A NOTE ON THE ASSOCIATED PRIMES OF THE THIRD POWER OF THE COVER IDEAL

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Abstract. An algebraic approach to graph theory involves the study of the edge ideal and the cover ideal of a given graph. While a lot is known for the associated primes of powers of the edge ideal, much less is known for the associated primes of the powers of the cover ideal. The associated primes of the cover ideal and its second power are completely determined. We show that the centered odd holes appear always among the associated primes of the third power of the cover ideal.

1. Introduction

We start the paper by introducing some definitions and notations, for which we follow [6] and [7]. In the following, a graph $G$ consists of two finite sets, the vertex set $V_G = \{x_1,\ldots,x_n\}$ and the edge set $E_G$ whose elements are unordered pairs of vertices. To conserve notation, for elements $x_i, x_j \in V_G$, we denote the element $\{x_i, x_j\} \in E_G$ by $x_ix_j$, we say that the vertices $x_i$ and $x_j$ are adjacent and the edge $x_ix_j$ is incident to $x_i$ or $x_j$. In the rest of the paper we assume that all graphs are simple, meaning that the only possible edges are $x_ix_j$ for $i \neq j$.

A subset $C \subseteq V_G$ is a vertex cover of $G$ if each edge in $E_G$ is incident to a vertex in $C$. A vertex cover $C$ is a minimal cover if there is no proper subset of $C$ which is a vertex cover of $G$.

The results of this paper are in the area of algebraic graph theory, where algebraic methods are used to investigate properties of graphs. Indeed, a graph $G$ with vertex set $V_G = \{x_1,\ldots,x_n\}$ can be related to the polynomial ring $R = k[x_1,\ldots,x_n]$, where $k$ is a field. In the following we take the liberty to refer to $x_i$ as a variable in the polynomial ring and as a vertex in the graph $G$, without any further specification. Given a ring $R$, we denote by $(f_1,\ldots,f_l)$ the ideal of $R$ generated by the elements $f_1,\ldots,f_l \in R$.

Two ideals of the polynomial ring $R = k[x_1,\ldots,x_n]$ that have proven most useful in studying the properties of a graph $G$ with vertex set $V_G = \{x_1,\ldots,x_n\}$ and edge set $E_G$ are the edge ideal

\[ I_G = (x_ix_j \mid x_ix_j \in E_G) \]

and the cover ideal

\[ J_G = (x_{i_1}\cdots x_{i_k} \mid x_{i_1},\ldots,x_{i_k} \text{ is a minimal cover of } G). \]

Note that both the edge ideal and the cover ideal of a graph are monomial square-free ideals, i.e. they are generated by monomials in which each variable appears at most one time.

Key words and phrases. Graph; polynomial ring; cover ideal; associated primes.
One of the most basic tools in commutative algebra to study an ideal $I$ of a noetherian ring $R$ is to compute the finite set of associated prime ideals of $I$, which is denoted by $\text{Ass}(R/I)$ (see for details [3]). In the case of a monomial ideal $L$ in a polynomial ring $S = k[x_1, \ldots, x_n]$, an element in $\text{Ass}(S/L)$ is a monomial prime ideal, which is an ideal generated by a subset of the variables. Because of this fact we can record the following definition.

**Definition.** Let $L$ be a monomial ideal in the polynomial ring $S = k[x_1, \ldots, x_n]$ and let $P = (x_{i_1}, \ldots, x_{i_r})$ a monomial prime ideal. If there exists a monomial $m$ such that $x_i m \in L$ for each $j = 1, \ldots, s$ and $x_i m \notin L$ for very $i \neq i_1, \ldots, i_s$ then $P$ is an associated prime to $L$. We denote by $\text{Ass}(S/L)$ the set of all associated (monomial) primes of $L$.

In [2], the authors give a constructive method for determining primes associated to the powers of the edge ideal, but much less is know for cover ideals. It is known that, given a graph $G$ and its cover ideal $J_G$, a monomial prime ideal $P$ is in $\text{Ass}(S/J_G)$ if and only if $P = (x_i, x_j)$ and $x_i x_j$ is an edge of $G$, (see for example [7]).

Before we can present the next result, we need some more definitions about graphs. Let $G$ be a graph with vertices $\{x_1, \ldots, x_n\}$. A path is a sequence of distinct vertices $x_{i_1}, x_{i_2}, \ldots, x_{i_k}$ such that $x_{i_j} x_{i_{j+1}} \in E_G$ for $j = 1, 2, \ldots, k - 1$. The length of a path $x_{i_1}, x_{i_2}, \ldots, x_{i_k}$ is given by the number of edges it includes, i.e. $k - 1$. A cycle is a path $x_{i_1}, x_{i_2}, \ldots, x_{i_k}$, where we assume that $k \geq 3$, together with the edge $x_{i_k} x_{i_1}$. If there exists $l$ such that $k = 2l + 1$, we call $s$ an odd cycle, or odd hole. Given a graph $G$ and a subset of the vertex set $W \subseteq V_G$, the graph generated by $W$ has $W$ has vertex set and for $x, y \in W$, $xy$ is an edge of $W$ if and only if it is an edge for $G$.

A recent result of Francisco, Ha and VanTuyl describes the associated primes of the ideal $(J_G)^2$, [4]. We state the result here as it is the initial point of our investigation.

**Theorem 1.1.** Let $G$ be a graph with vertex set $V_G = \{x_1, \ldots, x_n\}$, edge set $E_G$ and cover ideal $J_G$. A monomial prime ideal $P = (x_{i_1}, \ldots, x_{i_k})$ of the polynomial ring $S = k[x_1, \ldots, x_n]$ is in the set $\text{Ass}(S/J_G^2)$ if and only if one of the following cases hold:

- $k = 2$ and $x_{i_1} x_{i_2} \in E_G$;
- $k$ is odd and the graph generated by $x_{i_1}, \ldots, x_k$ is an odd hole.

Before we present our theorem we summarize the concepts above with an example. For the following graph $G$

![Graph Example](image)

we obtain the following

$\text{Ass}(J) = \{(x_1, x_2), (x_1, x_7), (x_2, x_3), (x_2, x_4), (x_3, x_4), (x_4, x_5), (x_4, x_6), (x_5, x_6), (x_6, x_7)\}$. 
The associated prime of $J$ is exactly all the primes generated by two variables which correspond to the edges of the graph.

$$\text{Ass}(J^2) = \{(x_1, x_2), (x_1, x_7), (x_2, x_3), (x_2, x_4), (x_3, x_4), (x_4, x_5), (x_4, x_6), (x_5, x_6),$$
$$ (x_6, x_7), (x_2, x_3, x_4), (x_4, x_5, x_6), (x_1, x_2, x_4, x_6, x_7)\}.$$

The associated prime of $J^3_G$ contains all the primes that are either generated by two variables corresponding to edges or that are generated by three variables corresponding to odd cycles of $G$. \(^1\)

In this paper we study the associated primes of the third power of the cover ideal, the ideal $J^3_G$, where $G$ is a graph. In particular we prove that the primes generated by the variables corresponding to the vertices of a centered odd hole always appear among the associated primes of $J^3_G$. The definition of centered odd holes and the proof of this statement will be given in Section 2.

The main result has connection with the coloring number of a graph, we discuss this at the end of Section 2.

2. Centered odd holes and the Main Theorem

**Definition.** A graph $C$ is said to be a centered odd hole if the following two conditions hold:

1. there is an odd hole $H$ and a vertex $y$, called the center of $C$, such that $V_C = V_H \cup \{y\}$;
2. the center $y$ is incident to at least three vertices of $H$ such that there are at least two and different odd cycles containing the center among the subgraphs of $C$.

In fact, there are three odd cycles containing the center, as $H$ is an odd hole. For a centered odd hole, we need to define some invariants.

**Definition.** Let $C$ be a centered odd hole, where $H$ is the odd hole and $y$ is the center. A vertex $x \in V_H$ is a radial vertex if $xy$ is an edge of the centered odd hole. The number of radial vertices is the radial number. Assume that $k$ is the radial number and that $x_1, \ldots, x_k$ are the radial vertices then $l_i$ will denote the length of the path in $H$ from $x_i$ to $x_{i+1}$ for $i = 1, \ldots, k-1$ and $l_k$ will denote the length of the path in $H$ from $x_k$ to $x_1$.

Recall that given a graph $G$, the number of vertices of $G$ is called the size of $G$ and it is denoted by $|G|$. In the main theorem we will use the following lemma.

**Lemma 2.1.** Let $C$ be a centered odd hole, where $H$ is the odd hole and $y$ is the center. Let $k$ be the radial number. If $W$ is a vertex cover for $C$ that contains $y$, then the inequality $|W| \geq \frac{|C|}{2} + 1$ holds. If $W$ is a vertex cover for $C$ that does not contain $y$, then the equality $|W| \geq k + \left\lfloor \frac{l_1 - 1}{2} \right\rfloor + \cdots + \left\lfloor \frac{l_k - 1}{2} \right\rfloor$ holds. Moreover the following inequality holds:

$$k + \left\lfloor \frac{l_1 - 1}{2} \right\rfloor + \cdots + \left\lfloor \frac{l_k - 1}{2} \right\rfloor \geq \frac{|C|}{2} + 1.$$  

\(^1\)All the computations in this paper are carried by the algebra system Macaulay2. Moreover Macaulay2 was used extensively to find the pattern that lead to the Main Theorem.
Proof. Let $V_H$ be the vertex set of $H$. Assume that $W$ contains the vertex $y$. The vertex set $W \cap V_H$ has to be a vertex cover for $H$. Moreover, since $H$ is an odd hole, the cardinality of $W \cap V_H$ has to be at least $\frac{|H|+1}{2}$, which is equal to $\frac{|C|}{2}$. Therefore the cardinality of $W$ is $\frac{|C|}{2} + 1$.

Assume now that $W$ does not contain the vertex $y$. Let $x_1, \ldots, x_k$ be the radial vertices. Since $y \notin W$, we obtain that all the radial vertices are in $W$. As $W \cap V_H$ is a cover of $H$, in the path from $x_i$ to $x_{i+1}$ we need at least $\lfloor \frac{k-1}{2} \rfloor$ vertices, for $i = 1, \ldots, k-1$, and we need $\lfloor \frac{k-1}{2} \rfloor$ vertices for the path $x_1, \ldots, x_k$.

To prove the last inequality consider the following chain of inequalities:

$$
k + \left\lfloor \frac{l_1 - 1}{2} \right\rfloor + \cdots + \left\lfloor \frac{l_k - 1}{2} \right\rfloor \geq \frac{k + l_1 - 1}{2} + \cdots + \frac{l_k - 1}{2} \geq \frac{l_1}{2} + \cdots + \frac{l_k}{2} + \frac{k}{2} \geq \frac{l_1 + \cdots + l_k + 1}{2} + \frac{k - 1}{2} \geq \frac{|C|}{2} + 1,$$

where in the last inequality we used the fact that $k \geq 3$. \hfill $\square$

In the following we will make an abuse of notation: if $G$ is a graph with vertices $x_1, \ldots, x_n$ and $H$ is a subgraph generated by the vertices $x_{i_1}, \ldots, x_{i_k}$, by $H$ we also denote the prime monomial ideal $(x_{i_1}, \ldots, x_{i_k})$ in the polynomial ring $k[x_1, \ldots, x_n]$. Here is our main theorem.

**Theorem 2.2.** Let $G$ be a graph with vertex set $V_G = \{x_1, \ldots, x_n\}$ and assume that there exists a subgraph $C$ which is a centered odd hole. Let $S = k[x_1, \ldots, x_n]$ and let $J$ be the cover ideal of $G$. Then the set $\text{Ass}(S/J^3)$ is not contained in the set $\text{Ass}(S/J^2)$ and in fact $C \in \text{Ass}(S/J^3) \setminus \text{Ass}(S/J^2)$.

**Proof.** By Lemma 2.11 in [5], we may assume that $G = C$ is a centered odd hole. Let $H$ be the odd hole and $y$ be the center of $C$. Let $x_1, x_2, \ldots, x_k$ be the radial vertices. Denote by $x_{i_1}, \ldots, x_{i_{k-1}}$ the vertices between $x_1$ and $x_{i+1}$, for $i = 1, \ldots, k-1$ and the vertices between $x_k$ and $x_1$ for $i = k$. We will show that the ideal $C = (x_1, \ldots, x_k, x_{i_1}, \ldots, y)$ is in $\text{Ass}(S/J^3)$ but not in $\text{Ass}(S/J^2)$. The prime ideal $C$ is not in $\text{Ass}(S/J^2)$ as it is not an odd hole nor an edge, see Theorem 1.1. To show that $C$ is in $\text{Ass}(S/J^3)$ we need to find a monomial $c$ such that $c \notin J^3$ and $xc \notin J^3$ for each vertex $x$ of $V_C$. Let $c$ be the monomial

$$c = y^2 \prod_{i=1}^{k} x_i^2 \prod_{i=1}^{k} \prod_{j=1}^{l} x_{ij}^a,$$

where $a = 1$ if $j$ is odd and $a = 2$ if $j$ is even.

We now prove that $c$ is the desired monomial, and to do so we first establish the following claim. Let $n$ be the size of $H$. For a monomial $m$ we denote by $\text{deg}(m)$ the degree of $m$.

**Claim 1.** The equality $\text{deg}(c) = k + 2 + n + \left\lfloor \frac{k-1}{2} \right\rfloor + \cdots + \left\lfloor \frac{k-1}{2} \right\rfloor$ holds.
Proof of Claim 1: In computing $\deg(c)$, the contribution from the variables $y$ and $x_i$, for $i = 1, \ldots, k$, is given by $2k + 2$. For $i = 1, \ldots, k - 1$, between $x_i$ and $x_{i+1}$, there are $l_i - 1$ vertices, and there are $l_k - 1$ vertices between $x_k$ and $x_1$. Given an integer $s$, there are $\lfloor \frac{s}{2} \rfloor$ even integers and $\lceil \frac{s}{2} \rceil$ odd integers between 1 and $s$. Therefore, in computing $\deg(c)$, the contribution from the variables $x_{ij}$ is given by

$$2\lfloor \frac{l_1 - 1}{2} \rfloor + \ldots + 2\lfloor \frac{l_k - 1}{2} \rfloor + \lfloor \frac{l_1 - 1}{2} \rfloor + \ldots + \lfloor \frac{l_k - 1}{2} \rfloor.$$

The degree of the monomial $c$ is therefore equal to

$$2k + 2 + 2\lfloor \frac{l_1 - 1}{2} \rfloor + \ldots + 2\lfloor \frac{l_k - 1}{2} \rfloor + \lfloor \frac{l_1 - 1}{2} \rfloor + \ldots + \lfloor \frac{l_k - 1}{2} \rfloor =$$

$$2k + 2 + (\lfloor \frac{l_1 - 1}{2} \rfloor + \lfloor \frac{l_1 - 1}{2} \rfloor) + \ldots + (\lfloor \frac{l_k - 1}{2} \rfloor + \lfloor \frac{l_k - 1}{2} \rfloor) + \lfloor \frac{l_1 - 1}{2} \rfloor + \ldots + \lfloor \frac{l_k - 1}{2} \rfloor =$$

$$k + 2 + k + (l_1 - 1) + \ldots + (l_k - 1) + \lfloor \frac{l_1 - 1}{2} \rfloor + \ldots + \lfloor \frac{l_k - 1}{2} \rfloor =$$

$$k + 2 + l_1 + \ldots + l_k + \lfloor \frac{l_1 - 1}{2} \rfloor + \ldots + \lfloor \frac{l_k - 1}{2} \rfloor =$$

$$k + 2 + n + \lfloor \frac{l_1 - 1}{2} \rfloor + \ldots + \lfloor \frac{l_k - 1}{2} \rfloor.$$

The last line of the equalities establishes Claim 1.

We now prove that $c$ does not belong to $J^3$. For this we will show the following strict inequality

Claim 2: $\deg(c) < 2(\lfloor \frac{|C|}{2} \rfloor + 1) + k + \lfloor \frac{l_1 - 1}{2} \rfloor + \ldots + \lfloor \frac{l_k - 1}{2} \rfloor$.

If the claim is true, then $c \notin J^3$. In fact if, by contradiction, we assume that $c \in J^3$ then we can write $c = hm_1m_2m_3$ with $m_i \in J$ for $i = 1, 2, 3$. Since $m_i \in J$, the variables that appear in $m_i$ correspond to a cover of the graph $C$. Moreover at least one cover must not contain the center $y$, we assume the monomial $m_3$ corresponds to such cover. By Lemma 2.1 the minimal cover has at least $\lfloor \frac{|C|}{2} \rfloor + 1$ vertices if the cover contains the center $y$ and at least $k + \lfloor \frac{l_1 - 1}{2} \rfloor + \ldots + \lfloor \frac{l_k - 1}{2} \rfloor$ if the cover does not contain $y$. Using the inequality from Lemma 2.1 to get the first inequality below, we obtain

$$\deg(c) = \deg(h) + \deg(m_1) + \deg(m_2) + \deg(m_3)$$

$$\geq \deg(h) + 2(\lfloor \frac{|C|}{2} \rfloor + 1) + k + \lfloor \frac{l_1 - 1}{2} \rfloor + \ldots + \lfloor \frac{l_k - 1}{2} \rfloor$$

$$\geq 2(\lfloor \frac{|C|}{2} \rfloor + 1) + k + \lfloor \frac{l_1 - 1}{2} \rfloor + \ldots + \lfloor \frac{l_k - 1}{2} \rfloor,$$

which will contradict Claim 2.

Proof of Claim 2: Assume by contradiction that

$$\deg(c) \geq 2(\lfloor \frac{|C|}{2} \rfloor + 1) + k + \lfloor \frac{l_1 - 1}{2} \rfloor + \ldots + \lfloor \frac{l_k - 1}{2} \rfloor,$$

then by Claim 1, we obtain that

$$k + 2 + n + \lfloor \frac{l_1 - 1}{2} \rfloor + \ldots + \lfloor \frac{l_k - 1}{2} \rfloor \geq 2(\lfloor \frac{|C|}{2} \rfloor + 1) + k + \lfloor \frac{l_1 - 1}{2} \rfloor + \ldots + \lfloor \frac{l_k - 1}{2} \rfloor,$$
which means that
\[ 2 + n \geq 2(\left\lceil \frac{|C|}{2} \right\rceil + 1). \]
As \( n \) is the size of \( H \), \(|C|\) is the number of vertices of the centered odd hole, and \( C = H \cup \{y\} \) we have that \(|C| = n + 1\). This implies that
\[ 2 + n \geq 2\left(\frac{n + 1}{2} + 1\right) = n + 2 + 1, \]
which is a contradiction.

To finish the proof we need to show that for every vertex \( x \in V_C \) we have \( xc \in J^3 \).

Let \( x \) be any vertex of \( H \) and relabel the vertices of \( H \) starting from \( x = t_1 \) clockwise \( t_2, \ldots, t_n \), where \( n \) is the size of \( H \). We can write \( xc = m_1 m_2 m_3 \) where
\[
\begin{align*}
m_1 &= y \prod_{i \text{ odd}} t_i, \\
m_2 &= y t_1 \prod_{i \text{ even}} t_i, \quad \text{and} \\
m_3 &= \prod_{i=1}^{k} x_i \prod_{i=1}^{k} x_{ij}.
\end{align*}
\]
Note that \( m_1 \) and \( m_2 \) correspond to covers as they contain \( y \) and every other vertex of the outside cycle. Also \( m_3 \) corresponds to a cover as all the \( x_i \) are included, and therefore all the edges connecting \( y \) to the outside cycle are covered, and every other vertex in the path from \( x_i \) to \( x_{i+1} \) is included.

Finally we need to write \( yc = m_1 m_2 m_3 \) with \( m_i \in J \) for \( i = 1, 2, 3 \). For this assume that \( x_1 \) is such that the path from \( x_k \) to \( x_1 \) is odd. Relabel the vertices \( x_1 = t_1 \) and then clockwise to \( t_n \). Let
\[ m_1 = y \prod_{i \text{ odd}} t_i. \]
Note that \( m_1 \) will give a cover as we are considering every other vertex in the odd cycle and the vertex \( y \). Now let \( l \) the least even number so that \( t_l \) corresponds to a radial vertex \( x_g \), for some \( g \). Set
\[ m_2 = y \prod_{l \leq i \leq n, i \text{ even}} t_i \prod_{1 \leq i \leq l, i \text{ odd}} t_i. \]
Because we are considering every other vertex from \( t_1 \) to \( t_{l-1} \), every other vertex from \( t_l \), and the center \( y \), the monomial \( m_2 \) corresponds to a cover of the centered odd hole.

Finally
\[ m_3 = y x_g x_{g+1} \ldots x_k \prod_{i=g, \ldots, k, j \text{ even}} x_{ij} \prod_{i=1, \ldots, l-1, i \text{ even}} t_i. \]
Also \( m_3 \) gives a cover as it contains every other vertex from \( t_2 \) to \( t_l = x_g \), every other vertex from \( x_i \) to \( x_{i+1} \), for \( i = g, \ldots, k - 1 \), every other vertex from \( x_k \) to \( x_1 \), and the center \( y \). Notice that \( x_1 \) is missing from the monomial \( m_3 \) but the vertex \( y \) is listed in the monomial as for the vertex preceding \( x_1 \), because of the assumption that the path \( x_k, \ldots, x_1 \) in \( H \) is odd. \( \square \)
For every ideal $I$ in a polynomial ring $S$ (and in fact in a more general ring), one can compute the sequence of sets $\text{Ass}(S/I^n)$ for $n \in \mathbb{N}$. It is a theorem in [1], proved in a much larger generality, that there exists a positive integer $a$ such that

$$\bigcup_{i=1}^{a} \text{Ass}(S/I^i) = \bigcup_{i=1}^{\infty} \text{Ass}(S/I^i).$$

(2.1)

Very little is known about the value of the positive integer $a_I$. In [5], the authors give an upper bound for $a_I$ in the case that $I$ is an edge ideal for some given graph.

The value of $a_J$, where $J$ is the cover ideal of a graph $G$, is related to the coloring number of $G$. More in detail, given a graph $G$, the coloring number for $G$ is the least number of colors that one needs to color the vertices of the graph so that two adjacent vertices have different colors. We denote the coloring number of a graph $G$ by $\chi(G)$. In [5], the authors show that, in equation 2.1, $a_J \geq \chi(G) - 1$ for $J$ being the cover ideal of the graph $G$. In the same paper, they look at some examples for which $a_J > \chi(G) - 1$. Centered odd holes give an infinite family of examples for which the inequality $a_J > \chi(G) - 1$ is satisfied.

**Corollary 2.3.** Let $C$ be a centered odd hole with cover ideal $J$. Assume that there exists a vertex of $C$ which is not radial, then $a_J \geq \chi(C)$.

**Proof.** Because $C$ contains an odd hole, one needs at least three colors for the vertexes of $C$. We first show that $\chi(C) = 3$. Let $\{a, b, c\}$ be a list of three colors. Assume that $x$ is a vertex of $C$ which is not radial. Color the vertex $x$ and the center $y$ with $c$, and finally color the remaining vertices alternating $a$ and $b$.

The main Theorem implies that $a_J \geq 3$. \qed

We finish the paper with an example that illustrates the idea behind the proof of the main theorem. Consider the centered odd hole $G$

![Diagram](image)

The monomial $c$ used in the proof of the Main Theorem, is given by

$$c = x_1^2x_2^2x_3^2x_4^2x_5^2x_6^2x_7^2x_8^2x_1x_2x_3x_4x_5x_6x_7x_8x_9x_{10}x_{11}x_{12}x_{13}x_{14}x_{15}. $$

We can write the monomial $yc = m_1m_2m_3$, where the monomial $m_1$ corresponds to the cover

![Diagram](image)

The monomial $m_2$ corresponds to the cover

![Diagram](image)
Finally, the monomial $m_3$ corresponds to the cover

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REFERENCES


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