.

Assume that (B) holds for $\{f_1, \ldots, f_{n-1}\}$. Let $\{1, \ldots, u\} = S'_1 \cup \ldots \cup S'_{n-1}$ be the partition asserted in conclusion (B). By Claim 4, for each $k \in \{1, \ldots, u\}$ we have either $x_k \in (f_i) : f_n$ or $x_k \in (f_n) : f_i$, where $i \in \{1, \ldots, n-1\}$ is such that $k \in S'_i$.

If the first situation occurs for all $k \in \{1, \ldots, u\}$, then Claim 3 shows that $x_k f_n = 0$ for all values of k except one, say k_0. Then conclusion A holds, with

$$\frac{f_n}{x_{k_0}} \in (f_1, \ldots, f_n) : (x_1, \ldots, x_u).$$

Assume that there exists a k_0 such that $x_{k_0} f_{i_0} \in (f_n)$ holds, where i_0 is such that $k_0 \in S'_{i_0}$. Note that $x_{k_0} f_{i_0} \neq 0$, so we have $N_{i_0 l} \geqslant N_{nl}$ for all $l \neq k_0$. By Claim 2(i), we may assume that S'_{i_0} has cardinality at least two. Let $k' \in S_{i_0}$, $k' \neq k$.

Since $N_{nk'} \leqslant N_{i_0 k'} < A_{k'} - 1$, it follows that $x_{k'} f_n \neq 0$ for all $k_0 \neq k' \in S'_{i_0}$. Also, by Claim 1, we cannot have $x_{k'} f_{i_0} \in (f_n)$. The only remaining possibility is that $0 \neq x_{k'} f_n \in (f_{i_0})$, and therefore $N_{nl} \geqslant N_{i_0 l}$ for all $l \neq k'$. In particular, $x_j f_n = 0$ for all $j \notin S'_{i_0}$. It follows that conclusion A holds, with

$$\frac{f_n}{x_{k_0}} \in (f_1, \ldots, f_n) : (x_1, \ldots, x_u).$$

Indeed, for $k' \in S_{i_0}$, $k' \neq k_0$ we have $x_{k'} f_n \in (f_{i_0})$, and $N_{nk_0} = N_{i_0 k_0} + 1$, from which we see that

$$\frac{f_n}{x_{k_0}} x_{k'} \in (f_{i_0}).$$

Claim 5 allows us to rename the variables so that we may assume that

$$x_1, \ldots, x_s \notin (f_n) : (f_1, \ldots, f_{n-1}, f_n) \tag{3.0.1}$$

$$x_{s+1}, \ldots, x_u \in (f_n) : (f_1, \ldots, f_{n-1}, f_n) \tag{3.0.2}$$

We apply the induction hypothesis to $\{f_1, \ldots, f_{n-1}\}$ with respect to the variables $\{x_1, \ldots, x_s\}$.

Assume that conclusion B holds for $\{f_1, \ldots, f_{n-1}\}$ with respect to $\{x_1, \ldots, x_s\}$, but A does not. Let $\{1, \ldots, s\} = S'_1 \cup \ldots \cup S'_{n-1}$ be the partition asserted by B.

We claim that

$$x_l f_1 = \dots x_l f_{n-1} = 0, \quad \text{for all} \quad s+1 \le l \le u. \tag{3.0.3}$$

Indeed, assume by way of contradiction that there exists an $l \in \{s+1, \dots, u\}$ and an $i \leqslant n-1$ such that $x_l f_i \neq 0$. Since $x_l f_i \in (f_n)$, we must have $N_{ik} \geqslant N_{nk}\, \forall k \neq l$. In particular, for $k \in S'_i$, we have $N_{ik} < A_k - 1$, and thus $N_{nk} < A_k - 1$, which means that $x_k f_n \neq 0$. Since we may assume that S'_i has cardinality at least two, Claim 4 shows that there exists a $k \in S'_i$ with $x_k f_i \in (f_n)$. The fact that both $x_j f_i$ and $x_k f_i$ are nonzero elements in (f_n) contradicts Claim 1.

Equation 3.0.1 and Claim 4 show that we have two possibilities:

(1) There exists a $k \in \{1, \dots, s\}$ with $0 \neq x_k f_n \in (f_i)$, where $k \in S'_i$, and $x_l f_n = 0$ for all $l \in \{1, \dots, s\}$, $l \neq k$. Then we also have $x_l f_n = 0$ for all $l \in \{s+1, \dots, u\}$, because $N_{nl} \geqslant N_{il}$, and Equation 3.0.3 shows that $N_{il} = A_l - 1$. It follows that conclusion A holds, as

$$\frac{f_n}{x_k} \in (f_1, \dots, f_n) : (x_1, \dots, x_u).$$

(2) $x_k f_n = 0$ for all $k \in \{1, , \dots, s\}$. If $x_l f_n \neq 0$ for all $l \in \{s+1, \dots, u\}$, then conclusion B holds for $\{f_1, \dots, f_n\}$, $\{x_1, \dots, x_u\}$, with $S_i = S'_i$ for $i \leqslant n-1$, and $S_n = \{s+1, \dots, u\}$. Otherwise, assume that $x_l f_n = 0$ for some $l \in \{s+1, \dots, u\}$. Use Equation 3.0.3 to see that $x_l f_i = 0$ for all $i \in \{1, \dots, n\}$, and thus we are done by induction on the number of variables, by Claim 3.

Now assume that conclusion A holds for $\{f_1, \dots, f_{n-1}\}$ with respect to the variables $\{x_1, \dots x_s\}$. Without loss of generality, we may assume that

$$\frac{f_1}{x_1} \in (f_1, f_2, \dots f_{n-1}) : (x_1, \dots, x_s). \tag{3.0.4}$$

If

$$x_l \frac{f_1}{x_1} \in (f_n), \quad \text{for every} \quad s+1 \le l \le u, \tag{3.0.5}$$

then conclusion A would hold for $\{f_1, \dots, f_n\}$, $\{x_1, \dots, x_u\}$, and we would be done. We know that $x_l f_1 \in (f_n)$ for all $s+1 \le l \le u$ by equation 3.0.2. If $x_l f_1 = 0$ for all $s+1 \le l \le u$, or if $N_{11} > N_{n1}$ then equation 3.0.5 holds. Without loss of generality we may assume that

$$N_{11} \le N_{n1} \tag{3.0.6}$$

and $x_l f_1 \neq 0$ for some $s+1 \le l \le u$. By Claim 1, there exists just one value of l, say $l = s+1$ such that $x_l f_1 \neq 0$ (since we have $x_l f_1 \in (f_n)$ for all $l \geqslant s+1$). So we may assume

$$x_{s+1} f_1 \neq 0, \;\; N_{11} = N_{n1} \text{ and } x_l f_1 = 0, \text{ for all } s+2 \le l \le u \tag{3.0.7}$$

Claim 6: With the above assumptions, the following holds:

$$x_2 f_1 = \dots x_s f_1 = 0 \tag{3.0.8}$$

If, say, $x_2 f_1 \neq 0$, then

$$0 \neq x_2 \frac{f_1}{x_1} \in (f_i)$$

for some $i \le n-1$, which implies that $N_{11} > N_{i1}$ and $N_{1s+1} \geqslant N_{is+1}$. As, by equation 3.0.2, $x_{s+1} f_i \in (f_n)$ then we obtain the following two possibilities:

(1) either $x_{s+1} f_i = 0$, which implies $x_{s+1} f_1 = 0$, contradicting 3.0.7; or

(2) $N_{i1} \geq N_{n1}$, which implies $N_{11} > N_{n1}$, contradicting 3.0.6.

This proves Claim 6.

Because of Claim 5, we may assume that there exists an index j, such that $1 \leq j \leq s$ and

$$x_j \in (f_1) : (f_2, \ldots, f_n) \tag{3.0.9}$$

We may assume that

$$x_1 f_1 \neq 0, \text{ and therefore } x_1 f_n \neq 0 (\text{since} N_{11} = N_{n1}). \tag{3.0.10}$$

Otherwise, by 3.0.7 and 3.0.8, $x_l f_1 = 0$ for all $l \in \{1, \ldots, s, s+2, \ldots, u\}$, and it follows that condition A holds:

$$\frac{f_1}{x_{s+1}} \in (f_1, \ldots, f_n) : (x_1, \ldots, x_u).$$

The following cases finish the proof of the theorem.

(1) Assume $j = 1$. Since $x_1 f_n \in (f_1)$ and $N_{11} = N_{n1}$, by 3.0.7, then $x_1 f_n = 0$ contradicting 3.0.10.

(2) Assume $j \geq 2$. We may assume $j = 2$. By 3.0.7 and 3.0.8 we have $x_l f_1 = 0$ for all $l \neq 1, s+1$. We may assume that $x_1 f_1 \neq 0$, by 3.0.10.

(a) Assume that $x_2 f_n \neq 0$. We know $x_1 \in (f_i) : (f_1, \ldots, f_n)$ for some $i \in \{1, \ldots, n-1\}$. As $0 \neq x_2 f_n \in (f_1)$ and $x_1 f_n \neq 0$, by Claim (1) it follows that $2 \leq i \leq n-1$ (because $N_{12} > N_{n2} \geqslant N_{i2}$, so $i \neq 1$). For such an i, we claim that

$$\frac{f_i}{x_1} \in (f_1, \ldots, f_n) : (x_1, \ldots, x_u). \tag{3.0.11}$$

First notice that $N_{i1} = N_{n1} + 1 = N_{11} + 1$, since $0 \neq x_1 f_n \in (f_i)$ and by 3.0.6. Moreover, as $x_2 f_n \neq 0$, by multiplying $x_1 f_n$ by x_2 we obtain that $0 \neq x_2 f_i \in (f_1)$ (we have $x_2 f_i \in (f_1)$ by equation 3.0.9, and we have $x_2 f_i \neq 0$ because $N_{i2} \leqslant N_{n2}$). Moreover, $\dfrac{x_2 f_i}{x_1} \in (f_1)$, since $N_{i1} > N_{11}$). If $x_l f_i \neq 0$ for some $l \notin \{1, 2, s+1\}$, then $x_l f_1 \neq 0$, contradicting 3.0.6 and 3.0.7. As $x_{s+1} \in (f_n) : (f_1, \ldots, f_n)$, we obtain $x_{s+1} f_i \in (f_n)$ and since $N_{i1} = N_{n1} + 1$ also $x_{s+1} \frac{f_i}{x_1} \in (f_n)$.

(b) Assume that $x_2 f_n = 0$. If $x_2 f_i = 0$, for all $i \in \{1, \ldots, u\}$ then we are done by Claim 3. So we may assume that there is a $t \notin \{1, n\}$ such that $x_2 f_t \neq 0$ and $x_2 f_t \in (f_1)$. Therefore $N_{12} = N_{t2} + 1$. As $x_l f_t \in (f_n)$ for every $s+1 \leq l \leq u$, if $x_l f_t \neq 0$ then $A_2 - 1 = N_{n2} \leq N_{t2}$ which contradicts $x_2 f_t \neq 0$. Therefore we have that $x_l f_t = 0$ for all $s+1 \leq t \leq u$. Also, as $0 \neq x_2 f_t \in (f_1)$, we have $N_{tk} \geq N_{1k}$ for all $k \neq 2$. As $x_l f_1 = 0$ for all l
$notin \{1, s+1\}$, it follows that $x_l f_t = 0$ for all $l \notin \{1, 2\}$. If also $x_1 f_t = 0$ then conclusion A holds as

$$\frac{f_t}{x_2} \in (f_1, \ldots, f_n) : (x_1, \ldots, x_u).$$

Assume that $x_1 f_t \neq 0$. Recall that $x_1 \in (f_i) : (f_1, \ldots, f_n)$ for some $i \leqslant n-1$. We claim that

$$\frac{f_i}{x_1} \in (f_1, \ldots, f_n) : (x_1, \ldots, x_u).$$

As $0 \neq x_1 f_t \in (f_i)$, we have $N_{il} \leq N_{tl}$ for all $l \neq 1$. As $x_2 f_t \neq 0$ this implies that $x_2 f_i \neq 0$. As $x_2 f_i \in (f_1)$ by equation 3.0.9, and since $x_l f_1 = 0$ for $l \notin \{1, s+1\}$, we obtain that $x_l f_i = 0$ for $l \notin \{1, 2, s+1\}$. To prove the claim, it is therefore enough to prove that $\frac{f_i}{x_1} x_2 \in (f_1)$ and $\frac{f_i}{x_1} x_{s+1} \in (f_n)$. As $0 \neq x_1 f_1 \in (f_i)$ we obtain $N_{i1} = N_{11} + 1 = N_{n1} + 1$, where the last equality follows from 3.0.7. This, together with the fact that $x_2 f_i \in (f_1)$ by equation 3.0.9, and $x_{s+1} f_i \in (f_n)$ by equation 3.0.2 concludes the claim.

□

Now we give the proof of Theorem 3.2

Proof. We may apply Theorem 3.3 to $\{f_1, \dots, f_n\}$, $\{x_1, \dots, x_d\}$.

Indeed, the assumption that $R = S/0 :_S (f_1, \dots, f_n)$ is strongly almost Gorenstein implies hypothesis (2) of Theorem 3.3 by Proposition 5.2 in [**8**]. We may assume without loss of generality that (1) holds by choosing $f_1, \dots, f_n$ to be a minimal set of generators for the ideal they generate. In order to establish hypothesis (3), note that R does not change if we replace S by $S' = \mathsf{k}[x_1, \dots, x_d]/(x_1^{A_1+1}, \dots, x_d^{A_d})$, and $f_1, \dots, f_n$ by $f'_1, \dots, f'_n$, where $f'_i = (x_1 \cdots x_d) f_i$.

If (A) holds, then we may apply Theorem 0.2 to conclude that a copy of the residue field k splits off the second syzygy of ω_R, and the conclusion follows from Lemma 2.1. Take $K = (f_1, \dots, f_n) \subset S$. From conclusion (A) of Theorem 3.3, we have

$$\frac{f_i}{x_j} \in (K : \mathfrak{m}) \setminus (\mathfrak{m}K : \mathfrak{m})$$

which, by Remark 1.3, implies $J : \mathfrak{m} \neq \mathfrak{m}J : \mathfrak{m}$, where $J = 0 :_S K$, and now Theorem 0.2 applies.

If (B) holds, we will check that k is a direct summand of the first syzygy of ω_R, and again the conclusion follows from Lemma 2.1. Let $S_1, \dots, S_n$ be the sets asserted in Conclusion (B). We have

$$(x_1^{A_1}, \dots, x_d^{A_d}) : (f_1, \dots, f_n) = (x_1^{A_1}, \dots, x_d^{A_d}) + (x_j x_{j'} \,|\, j, j' \text{ not in the same } S_i).$$

The relations on the generators $f_1, \dots, f_n$ of ω_R are $x_j f_i = 0$ for $j \notin S_i$, and $(\Pi_{j \in S_i} x_j^{A_j - 1}) f_i - (\Pi_{j \in S_{i'}} x_j^{A_j - 1}) f_{i'} = 0$. Note that the latter relations are killed by the maximal ideal, thus each of them generates a copy of k which splits off the first syzygy.

□

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