A NOTE ON THE ASSOCIATED PRIMES OF THE THIRD POWER OF THE COVER IDEAL

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Abstract. An algebraic approach to graph theory involves the study of the edge ideal and the cover ideal of a given graph. While a lot is known for the associated primes of powers of the edge ideal, much less is known for the associated primes of the powers of the cover ideal. The associated primes of the cover ideal and its second power are completely determined. We show that the *centered odd holes* appear always among the associated primes of the third power of the cover ideal.

1. Introduction

We start the paper by introducing some definitions and notations, for which we follow [6] and [7]. In the following, a *graph* G consists of two finite sets, the vertex set $V_G = \{x_1, \dots, x_n\}$ and the edge set E_G whose elements are unordered pairs of vertices. To conserve notation, for elements $x_i, x_j \in V_G$, we denote the element $\{x_i, x_j\} \in E_G$ by $x_i x_j$, we say that the vertices x_i and x_j are *adjacent* and the edge $x_i x_j$ is *incident* to x_i or x_j. In the rest of the paper we assume that all graphs are *simple*, meaning that the only possible edges are $x_i x_j$ for $i \neq j$.

A subset $C \subseteq V_G$ is a *vertex cover* of G if each edge in E_G is incident to a vertex in C. A vertex cover C is a *minimal cover* if there is no proper subset of C which is a vertex cover of G.

The results of this paper are in the area of algebraic graph theory, where algebraic methods are used to investigate properties of graphs. Indeed, a graph G with vertex set $V_G = \{x_1, \dots, x_n\}$ can be related to the polynomial ring $R = \mathsf{k}[x_1, \dots, x_n]$, where k is a field. In the following we take the liberty to refer to x_i as a variable in the polynomial ring and as a vertex in the graph G, without any further specification. Given a ring R, we denote by $(f_1, \dots, f_l)$ the ideal of R generated by the elements $f_1, \dots, f_l \in R$.

Two ideals of the polynomial ring $R = \mathsf{k}[x_1, \dots, x_n]$ that have proven most useful in studying the properties of a graph G with vertex set $V_G = \{x_1, \dots, x_n\}$ and edge set E_G are the *edge ideal*

$$I_G = (x_i x_j \mid x_i x_j \in E_G)$$

and the *cover ideal*

$$J_G = (x_{i_1} \cdots x_{i_k} \mid x_{i_1}, \dots, x_{i_k} \text{ is a minimal cover of } G).$$

Note that both the edge ideal and the cover ideal of a graph are monomial square-free ideals, i.e. they are generated by monomials in which each variable appears at most one time.

Key words and phrases. Graph; polynomial ring; cover ideal; associated primes.

One of the most basic tools in commutative algebra to study an ideal I of a notherian ring R is to compute the finite set of *associated prime ideals* of I, which is denoted by $\mathrm{Ass}(R/I)$ (see for details [3]). In the case of a monomial ideal L in a polynomial ring $S = \mathsf{k}[x_1, \dots, x_n]$, an element in $\mathrm{Ass}(S/L)$ is a *monomial prime ideal*, which is an ideal generated by a subset of the variables. Because of this fact we can record the following definition.

Definition. Let L be a monomial ideal in the polynomial ring $S = \mathsf{k}[x_1, \dots, x_n]$ and let $P = (x_{i_1}, \dots, x_{i_s})$ a monomial prime ideal. If there exists a monomial m such that $x_{i_j} m \in L$ for each $j = 1, \dots, s$ and $x_i m \notin L$ for very $i \neq i_1, \dots, i_s$ then P is an *associated* prime to L. We denote by $\mathrm{Ass}(S/L)$ the set of all associated (monomial) primes of L.

In [2], the authors give a constructive method for determining primes associated to the powers of the edge ideal, but much less is know for cover ideals. It is known that, given a graph G and its cover ideal J_G, a monomial prime ideal P is in $Ass(S/J_G)$ if and only if $P = (x_i, x_j)$ and $x_i x_j$ is an edge of G, (see for example [7]).

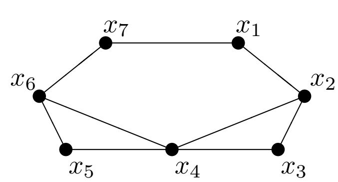
Before we can present the next result, we need some more definitions about graphs. Let G be a graph with vertices $\{x_1, \dots, x_n\}$. A *path* is a sequence of distinct vertices $x_{i_1}, x_{i_2}, \dots, x_{i_k}$ such that $x_{i_j} x_{i_{j+1}} \in E_G$ for $j = 1, 2, \dots, k-1$. The length of a path $x_{i_1}, x_{i_2}, \dots, x_{i_k}$ is given by the number of edges it includes, i.e. $k-1$. A *cycle* s is a path $x_{i_1,} x_{i_2}, \dots, x_{i_k}$, where we assume that $k \geq 3$, together with the edge $x_{i_k} x_{i_1}$. If there exists and l such that $k = 2l+1$, we call s an odd cycle, or *odd hole*. Given a graph G and a subset of the vertex set $W \subseteq V_G$, the *graph generated by* W has W has vertex set and for $x, y \in W$, xy is an edge of W if and only if it is an edge for G.

A recent result of Francisco, Ha and VanTuyl describes the associated primes of the ideal $(J_G)^2$, [4]. We state the result here as it is the initial point of our investigation.

Theorem 1.1. *Let G be a graph with vertex set $V_G = \{x_1, \dots, x_n\}$, edge set E_G and cover ideal J_G. A monomial prime ideal $P = (x_{i_1}, \dots, x_{i_k})$ of the polynomial ring $S = \mathsf{k}[x_1, \dots, x_n]$ is in the set $Ass(S/J_G^2)$ if and only if one of the following cases hold:*

- *$k = 2$ and $x_{i_1} x_{i_2} \in E_G$;*
- *k is odd and the graph generated by $x_{i_1}, \dots, x_k$ is an odd hole.*

Before we present our theorem we summarize the concepts above with an example. For the following graph G



we obtain the following

$$Ass(J) = \{(x_1, x_2), (x_1, x_7), (x_2, x_3), (x_2, x_4), (x_3, x_4), (x_4, x_5), (x_4, x_6), (x_5, x_6), (x_6, x_7)\}.$$

The associated prime of J is exactly all the primes generated by two variables which correspond to the edges of the graph.

$$Ass(J^2) = \{(x_1, x_2), (x_1, x_7), (x_2, x_3), (x_2, x_4), (x_3, x_4), (x_4, x_5), (x_4, x_6), (x_5, x_6), (x_6, x_7), (x_2, x_3, x_4), (x_4, x_5, x_6), (x_1, x_2, x_4, x_6, x_7)\}.$$

The associated prime of J_G^2 contains all the primes that are either generated by two variable corresponding to edges or that are generated by three variables corresponding to odd cycles of G. [1]

In this paper we study the associated primes of the third power of the cover ideal, the ideal J_G^3, where G is a graph. In particular we prove that the primes generated by the variables corresponding to the vertices of a *centered odd hole* always appear among the associated primes of J_G^3. The definition of centered odd holes and the proof of this statement will be given in Section 2.

The main result has connection with the coloring number of a graph, we discuss this at the end of Section 2.

2. Centered odd holes and the Main Theorem

Definition. A graph C is said to be a *centered odd hole* if the following two conditions hold:

(1) there is an odd hole H and a vertex y, called the *center of* C, such that $V_C = V_H \cup \{y\}$;
(2) the center y is incident to at least three vertices of H such that there are at least two and different odd cycles containing the center among the subgraphs of C.

In fact, there are three odd cycles containing the center, as H is an odd hole. For a centered odd hole, we need to define some invariants.

Definition. Let C be a centered odd hole, where H is the odd hole and y is the center. A vertex $x \in V_H$ is a *radial vertex* if xy is an edge of the centered odd hole. The number of radial vertices is the *radial number*. Assume that k is the radial number and that $x_1, \dots, x_k$ are the radial vertices then l_i will denote the length of the path in H from x_i to x_{i+1} for $i = 1, \dots, k-1$ and l_k will denote the length of the path in H from x_k to x_1.

Recall that given a graph G, the number of vertices of G is called the *size* of G and it is denoted by $|G|$. In the main theorem we will use the following lemma.

Lemma 2.1. *Let C be a centered odd hole, where H is the odd hole and y is the center. Let k be the radial number. If W is a vertex cover for C that contains y, then the inequality $|W| \geq \frac{|C|}{2} + 1$ holds. If W is a vertex cover for C that does not contain y, then the equality $|W| \geq k + \lfloor \frac{l_1-1}{2} \rfloor + \cdots + \lfloor \frac{l_k-1}{2} \rfloor$ holds. Moreover the following inequality holds:*

$$k + \lfloor \frac{l_1 - 1}{2} \rfloor + \cdots + \lfloor \frac{l_k - 1}{2} \rfloor \geq \frac{|C|}{2} + 1.$$

[1]All the computations in this paper are carried by the algebra system Macaulay2. Moreover Macaulay2 was used extenstensevely to find the patter that lead to the Main Theorem.

Proof. Let V_H be the vertex set of H. Assume that W contains the vertex y. The vertex set $W \cap V_H$ has to be a vertex cover for H. Moreover, since H is an odd hole, the cardinality of $W \cap V_H$ has to be at least $\frac{|H|+1}{2}$, which is equal to $\frac{|C|}{2}$. Therefore the cardinality of W is $\frac{|C|}{2} + 1$.

Assume now that W does not contain the vertex y. Let $x_1, \dots, x_k$ be the radial vertices. Since $y \notin W$, we obtain that all the radial vertices are in W. As $W \cap V_H$ is a cover of H, in the path from x_i to x_{i+1} we need at least $\lfloor \frac{l_i-1}{2} \rfloor$ vertices, for $i = 1, \dots, k-1$, and we need $\lfloor \frac{l_k-1}{2} \rfloor$ vertices for the path $x_1, \dots, x_k$.

To prove the last inequality consider the following chain of inequalities:

$$
\begin{aligned}
k + \lfloor \frac{l_1 - 1}{2} \rfloor + \cdots + \lfloor \frac{l_k - 1}{2} \rfloor &\geq \\
k + \frac{l_1 - 1}{2} + \cdots + \frac{l_k - 1}{2} &\geq \\
\frac{l_1}{2} + \cdots + \frac{l_k}{2} + \frac{k}{2} &\geq \\
\frac{l_1 + \cdots + l_k + 1}{2} + \frac{k-1}{2} &\geq \\
\frac{|C|}{2} + 1,
\end{aligned}
$$

where in the last inequality we used the fact that $k \geq 3$. □

In the following we will make an abuse of notation: if G is a graph with vertices $x_1, \dots, x_n$ and H is a subgraph generated by the vertices $x_{i_1}, \dots, x_{i_k}$, by H we also denote the prime monomial ideal $(x_{i_1} \dots, x_{i_k})$ in the polynomial ring $\mathsf{k}[x_1, \dots, x_n]$. Here is our main theorem.

Theorem 2.2. *Let G be a graph with vertex set $V_G = \{x_1, \dots, x_n\}$ and assume that there exists a subgraph C which is a centered odd hole. Let $S = \mathsf{k}[x_1, \dots, x_n]$ and let J be the cover ideal of G. Then the set $Ass(S/J^3)$ is not contained in the set* $\operatorname{Ass}(S/J^2)$ *and in fact* $C \in \operatorname{Ass}(S/J^3) \setminus \operatorname{Ass}(S/J^2)$.

Proof. By Lemma 2.11 in [5], we may assume that $G = C$ is a centered odd hole. Let H be the odd hole and y be the center of C. Let $x_1, x_2, \dots, x_k$ be the radial vertices. Denote by $x_{i_1}, \dots, x_{i_{l_k-1}}$ the vertices between x_i and x_{i+1}, for $i = 1, \dots, k-1$ and the vertices between x_k and x_1 for $i = k$. We will show that the ideal $C = (x_1, \dots, x_k, x_{i_j}, \dots, y)$ is in $Ass(S/J^3)$ but not in $Ass(S/J^2)$. The prime ideal C is not in $Ass(S/J^2)$ as it is not an odd hole nor an edge, see Theorem 1.1. To show that C is in $Ass(S/J^3)$ we need to find a monomial c such that $c \notin J^3$ and $xc \in J^3$ for each vertex x of V_C. Let c be the monomial

$$c = y^2 \prod_{i=1,\dots,k} x_i^2 \prod_{i=1,\dots,k;\, j=1,\dots,l_i-1} x_{ij}^a,$$

where $a = 1$ if j is odd and $a = 2$ if j is even.

We now prove that c is the desired monomial, and to do so we first establish the following claim. Let n be the size of H. For a monomial m we denote by $\deg(m)$ the degree of m.

Claim 1: The equality $\deg(c) = k + 2 + n + \lfloor \frac{l_1-1}{2} \rfloor + \cdots + \lfloor \frac{l_k-1}{2} \rfloor$ holds.

Proof of Claim 1: In computing $\deg(c)$, the contribution from the variables y and x_i, for $i = 1, \dots, k$, is given by $2k+2$. For $i = 1, \dots, k-1$, between x_i and x_{i+1}, there are $l_i - 1$ vertices, and there are $l_k - 1$ vertices between x_k and x_1. Given an integer s, there are $\lfloor \frac{s}{2} \rfloor$ even integers and $\lceil \frac{s}{2} \rceil$ odd integers between 1 and s. Therefore, in computing $\deg(c)$, the contribution from the variables x_{ij} is given by

$$2\lfloor \frac{l_1-1}{2} \rfloor + \cdots + 2\lfloor \frac{l_k-1}{2} \rfloor + \lceil \frac{l_1-1}{2} \rceil + \cdots + \lceil \frac{l_k-1}{2} \rceil.$$

The degree of the monomial c is therefore equal to

$$\begin{aligned}
2k+2+2\lfloor \frac{l_1-1}{2} \rfloor + \cdots + 2\lfloor \frac{l_k-1}{2} \rfloor + \lceil \frac{l_1-1}{2} \rceil + \cdots + \lceil \frac{l_k-1}{2} \rceil &= \\
2k+2+(\lfloor \frac{l_1-1}{2} \rfloor + \lceil \frac{l_1-1}{2} \rceil) + \cdots + (\lfloor \frac{l_k-1}{2} \rfloor + \lceil \frac{l_k-1}{2} \rceil) + \lfloor \frac{l_1-1}{2} \rfloor + \cdots + \lfloor \frac{l_k-1}{2} \rfloor &= \\
k+2+k+(l_1-1)+\cdots+(l_k-1) + \lfloor \frac{l_1-1}{2} \rfloor + \cdots + \lfloor \frac{l_k-1}{2} \rfloor &= \\
k+2+l_1+\cdots+l_k + \lfloor \frac{l_1-1}{2} \rfloor + \cdots + \lfloor \frac{l_k-1}{2} \rfloor &= \\
k+2+n + \lfloor \frac{l_1-1}{2} \rfloor + \cdots + \lfloor \frac{l_k-1}{2} \rfloor&.
\end{aligned}$$

The last line of the equalities establishes *Claim 1*.

We now prove that c does not belong to J^3. For this we will show the following strict inequality

Claim 2: $\deg(c) < 2(\frac{|C|}{2}+1) + k + \lfloor \frac{l_1-1}{2} \rfloor + \cdots + \lfloor \frac{l_k-1}{2} \rfloor$.

If the claim is true, then $c \notin J^3$. In fact if, by contradiction, we assume that $c \in J^3$ then we can write $c = hm_1m_2m_3$ with $m_i \in J$ for $i = 1, 2, 3$. Since $m_i \in J$, the variables that appear in m_i correspond to a cover of the graph C. Moreover at least one cover must not contain the center y, we assume the monomial m_3 corresponds to such cover. By Lemma 2.1 the minimal cover has at least $\frac{|C|}{2}+1$ vertices if the cover contains the center y and at least $k + \lfloor \frac{l_1-1}{2} \rfloor + \cdots + \lfloor \frac{l_k-1}{2} \rfloor$ if the cover does not contain y. Using the inequality from Lemma 2.1 to get the first inequality below, we obtain

$$\begin{aligned}
\deg(c) &= \deg(h) + \deg(m_1) + \deg(m_2) + \deg(m_3) \\
&\geq \deg(h) + 2(\frac{|C|}{2}+1) + k + \lfloor \frac{l_1-1}{2} \rfloor + \cdots + \lfloor \frac{l_k-1}{2} \rfloor \\
&\geq 2(\frac{|C|}{2}+1) + k + \lfloor \frac{l_1-1}{2} \rfloor + \cdots + \lfloor \frac{l_k-1}{2} \rfloor,
\end{aligned}$$

which will contradict *Claim 2*.

Proof of Claim 2: Assume by contradiction that

$$\deg(c) \geq 2(\frac{|C|}{2}+1) + k + \lfloor \frac{l_1-1}{2} \rfloor + \cdots + \lfloor \frac{l_k-1}{2} \rfloor,$$

then by *Claim 1*, we obtain that

$$k+2+n+\lfloor \frac{l_1-1}{2} \rfloor + \cdots + \lfloor \frac{l_k-1}{2} \rfloor \geq 2(\frac{|C|}{2}+1) + k + \lfloor \frac{l_1-1}{2} \rfloor + \cdots + \lfloor \frac{l_k-1}{2} \rfloor,$$

which means that

$$2 + n \geq 2(\frac{|C|)}{2} + 1).$$

As n is the size of H, $|C|$ is the number of vertices of the centered odd hole, and $C = H \cup \{y\}$ we have that $|C| = n + 1$. This implies that

$$2 + n \geq 2(\frac{n+1}{2} + 1) = n + 2 + 1,$$

which is a contradiction.

To finish the proof we need to show that for every vertex $x \in V_C$ we have $xc \in J^3$. We do so in the following claims.

Let x be any vertex of H and relabel the vertices of H starting from $x = t_1$ clockwise $t_2, \ldots, t_n$, where n is the size of H. We can write $xc = m_1 m_2 m_3$ where

$$\begin{aligned} m_1 &= y \prod_{i \text{ odd}} t_i, \\ m_2 &= y t_1 \prod_{i \text{ even}} t_i, \quad \text{and} \\ m_3 &= \prod_{i=1,\ldots,k} x_i \prod_{i=1\ldots,k;\ j \text{ even}} x_{ij}. \end{aligned}$$

Note that m_1 and m_2 correspond to covers as they contain y and every other vertex of the outside cycle. Also m_3 corresponds to a cover as all the x_i are included, and therefore all the edges connecting y to the outside cycle are covered, and every other vertex in the path from x_i to x_{i+1} is included.

Finally we need to write $yc = m_1 m_2 m_3$ with $m_i \in J$ for $i = 1, 2, 3$. For this assume that x_1 is such that the path from x_k to x_1 is odd. Relabel the vertices $x_1 = t_1$ and then clockwise to t_n. Let

$$m_1 = y \prod_{i \, \text{odd}} t_i.$$

Note that m_1 will give a cover as we are considering every other vertex in the odd cycle and the vertex y. Now let l the least even number so that t_l corresponds to a radial vertex x_g, for some g. Set

$$m_2 = y \prod_{l \leq i \leq n,\, i \, \text{even}} t_i \prod_{1 \leq i \leq l,\, i \, \text{odd}} t_i.$$

Because we are considering every other vertex from t_1 to t_{l-1}, every other vertex from t_l, and the center y, the monomial m_2 corresponds to a cover of the centered odd hole.

Finally

$$m_3 = y x_g x_{g+1} \ldots x_k \prod_{i=g,\ldots k, j \, \text{even}} x_{ij} \prod_{i=1,\ldots l-1,\, i \, \text{even}} t_i.$$

Also m_3 gives a cover as it contains every other vertex from t_2 to $t_l = x_g$, every other vertex from x_i to x_{i+1}, for $i = g, \ldots, k-1$, every other vertex from x_k to x_1, and the center y. Notice that x_1 is missing from the monomial m_3 but the vertex y is listed in the monomial as for the vertex preceding x_1, because of the assumption that the path $x_k, \ldots, x_1$ in H is odd. $\square$

For every ideal I in a polynomial ring S (and in fact in a more general ring), one can compute the sequence of sets $\mathrm{Ass}(S/I^n)$ for $n \in \mathbb{N}$. It is a theorem in [1], proved in a much larger generality, that there exists a positive integer a such that

$$\cup_{i=1}^{a_I} \mathrm{Ass}(S/I^i) = \cup_{i=1}^{\infty} \mathrm{Ass}(S/I^i). \tag{2.1}$$

Very little is known about the value of the positive integer a_I. In [5], the authors give an upper bound for a_I in the case that I is an edge ideal for some given graph.

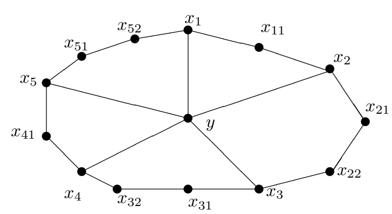
The value of a_J, where J is the cover ideal of a graph G, is related to the coloring number of G. More in detail, given a graph G, the *coloring number* for G is the least number of colors that one needs to color the vertices of the graph so that two adjacent vertices have different colors. We denote the coloring number of a graph G by $\chi(G)$. In [5], the authors show that, in equation 2.1, $a_J \geq \chi(G) - 1$ for J being the cover ideal of the graph G. In the same paper, they look at some examples for which $a_J > \chi(G) - 1$. Centered odd holes give an infinite family of examples for which the inequality $a_J > \chi(G) - 1$ is satisfied.

Corollary 2.3. *Let C be a centered odd hole with cover ideal J. Assume that there exists a vertex of C which is not radial, then $a_J \geq \chi(C)$.*

Proof. Because C contains an odd hole, one needs at least three colors for the vertexes of C. We first show that $\chi(C) = 3$. Let $\{a, b, c\}$ be a list of three colors. Assume that x is a vertex of C which is not radial. Color the vertex x and the center y with c, and finally color the remaining vertices alternating a and b.

The main Theorem implies that $a_J \geq 3$. □

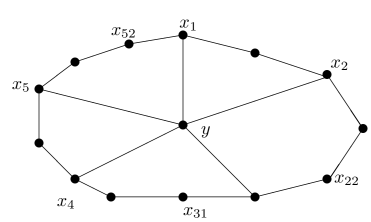
We finish the paper with an example that illustrates the idea behind the proof of the main theorem. Consider the centered odd hole G



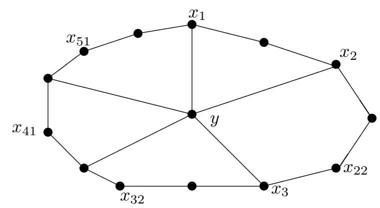
The monomial c used in the proof of the Main Theorem, is given by

$$c = x_1^2 x_2^2 x_3^2 x_4^2 x_5^2 x_{22}^2 x_{32}^2 x_{52}^2 x_{11} x_{21} x_{31} x_{41} x_{51}.$$

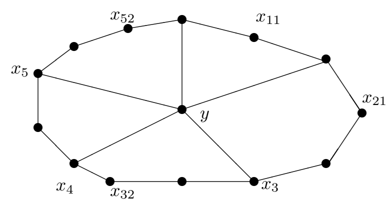
We can write the monomial $yc = m_1 m_2 m_3$, where the monomial m_1 corresponds to the cover



The monomial m_2 corresponds to the cover



Finally, the monomial m_3 corresponds to the cover



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